Normal Modes

Normal Modes occur as solutions to coupled linear differential equations, and are *defined* as a solution for the variables $\mathbf{x}$ in which all variables have the same dependence $f(t)$:

$$\boxed{\mathbf{x} = f(t)\,\mathbf{c}. \quad \textit{Definition of a Normal Mode Solution} \qquad \text{(A)}}$$

Here $\mathbf{c}$ is a constant vector — in other words: all the variables that make up the vector $\mathbf{x}$ vary in the *same* way with time, but with different amplitudes given by the corresponding component of $\mathbf{c}$. Note that box (A) only defines $f(t)$ up to a constant: any constant factor can be moved from the definition of $f(t)$ into the definition of $\mathbf{c}$. It follows that we are free to select a normalisation for $\mathbf{c}$. Since the role of $\mathbf{c}$ is to solely to define the relative amplitudes of oscillation of the different masses, the only sensible normalisation is $\mathbf{c}^T\mathbf{c} = 1$, and we shall adopt that throughout. Furthermore, since the elements of $\mathbf{c}$ are pure dimensionless numbers, it is the function $f(t)$ which gives the solution its required dimension of length.[1]

We shall be looking at examples of coupled *second order* differential equations, and in these cases the function $f(t)$ is a sine or cosine function, but similar techniques can be applied to *first order* systems, where $f(t)$ would be an exponential.

1 Setting up the problem.

We shall use two sample systems as examples. The first involves two equal masses m oscillating horizontally, with equal springs k joining them to fixed side walls, and a coupling spring k' between them (see Figure 1). We measure both displacements from equilibrium in the same direction: from the position of mass 1 towards the position of mass 2. Thus a displacement x_1 lengthens the side spring and compresses the coupling spring, whereas a displacement x_2 lengthens the coupling spring and compresses the side spring. Then the equations of motion are

$$m\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = -kx_1 + k'(x_2 - x_1) \qquad (1)$$

$$m\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = -kx_2 + k'(x_1 - x_2). \qquad (2)$$

If we write x_1 and x_2 as elements of a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then we can re-write the equations of motion as

[1] This is not always possible: for example it is conceivable that the variables themselves do not all have the same dimensions, so that their ratios cannot be pure numbers.

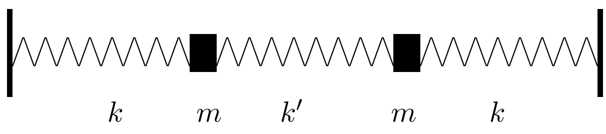

Figure 1: First Normal Modes Example

$$\boxed{m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x} \quad \textit{Normal Mode Problem: Equal Mass Case} \qquad \text{(B)}}$$

where the matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}.$$

The second example is two equal masses hanging on two springs in the order (from the top) spring k_1, mass m, spring k_2, mass m (see Figure 2). We have two choices about the origin of displacements; the simplest is to leave the system to settle under gravity, so that the weights of the two masses are balanced by the forces exerted by the springs, and then measure the displacements from that sagged position. The forces that appear are then the additional forces, over and above those necessary to balance the weights. (This is in fact exactly the approach we took with the horizontal spring system, where the three springs may well be under tension in the $\mathbf{x} = 0$ position, but in this case the spring forces on each mass cancel out.) However an alternative is to measure the displacements from the unstretched position.

Taking the simpler option of measuring from the sagged position, a diplacement x_1 extends spring k_1 and compresses spring k_2, whereas an extension x_2 extends spring k_2. Thus the equations of motion are

$$m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -k_1 x_1 + k_2(x_2 - x_1) \qquad (3)$$

$$m\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = k_2(x_1 - x_2). \qquad (4)$$

We can write this in the same form as box (B):

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x}$$

where now the matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}. \qquad (5)$$

If instead we measure displacements from the unstretched spring positions then we must include the weights in the equations of motion:

$$m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -k_1 x_1 + k_2(x_2 - x_1) + mg \qquad (6)$$

