Geosci 343 Problem Set 1

January 28, 2009

Problem 1.1 Consider rigid body rotation in 2D, defined by the velocity field $(u, v) = (-\Omega y, \Omega x)$

a) Show that this flow is nondivergent

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Omega - \Omega = 0$$

b) Write down the Poisson equation which determines the pressure field which keeps this flow nondivergent.
Starting from the momentum equation from Newton's 2nd law:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} \cdot p \tag{1}$$

Writing this in terms of the given velocity field and remembering that the flow is nondivergent and time-independent

$$\Omega^2 x\hat{x} + \Omega^2 y\hat{y} = -\frac{1}{\rho}(\frac{\partial p}{\partial x}\hat{x} + \frac{\partial p}{\partial y}\hat{y}) \tag{2}$$

To obtain the Laplacian, take the divergence of equation (2) to obtain

$$2\Omega^2 \rho = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) \tag{3}$$

which is the Poisson equation, where the right hand side of (3) is the Laplacian in polar coordinates. Integrating (3) twice gives

$$p(r) = \frac{\Omega^2 \rho}{2} r^2 + p_0 \tag{4}$$

where we have set the constant of first integration equal to 0 because there is no pressure gradient at r=0, and the constant of second integration is the pressure at r=0 (p_0).

The centrifugal acceleration points radially outward and is proportional to the tangential velocity squared and inversely proportional to the radius ($\Omega^2 r$, where Ωr is the tangential velocity). This is equal and opposite to the pressure gradient force obtained from the radial derivative of equation (4) plugged into Newton's law (1). Therefore the pressure gradient force is what keeps the fluid moving in a circle.

Problem 1.2 The flow around a point vortex is given by the streamfunction $\psi = -(\Gamma/2\pi)\ln r$. Explicitly compute the circulation around a circle of radius R with center at the point (x_0, y_0). Show by explicit calculation that the circulation is zero if the circle does not contain the origin, and is Γ otherwise.

One approach is to write the streamfunction in Cartesian coordinates

$$\psi = \frac{\Gamma}{2\pi}\ln(\sqrt{x^2+y^2}) \tag{5}$$

The velocities become

$$\vec{v} = -\frac{\Gamma}{2\pi}\frac{y}{(x^2+y^2)}\hat{x} + \frac{\Gamma}{2\pi}\frac{x}{(x^2+y^2)}\hat{y} \tag{6}$$

The circulation is given by the line integral

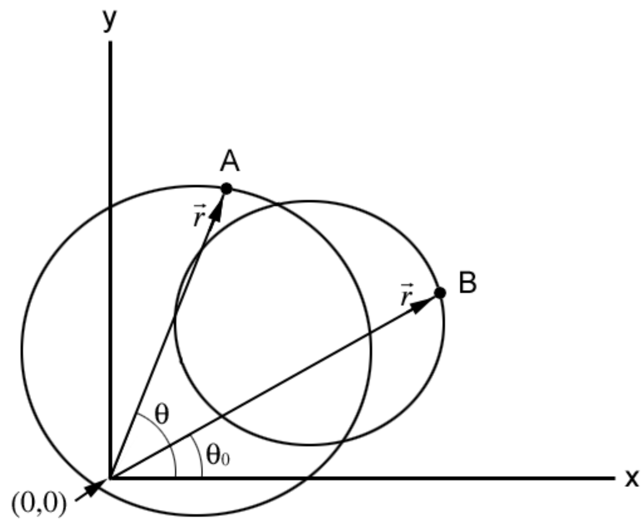
$$\oint \vec{v}\cdot d\vec{r} \tag{7}$$

where $d\vec{r}$ is the differential of curve length evaluated at a point (x,y) on any curve with continuous first derivatives. If we take the center of the point vortex at (0,0) then any position on the curve is given by the position vector $\vec{r} = x\hat{x} + y\hat{y}$. The differential length becomes $d\vec{r} = dx\hat{x} + dy\hat{y}$ and the line integral (7) becomes

$$\oint \vec{v}\cdot d\vec{r} = \oint(-\frac{\Gamma}{2\pi}\frac{y}{(x^2+y^2)}\hat{x} + \frac{\Gamma}{2\pi}\frac{x}{(x^2+y^2)}\hat{y})\cdot(dx\hat{x} + dy\hat{y}) \tag{8}$$

$$= \oint -\frac{\Gamma}{2\pi}\frac{xdy - ydx}{(x^2+y^2)} \tag{9}$$

An angle θ is formed between the position vector and the Cartesian x coordinate. The length of the position vector ($||\vec{r}||$, denoted as the scalar r here)



along with θ gives $x = r\cos\theta$ and $y = r\sin\theta$. For the differential length we have $dx = -r\sin\theta d\theta + dr\cos\theta d\theta$ and $dy = r\cos\theta d\theta + dr\sin\theta d\theta$. Using these relations and some minor algebra we can integrate equation (9) counter clockwise around the curve containing point B to obtain

$$\oint \vec{v}\cdot d\vec{r} = \int_{\theta_0}^{\theta_0} \frac{\Gamma}{2\pi} d\theta = \frac{\Gamma}{2\pi}(\theta_0 - \theta_0) = 0 \tag{10}$$

where the integration ends at the same angle as it started, so that there is no contribution to the circulation. For the integral around the curve containing point A we have

$$\oint \vec{v}\cdot d\vec{r} = \int_{0}^{2\pi} \frac{\Gamma}{2\pi} d\theta = \Gamma \tag{11}$$

because θ goes through one complete revolution. Note curve A and curve B could be circles centered at a given (x_0, y_0) but this is not necessary for the above argument.

