Equations with separation of variables

To solve equations

$$y' = f(x)g(y) \tag{1}$$

and

$$M(x)N(y)dx + P(x)Q(y)dy = 0 \tag{2}$$

you need to find such an expression multiplying or dividing by which both sides you come to equation with x in one side and y in the other.

Example

$$x^2y^2y' + 1 = y \tag{3}$$

$$x^2y^2\frac{dy}{dx} = y - 1 \quad x^2y^2dy = (y-1)dx \tag{4}$$

Dividing both sides of the equation by $x^2(y-1)$ you get

$$\frac{y^2}{y-1}dy = \frac{dx}{x^2} \tag{5}$$

The variables are separated. Integrating both side of equation you get

$$\int \frac{y^2}{y-1}dy = \int \frac{dx}{x^2}; \quad \frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{x} + C \tag{6}$$

Equations like

$$y' = f(ax + by)$$

are reduced to equations with separation of variables by substitution

$$z = ax + by$$

or

$$z = ax + by + c$$

Homogeneous equations

Homogeneous equation can be written as

$$y' = \frac{y}{x} \tag{7}$$

or

$$M(x,y)dx + N(x,y)dy = 0, \tag{8}$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same order {Function $M(x,y)$ is homogeneous of order n, if $M(kx,ky) = k^n M(x,y)$ }. To solve this problem you make a substitution

$$y = tx \tag{9}$$

Example

$$xdy = (x+y)dx \tag{10}$$

This is homogeneous equation. Let

$$y = tx$$

then

$$dy = tdx + xdt \tag{11}$$

substituting

$$x(xdt + tdx) = (x + tx)dx; \quad xdt = dx \tag{12}$$

Solving this equation with separated variables

$$dt = \frac{dx}{x}; \quad t = ln|x| + C. \tag{13}$$

Coming back to the original variable y, you get

$$y = x\,(ln|x| + C) \tag{14}$$

Equations like

$$y' = f\left(\frac{a_1x + b_1y + c_1}{ax + by + c}\right) \tag{15}$$

are reduced to homogenous equations by moving the origin of coordinates to the intersection point of lines

$$ax + by + c = 0; \quad a_1x + b_1y + c = 0$$

If these lines do not intersect, then

$$a_1x + b_1y = k(ax + by),$$

hence these equations are

$$y' = F(ax + by). \tag{16}$$

It's reduced to an equation with separated variables by substitution

$$z = ax + by$$

or

$$z = ax + by + c$$

Some equations can be reduced to homogeneous by substitution

$$y = z^m$$

Usually m is known. In order to find it, you make substitution $y = z^m$. The requirement of homogeneity of the equation gives you m, if it exists. If it doesn't exist, then you can't reduce the equation to homogeneous.

Example

$$2x^4yy' + y^4 = 4x^6 \tag{17}$$

Let's make a substitution

$$y = z^m,$$

then the eq(17) is

$$2mx^4yy' + y^4z^{2m-1}z' + z^(4m) = 4x^6 \tag{18}$$

This equation is homogeneous if orders of all terms are equal, *e.i.*

$$4 + (2m - 1) = 4m = 6.$$

That could be satisfied if $m = \frac{3}{2}$. Hence, you can reduce (17) to homogeneous equation by substitution

$$y = z^{3/2}$$

First order linear equations

An equation is linear if it has form

$$y' + a(x)y = b(x) \tag{19}$$

To solve it, you need to solve

$$y' + a(x) = 0. \tag{20}$$

first. Then replace a constant by unknown function $C(x)$, and substitute found solution into (20).
Some equations become linear if you interchange $y(x)$ with independent variable x.

Example

$$y(x) = \left(2x + y(x)^3\right) y(x)' \tag{21}$$

Let's rewrite it like

$$ydx - (2x + y^3)dy = 0. \tag{22}$$

Since x and dx are linear, then the equation is linear, if you look for solution $x(y)$. Then

$$\frac{dx}{dy} - \frac{2}{y}x = y^2 \tag{23}$$

This equation is solved similar to (19).

Bernoulli equation

$$y' + a(x)y = b(x)y^n \tag{24}$$

You have to divide both sides by y^n and make substitution

$$1/y^{n-1} = z$$

After that you get a linear equation which you already know how to solve.

Linear equations with constant coefficients

Homogeneous equation

To solve equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0 \tag{25}$$

construct the characteristic equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n = 0 \tag{26}$$

and find all his roots. The general solution of (30) is

$$y = \sum_{k=1}^{k=N} C_k e^{\lambda_i x} + \sum_{p=1}^{p=M} \left(C_{p+1} + C_{p+2} x + C_{p+3} x^2 + \ldots + C_{p+m} x^{m-1} \right) e^{\lambda x} \tag{27}$$

where k is a single root and p is a multiple root of order m.

Example

$$y^V - 2y^{VI} - 16y' + 32y = 0 \tag{28}$$

The characteristic equation is

$$\lambda^5 - 2\lambda^4 - 16\lambda + 32 = 0. \tag{29}$$

Five roots of this equation are

$$\lambda_1 = \lambda_2 = 2, \quad \lambda_3 = -2, \quad \lambda_4 = 2i, \quad \lambda_5 = -2i.$$

Hence, the solution for (28) is

$$y = (C_1 + C_2 x) e^{2x} + C_3 e^{-2x} + C_4 \cos 2x + C_5 \sin 2x.$$

Nonhomogeneous linear equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = f(x). \tag{30}$$

A solution of this equation with any $f(x)$ is found by method of constant variation. First, find a solution of corresponding homogeneous equation (30), and replace your unknown constants by functions of x

$$y = C_1(x) + y_1 + \ldots + C_n(x) y_n$$

. Unknown functions $C_i(x)$ are defined from the system of equation

$$\begin{aligned}
C_1' y + \ldots + C_n' y_n &= 0 \\
C_1' y' + \ldots + C_n' y_n' &= 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
C_1' y_1^{(n-2)} + \ldots + C_n' y_n^{(n-2)} &= 0 \\
a_0 \left(C_1' y_1^{(n-1)} + \ldots + C_n' y_n^{(n-1)} \right) &= 0
\end{aligned}$$