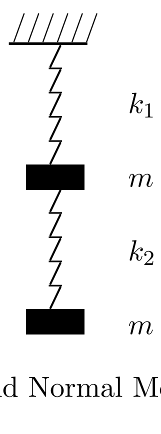

Figure 2: Second Normal Modes Example

$$m\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = k_2(x_1 - x_2) + mg, \tag{7}$$

which we can write as

$$\boxed{m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x} + \mathbf{w} \quad \textit{Inhomogeneous Normal Mode Problem} \qquad \text{(C)}}$$

where the weight vector is

$$\mathbf{w} = \begin{pmatrix} mg \\ mg \end{pmatrix}.$$

We shall return to this formulation in section 5. We have thus cast both of these normal mode problems in the same form as each other, and in the same form as a simple mass on a spring problem

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx$$

with the single variable x replaced by a vector $\mathbf{x}$ and the spring constant k replaced by a matrix $\mathbf{K}$. We shall solve this equation in section 3, after introducing the relevant mathematics in the folllowing section.

We first note some features of the matrix $\mathbf{K}$. In both cases the matrix is *symmetric*: $\mathbf{K} = \mathbf{K}^T$. This is generally true: in fact it is *always* true in any normal mode problem where the forces can be expressed as the gradients of a potential energy function, which covers almost every case except those where magnetic forces are present. We can thus use this feature to structure our solution. However the first matrix $\mathbf{K}$ has an additional symmetry not posessed by the second: if we write the components of $\mathbf{K}$ in the standard way

$$\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

then in addition to the symmetry $k_{12} = k_{21}$ noted above, the first $\mathbf{K}$ matrix has the additional symmetry $k_{11} = k_{22}$, which the second one lacks. We shall see that this affects what other options are available to us apart from the normal mode method of solution, but does not affect the normal mode method at all.

2 Eigenvalues and Eigenvectors

2.1 Definitions

For any square matrix $\mathbf{V}$ we define the eigenvalue λ and right eigenvector $\mathbf{c}$ by

$$\boxed{\mathbf{V}\mathbf{c} = \lambda\mathbf{c}. \quad \textit{Right Eigenvector Definition} \qquad \text{(D)}}$$

The vector $\mathbf{c}$ is clearly special with respect to $\mathbf{V}$, because when we multiply it by $\mathbf{V}$ we get a vector which is *parallel to* $\mathbf{c}$, whereas in general the resulting vector would not be. However, note that any multiple of $\mathbf{c}$, $\mathbf{c}' = \alpha\mathbf{c}$ is also an eigenvector with the same eigenvalue. Thus the eigenvectors of $\mathbf{V}$ define *directions* in $\mathbf{x}$-space, but have *no natural length.* The $\mathbf{c}$ vector introduced in box (A) will turn out to be an eigenvector, but the normalisation we have adopted for it follows from its use and not because it is an eigenvector.

In similar fashion we define the left eigenvector $\mathbf{d}$ by

$$\boxed{\mathbf{d}^T\mathbf{V} = \lambda\mathbf{d}^T. \quad \textit{Left Eigenvector Definition} \qquad \text{(E)}}$$

In each case the eigenvalues are defined by the *characteristic equation:*

$$\boxed{\det|\mathbf{V} - \lambda\mathbf{I}| = 0 \quad \textit{Characteristic Equation} \qquad \text{(F)}}$$

which shows that both sets of eigenvectors have a common set of eigenvalues. All square matrices possess at least one such eigenvalue and left and right eigenvectors.

2.2 Orthogonality

Eigenvectors have certain important orthogonality properties: consider the construct $\mathbf{d}_i^T\mathbf{V}\mathbf{c}_j$ for vectors belonging to different eigenvalues λ_i, λ_j. We can use either the left or right eigenvector definition to deduce

$$\mathbf{d}_i^T\mathbf{V}\mathbf{c}_j = \lambda_i\mathbf{d}_i^T\mathbf{c}_j = \lambda_j\mathbf{d}_i^T\mathbf{c}_j$$

and hence $(\lambda_i - \lambda_j)\mathbf{d}_i^T\mathbf{c}_j = 0$. Thus left eigenvectors are orthogonal to all right eigenvectors belonging to different eigenvalues:

$$\mathbf{d}_i^T\mathbf{c}_j = 0 \quad \text{if} \quad i \neq j. \tag{8}$$

While I have stressed that eigenvectors have no natural length, it is almost always convenient to use a relative normalisation of the left and right eigenvectors such that the inner product of $\mathbf{d}_i$ with the corresponding $\mathbf{c}_i$ is one. With this choice we can summarise the orthogonality relations as

$$\boxed{\mathbf{d}_i^T\mathbf{c}_j = \delta_{ij} \quad \textit{Dual Vectors.} \qquad \text{(G)}}$$

This set of inner products is the defining property of the $\mathbf{d}_i$ as *dual vectors* to the $\mathbf{c}_i$.