Problem 1.3 Find the streamfunction and the velocity field which is associated with the vorticity field $\omega = cosx + cos10x$. Plot the vorticity, y-velocity and streamfunction on the same graph. What do you notice about the degree of smoothness of the three fields?

We can get the streamfunction from the vorticity field if the flow is non-divergent, by using

$$\nabla^2\psi = \omega$$

This equation is linear, so we can use the principle of superposition to first solve the homogeneous problem

$$\nabla^2\psi = 0 \tag{12}$$

and add the result to any particular solution for the given nonhomogeneous problem

$$\nabla^2\psi = cosx + cos10x \tag{13}$$

to arrive at a general solution to equation (2)

One way to solve the homogeneous problem (1) is to use separation of variables

$$\psi = f(x)g(y)$$

Substituting the trial solution into

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

gives

$$\frac{d^2f}{dx^2}g(y) + \frac{d^2g}{dy^2}f(x) = 0 \tag{14}$$

which requires

$$\frac{d^2f}{dx^2}\frac{1}{f(x)} = \lambda \tag{15}$$

$$\frac{d^2g}{dy^2}\frac{1}{g(y)} = -\lambda \tag{16}$$

where λ is a separation constant

Case 1) $\lambda = 0$

$$f(x) = A_1x^2 + B_1x + C_1$$

$$g(y) = A_2y^2 + B_2y + C_2$$

Case 2) $\lambda = \alpha^2 > 0$

$$f(x) = A_1e^{\alpha x}$$

$$g(y) = A_2 e^{\alpha y} + B_2 e^{-\alpha y}$$

Case 3) $\lambda = \alpha^2 < 0$

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$g(y) = A_2 e^{\alpha y}$$

However we know from equation (2) that ψ does not have exponential solutions, which leaves us with case 1, $\lambda = 0$, so we can use the solution to equation (12) given by

$$\psi = Axy + Bx + Cy \tag{17}$$

To find a particular solution to (13) we can note the form of the right hand side and try

$$\psi = -cosx - \frac{1}{100} cos10x$$

We add this particular solution to the solution for the homogeneous problem (equation 17) to obtain a general solution

$$\psi = -cosx - \frac{1}{100} cos10x + Axy + Bx + Cy \tag{18}$$

Problem 1.4 Show that the flow defined by the streamfunction $\psi = A(\cos kx \cos ly)$ is an exact steady solution to the 2D Euler equations...

There is no time dependence, so the 2D Euler equation becomes

$$\frac{\partial \omega}{\partial t} + (\vec{v} \cdot \vec{\nabla})\omega = (\vec{v} \cdot \vec{\nabla})\omega = 0 \tag{19}$$

Forming the vorticity from the streamfunction gives

$$\omega = \nabla^2 \psi = -Ak^2 \cos kx \cos ly - Al^2 \cos kx \cos ly \tag{20}$$

Forming the velocities

$$v = \frac{\partial \psi}{\partial x} = -Ak \sin kx \cos ly \tag{21}$$

$$u = \frac{-\partial \psi}{\partial y} = Al \cos kx \sin ly \tag{22}$$

Using equations (22,21,20) in equation(19) shows that the given streamfunction satisfies equation (19).

Problem 1.5 Consider three point vortices with identical circulation Γ, which are initially equally spaced on a line. Show that this configuration remains linear as time progresses, and rotates about the center vortex at constant angular velocity. Compute the rotation rate.

The circulation of each vortex is given by

$$\oint \vec{v} \cdot d\vec{r} = \Gamma \tag{23}$$

Integrating this around a circle of radius R is the same as multiplying the velocity (which is constant along a circle centered about the vortex) by the circumference. We can obtain the tangential velocity for a given vortex by dividing by the circumference

$$v_\theta = \frac{\Gamma}{2\pi R} \tag{24}$$

The velocity field is determined by the streamfunction and we can add streamfunction contributions from each point vortex. The velocity field is linear in streamfunction, so we can also add the contributions of each point vortex to the velocity. The tangential velocity is proportional to the y component of the velocity in Cartesian coordinates by a factor $\cos\theta$. So we know that the velocity is positive or negative depending on whether we are left $\cos(0)$ or right $\cos(-\pi)$ of the vortex as we move along the x axis. For three vortices equally spaced (by a distance R) on a line (say the x axis) we see that the tangential velocities from the left and right vortices cancel over the center vortex.

For the left vortex we add the negative contributions of the center and right vortices while keeping in mind the distance between the left and right vortices is twice the distance (2R) between the left and center vortices (R)

$$v = -\frac{\Gamma}{2\pi R} - \frac{\Gamma}{4\pi R} = -\frac{3\Gamma}{4\pi R} \tag{25}$$

For the right vortex we have

$$v = \frac{\Gamma}{2\pi R} + \frac{\Gamma}{4\pi R} = \frac{3\Gamma}{4\pi R} \tag{26}$$

So the vortices will spin counterclockwise about the center which is fixed in time. The angular velocity is given by the tangential velocity divided by R, so the rotation rate about the center vortex (in cycles per unit time) is

$$\frac{3\Gamma}{4\pi R^2} \tag{27}$$