Autumn Quarter 2005
Math. Methods Problem Set 4: ODE's in 1 dimension

November 30, 2005

1 Approach to radiative equilibrium

For a complete solution in full detail, see the separate solution writeup for this problem, and the accompanying Python script `radcool.py`. *Here I'll just provide answers to the simpler parts of the problem.*

The temperature of a planet heated by constant Solar radiation S, and subject to cooling by infrared radiation to space, is governed by

$$M\frac{dT}{dt} = S - \sigma T^4 \tag{1}$$

where M is a constant "thermal inertia" factor and σ is the Stefan-Boltzmann constant. Answer the following:

- Find the equilibrium point(s) (steady states) of the system
- Determine the stability of the equilibria
- Find an approximate form of the solution if T is initially much smaller than the equilibrium temperature (but still positive), valid during the time over which the temperature remains small.
- Find an approximate form of the solution if T is initially much larger than the equilibrium temperature, valid during the time before which the temperature gets too close to the equilibrium

- Using a few qualitative sketches, explain what you think the general behavior of the system is like.

Solution: The equilibrium point(s) are determined by setting the time derivative to zero. There is only one (real) equilibrium, which is $T_o = (S/\sigma)^{1/4}$. To get the stability we write $T = T_o + T'$ where T' is assumed small. Substituting into the equation and keeping only the linear term in T' we find

$$M\frac{dT'}{dt} = S - \sigma(T_o + T')^4 \approx S - \sigma T_o^4 - 4\sigma T_o^3 T' = -4\sigma T_o^3 T' \qquad (2)$$

Thus, the equilibrium is stable, with a time constant given by $M/4\sigma T_o^3$.

When T is very small, $\sigma T^4 \ll S$ so the equation becomes simply

$$M\frac{dT}{dt} = S \qquad (3)$$

and the temperature grows linearly.

When T is very large, $\sigma T^4 \gg S$ so the equation becomes simply

$$M\frac{dT}{dt} = -\sigma T^4 \qquad (4)$$

which has the solution derived in class (with temperature decaying like $1/t^{1/3}$ at large times).

So, to put it all together we know that:

- If the system is started near its equilibrium, it approaches the equilibrium exponentially.
- If the system is started at temperature much larger than the equilibrium, it will decay like $1/t^{1/3}$ until it starts to get close to the equilibrium, whereafter it will approach the equilibrium exponentially.
- If the system is started at a temperature much smaller than the equilibrium, the temperature will grow linearly until it starts to get close to the equilibrium, whereafter it approaches the equilibrium exponentially.

For sketches, see the write-up of the complete solution, and the accompanying Python script.

2 Condorcet's equation

Discuss the behavior of the solutions to the equation

$$\frac{dP}{dt} = g_o P(1+P) \tag{5}$$

assuming that P is initially positive. What is the approximate form of the solution when $P \ll 1$? How does the system behave when $g_o > 0$? When $g_o < 0$?

Solution: We solve this just like we did the logistic equation:

$$\int \frac{dP}{P(1+P)} = g_0 t + const. \tag{6}$$

Then, using partial fractions $1/(P(1+P)) = 1/P - 1/(1+P)$, so

$$\ln P - \ln{(1+P)} = g_o t + const. \tag{7}$$

or

$$\frac{P}{1+P} = \frac{P_o}{1+P_o} e^{g_o t} \tag{8}$$

where P_o is the initial value of P. If $g_o < 0$ then the right hand side decays to zero and so P ultimately decays exponentially to zero as t gets large. This is true whatever the initial value of P, so long as it is non-negative. On the other hand, if $g_o > 0$ then the right hand side approaches 1 at the time when $\exp g_o t = (1+P_o)/P_o$. The only way the equation can be satisfied is if P becomes infinite at this time. Thus, Condorcet's equation has finite-time blowup if the growth rate is positive.

Now for the approximate form of the solutions. For $P \ll 1$ the quadratic term is negligable and the equation becomes simply

$$\frac{dP}{dt} = g_o P \tag{9}$$

which exhibits exponential growth or decay, according to whether g_o is positive or negative. In the exponentially growing case,P will eventually get large enough that the quadratic term is no longer negligable. Then, the nonlinear equation must be used, and we get the finite-time blowup again. On the other hand, in the exponentially decaying case, P gets smaller and smaller, and the linearized approximation remains valid. You can easily verify that the exact solution has the same behavior in the limit of small P.

When P is very large, it is the quadratic term that dominates, and the equation becomes

$$\frac{dP}{dt} = g_o P^2 \tag{10}$$

We already analyzed this equation in class. If $g_o > 0$, then P stays large and in fact becomes infinite when $g_o t = 1/P_o$. This is the same result as we got for the exact solution in the limit of large P_o (you need to Taylor-expand the exponential in t to see this; t is small if P_o is large). Now, if $g_o < 0$, the approximate solution instead decays like $P_o/(P_o - g_o t)$. Eventually it will get small enough that the linear term is no longer negligable in comparison to the quadratic term. At that point the exact solution starts to look different from the approximate solution, because the former decays exponentially while the latter decays much more slowly (like $1/t$).

3 Logistic equation with time-varying growth rate

Find the solution to the equation

$$\frac{dP}{dt} = g_o(t)P(1-P) \tag{11}$$

where $g_o(t) = g_1 \cdot (1 + \sin \omega t)$

Solution: Following the solution technique for the Logistic equation, we write

$$\int \frac{dP}{P(1-P)} = \int g_o(t)dt = g_1(t - \omega^{-1}\cos(\omega t)) + const. \tag{12}$$

The left hand side is integrated by partial fractions exactly as we did for the Logistic equation. If we let $h(t) = g_1(t - \omega^{-1}\cos(\omega t))$ then

$$\frac{P}{1-P} = \frac{P_o}{1-P_o} e^{h(t)} \tag{13}$$

which you can easily solve for $P(t)$ if you want. If $g_1 > 0$, then $h(t)$ becomes infinite at large t and $P \to 1$ as for the usual Logistic equation. In fact, since $h(t) \approx g_1 t$ for large t, the approach to the stable equilibrium is just the same as in the case of a constant growth rate. The periods of stronger than average growth cancel out the effects of the periods of weaker than average growth.

4 Programming Project: A polynomial object

See the solution script `PolynomialClass.py` to see how to define the polynomial object. The class defined in this script supports multiplication and addition (including multiplication and addition by numbers), as well as evaluation.