2.3 Special properties of a symmetric matrix

If $\mathbf{V}$ is a symmetric matrix, such as $\mathbf{K}$, then we can prove a number of additional properties.

(a) **Identity of left and right eigenvectors.** Taking the transpose of the *left* eigenvector definition, box (E), we obtain

$$\mathbf{K}\mathbf{d} = \lambda\mathbf{d},$$

which is the *right* eigenvector definition. Thus the first special property of a symmetric matrix is that *the left and right eigenvectors are identical*:

$$\mathbf{d}_i = \mathbf{c}_i. \tag{9}$$

(b) **Mutual orthogonality of eigenvectors.** The identity of left and right eigenvectors, together with the dual property of eigenvectors given in box (G), give us the second special property of a symmetric matrix: *the eigenvectors are all orthogonal.*

$$\mathbf{c}_i^T\mathbf{c}_j = \delta_{ij}.$$

(c) **Reality of eigenvalues.** The third special property of a real symmetric matrix is that *the eigenvalues are all real.* (Note that, for an $N \times N$ matrix, the characteristic equation, box (F), is an N'th order polynomial, which in general may have either real or complex roots.) To prove this, consider the construct $\mathbf{c}_i^\dagger \mathbf{K} \mathbf{c}_i$. This can be evaluated by using either the right eigenvector definition, or the complex conjugate of the left eigenvector equation $\mathbf{c}_i^\dagger \mathbf{K}^\star = \lambda_i^\star \mathbf{c}_i^\dagger = \mathbf{c}_i^\dagger \mathbf{K}$ since $\mathbf{K}$ is real. These alternatives give

$$\mathbf{c}_i^\dagger \mathbf{K} \mathbf{c}_i = \lambda_i \mathbf{c}_i^\dagger \mathbf{c}_i = \lambda_i^\star \mathbf{c}_i^\dagger \mathbf{c}_i$$

so that $(\lambda_i - \lambda_i^\star)\mathbf{c}_i^\dagger \mathbf{c}_i = 0$. The product of vectors is the sum of the square moduli of the elements, and thus positive definite, so that $\lambda_i = \lambda_i^\star$: λ_i *is real.*

Since $\mathbf{K}$ is real and λ is real, we are free to choose real eigenvectors $\mathbf{c}$.

(d) **Completeness.** The fourth and final special property of a symmetric matrix is that *the eigenvectors form a complete set*; that is, for an $N \times N$ matrix there are N of them. This sounds obvious, until we consider that the characteristic equation in box (F) may have repeated roots. In such a case we require multiple eigenvectors to belong to this eigenvalue, and while this is exactly what happens for a symmetric matrix, it cannot be guaranteed in general. Suppose we have a repeated root, and have found only $N - 1$ eigenvectors $\mathbf{c}_1 \ldots \mathbf{c}_{N-1}$. We then complete the set by adding the unique vector $\mathbf{f}$ which is orthogonal to all the $\mathbf{c}$: $\mathbf{f}^T \mathbf{c}_i = 0$. Now consider the construct $\mathbf{c}_i^T \mathbf{K} \mathbf{f}$. Using the left eigenvector relation we deduce

$$\mathbf{c}_i^T \mathbf{K} \mathbf{f} = \lambda_i \mathbf{c}_i^T \mathbf{f} = 0.$$

This shows that the vector $\mathbf{K}\mathbf{f}$ is orthogonal to all the $\mathbf{c}_i$, and hence can only consist of a multiple of $\mathbf{f}$: in other words $\mathbf{f}$ is the missing eigenvector. This argument can generalised to any number of missing eigenvectors. Thus we can always find a complete set of eigenvectors: $L = N$.

3 Solution of the Normal Modes Equation

3.1 Methods of Solution

We noted above the structural similarity of the equation of motion in box (B) to the equation for a mass on a spring, and we use this similarity to suggest the next step: divide by m. For the mass on a spring problem this gives

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m} x = -\omega^2 x.$$

This is often termed the SHM equation, since it is the defining equation of simple harmonic motion, independent of the specific details (k and m in this case) of the oscillating system.

We apply this approach to the coupled equations of motion in box (B):

$$\boxed{\frac{\mathrm{d}^2 \mathbf{x}}{\mathrm{d}t^2} = -\frac{1}{m}\mathbf{K}\mathbf{x} = -\mathbf{G}\mathbf{x}. \quad \textit{Eq. of Motion: SHM form.} \qquad \text{(H)}}$$

The product of the scalar $1/m$ and the matrix $\mathbf{K}$ produces a matrix in which every element is multiplied by $1/m$, which we define to be $\mathbf{G}$. The second step of defining it to be ω^2 is not so straightforward since ω would have to be the square root of $\mathbf{G}$, so we leave the equation with $\mathbf{G}$ in it.

This represents a system of N coupled linear homogeneous second order differential equations. We shall restict ourselves to $N = 2$, but the generalisation to arbitrary N should be obvious. We expect the solution to consist of a sum of $2N = 4$ functions each with an arbitrary multiplicative constant. There are (at least) three solution methods:

(a) Elimination;

(b) Uncoupling, or Normal Coordinates;

(c) Normal Modes;

and we shall discuss the first two briefly before returning to the third, Normal Modes.

Elimination. We shall take the second example, equations (3) and (4). Re-arranging equation (4) we get

$$x_1 = x_2 + \frac{m}{k_2}\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2}, \tag{10}$$

which we can differentiate to give

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \frac{m}{k_2}\frac{\mathrm{d}^4 x_2}{\mathrm{d}t^4}. \tag{11}$$

These allow us to eliminate x_1 in favour of x_2. Substituting into equation (3) we get

$$\frac{\mathrm{d}^4 x_2}{\mathrm{d}t^4} + \left(2\frac{k_2}{m} + \frac{k_1}{m}\right)\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \frac{k_1 k_2}{m^2} x_2 = 0.v \tag{12}$$

This is now a fourth-order linear differential equation which we can solve in the usual way to give four functions each with a multiplicative constant. We can then complete the solution by using equation (10) to find x_1. This approach clearly gives an answer in the expected form, but is fairly lengthy, and gives no insight into the problem whatever.

Normal Co-ordinates. These are defined to be linear combinations of the co-ordinates x_1 and x_2 which obey uncoupled equations. In the case of a $\mathbf{K}$ matrix with the extra symmetry $k_{11} = k_{22}$ referred to in section 1 they are easy to find. We therefore consider the first example, equations (1) and (2). If we add these we obtain

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1 + x_2) = -k(x_1 + x_2);$$

while if we subtract them we obtain

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1 - x_2) = -(k + 2k')(x_1 - x_2).$$

Thus the linear combinations $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$ are normal co-ordinates, satisfing uncoupled equations that we can solve in the usual way. We can then invert the relations and find x_1 and x_2 in terms of q_1 and q_2. If the factors (in this case $1, 1$ and $1, -1$) with which the original equations have to be combined are obvious, then this method is very straightforward.

It also gives more insight than the elimination: we learn that certain *combinations* of the co-ordinates perform simple harmonic motion. However, although the factors for forming the normal co-ordinates always exist, they cannot usually be spotted easily — try spotting them for the second example problem! Thus the method of choice in general is the third alternative, to which we now turn.

3.2 Normal Mode Solution.

This method begins by simply substituting the normal mode form, box (A), into the SHM form of the equation of motion, box (H):

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\mathbf{c} = -f(t)\mathbf{G}\mathbf{c}.$$

Requiring that f is not always zero (a modest requirement for a non-trivial solution!) we can divide by f to give

$$\left(\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right)\mathbf{c} = -\mathbf{G}\mathbf{c}. \tag{13}$$

We now observe that the right side of this equation is constant, not a function of t. Thus the bracket on the left-hand side must also be constant, and we call this constant $-\lambda$. Thus we have

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -\lambda f \quad \text{and} \quad \mathbf{G}\mathbf{c} = \lambda\mathbf{c}. \tag{14}$$

The right-hand equation tells us that λ is an eigenvalue of $\mathbf{G}$ and $\mathbf{c}$ is the corresponding eigenvector. The theory of the last section immediately tells us that there are two eigenvalues λ_i, with orthogonal eigenvectors $\mathbf{c}_i$. Thus we have found not just one but two solutions in the specified form, and the linearity of the equations of motion allows us to superpose them with arbitrary constants.

The type of solution f_i depends on the value of λ_i: the left-hand equation tells us that f_i is oscillatory (complex-exponential or trignometric) if $\lambda_i > 0$, real-exponential or hyperbolic if $\lambda_i < 0$, and linear if $\lambda_i = 0$. Of course we do not expect unbounded motion to occur in these problems, so we expect only oscillatory solutions, and we shall therefore write $\lambda_i = \omega_i^2$. Thus each f_i is given by

$$f_i(t) = A_i \cos(\omega_i t) + B_i \sin(\omega_i t).$$

Thus the full solution can be written

$$\mathbf{x}(t) = \Big(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t\Big)\mathbf{c}_1 + \Big(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t\Big)\mathbf{c}_2 \tag{15}$$

containing, as expected, four constants. These can be determined by initial conditions, the positions $\mathbf{x}(0)$ and velocities $\mathbf{v}(0)$ at $t = 0$. Evaluating equation (15) at $t = 0$ we find

$$\mathbf{x}(0) = A_1\mathbf{c}_1 + A_2\mathbf{c}_2. \tag{16}$$

At this point the dual vectors $\mathbf{d}_i$ come into their own: if we dot equation (16) with one of them, a single term on the right-hand side is picked out, and the dot product for that term is just one:

$$\mathbf{d}_1^T\mathbf{x}(0) = A_1.$$

Similarly the velocity at $t = 0$ is given by

$$\mathbf{v}(0) = B_1\omega_1\mathbf{c}_1 + B_2\omega_2\mathbf{c}_2$$

which we can solve in a similar fashion. For example if we dot with $\mathbf{d}_1$:

$$\mathbf{d}_1^T\mathbf{v}(0) = B_1\omega_1.$$

This gives a complete solution in terms of the initial conditions for all normal mode problems:

$$\boxed{\mathbf{x}(t) = \sum_{\text{modes } i}\left[[\mathbf{d}_i^T\mathbf{x}(0)]\,\cos(\omega_i t) + [\mathbf{d}_i^T\mathbf{v}(0)/\omega_i]\,\sin(\omega_i t)\right]\mathbf{c}_i. \quad \textit{Complete Solution} \qquad \text{(I)}}$$

3.3 Normal Co-ordinates re-visited.

The method of normal co-ordinates introduced above is, naturally, related to the normal mode method. If we take the SHM equation of motion from box (H) and dot it with one of the left eigenvectors $\mathbf{d}_i$ we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{d}_i^T\mathbf{x} = -\mathbf{d}_i^T\mathbf{G}\mathbf{x}.$$

On the right-hand side we use the left eigenvector equation from box (E) to give

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{d}_i^T\mathbf{x} = -\lambda_i\mathbf{d}_i^T\mathbf{x} = -\omega_i^2\mathbf{d}_i^T\mathbf{x}. \tag{17}$$

Thus, for every i the linear combination $\mathbf{d}_i^T\mathbf{x}$ satifies an uncoupled equation: *the products* $q_i = \mathbf{d}_i^T\mathbf{x}$ *are normal co-ordinates.* We also note that the equation satified by q_i is the SHM equation.

This by no means exhausts the meaning of the q_i. In the discussion preceding box (I) above we found that the initial position vector can be written $\mathbf{x}(0) = \sum_i[\mathbf{d}_i^T\mathbf{x}(0)]\mathbf{c}_i$, but this is obviously true for $\mathbf{x}$ at any time:

$$\boxed{\mathbf{x}(t) = \sum_{\text{modes } i}\left[\mathbf{d}_i^T\mathbf{x}(t)\right]\mathbf{c}_i = \sum_{\text{modes } i} q_i(t)\mathbf{c}_i. \quad \textit{Modal decomposition of } \mathbf{x} \qquad \text{(J)}}$$

Thus the $q_i(t)$ are the components of $\mathbf{x}(t)$ in the $\mathbf{c}$-basis.

Finally, comparing this with the opening equation in box (A) we see that the normal co-ordinates $q_i(t)$, viewed as functions of time, are in fact the functions $f_i(t)$ in the normal mode definition.

4 Energy in the Equal Mass case.

4.1 Kinetic Energy.

We are still dealing with the rather special case when all the oscillating masses are equal. The final topic to be looked at is the energy of the oscillating system. The kinetic energy of the oscillating masses m is obviously

$$T = \sum_i \frac{1}{2}mv_i^2.$$

We can write this as a matrix product using the velocity vector $\mathbf{v}$:

$$T = \frac{1}{2}m\mathbf{v}^T\mathbf{v}. \tag{18}$$

4.2 Potential Energy.

We can derive the forces from a potential as long as the stiffness matrix $\mathbf{K}$ is symmetric. We shall find the potential energy at extension $\mathbf{x}$ by integrating the work done to get to $\mathbf{x}$ from zero extension (also the zero of potential energy) along a path $\mathbf{y} = \alpha\mathbf{x}$ from $\alpha = 0$ to $\alpha = 1$. The forces exerted by the mechanical system at extension $\mathbf{y}$ are given by $\mathbf{f} = -\mathbf{K}\mathbf{y}$; thus the external source of energy must supply forces $\mathbf{f}_{\mathrm{ext}} = +\mathbf{K}\mathbf{y}$. The work done is found by integrating $\mathbf{f}_{\mathrm{ext}} \cdot \mathrm{d}\mathbf{y}$:

$$V(\mathbf{x}) = \int_0^{\mathbf{x}} (\mathbf{K}\mathbf{y}) \cdot \mathrm{d}\mathbf{y} = \mathbf{x}^T\mathbf{K}\mathbf{x} \int_0^1 \alpha\, d\alpha = \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}.$$

If we now differentiate to find the forces (force on mass i given by $-\partial V/\partial \mathbf{x}_i$) we find

$$\mathbf{f} = -\frac{1}{2}\left(\mathbf{K}\mathbf{x} + \mathbf{K}^T\mathbf{x}\right) = -\mathbf{K}\mathbf{x}$$

provided $\mathbf{K}$ is symmetric.

4.3 Conservation of Energy.

Putting these two results together we find the energy is given by

$$\boxed{E = \frac{1}{2}\Big(m\mathbf{v}^T\mathbf{v} + \mathbf{x}^T\mathbf{K}\mathbf{x}\Big). \qquad \text{(K)}}$$

Differentiating with respect to t:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\mathbf{v}^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}^T\mathbf{K}\mathbf{x} = \mathbf{v}^T\Big(m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{K}\mathbf{x}\Big).$$

However the equation of motion in box (B) ensures that the bracket is zero and so energy is conserved

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$

4.4 Division of the Energy between Normal Modes

If we substitute the decomposition of $\mathbf{x}$ into normal coordinates from box (J) into box (K) we can find how the energy is divided between the normal modes:

$$E = \frac{1}{2}\sum_{\text{modes } i}\sum_{\text{modes } j}\Big(m\dot{q}_i\mathbf{c}_i^T\mathbf{c}_j\dot{q}_j + q_i\mathbf{c}_i^T\mathbf{K}\mathbf{c}_j q_j\Big)$$

where we have used the dot notation for compactness to denote $\mathrm{d}q/\mathrm{d}t$. In the second term we would like to use the eigenvector equation to simplify $\mathbf{K}\mathbf{c}_j$. However, the $\mathbf{c}$ have been defined as eigenvectors of G, not $\mathbf{K}$. Fortunately, it is easy to show that $\mathbf{K}$ has the same eigenvectors as $\mathbf{G}$. We start from

$$\mathbf{G}\mathbf{c} = \lambda\mathbf{c}$$

and multiply by m

$$\mathbf{K}\mathbf{c} = m\lambda\mathbf{c}.$$

Thus **K** has the same eigenvectors as **G**, but the eigenvalues are all multiplied by m. Using the right eigenvector definition from box (D) we obtain

$$\mathbf{c}_i^T \mathbf{K} \mathbf{c}_j = m\lambda_j \mathbf{c}_i^T \mathbf{c}_j = m\omega_j^2 \mathbf{c}_i^T \mathbf{c}_j.$$

Both terms now contain the dot product of two eigenvectors, $\mathbf{c}_i^T \mathbf{c}_j$, which is just δ_{ij}. The effect of this Kronecker delta is that all the cross terms that connect different normal modes are zero. Thus the energy splits up into a sum of modal energies:

$$E = \sum_{\text{modes } i} E_i = \sum_{\text{modes } i} \frac{m}{2}\left[(\dot{q}_i)^2 + \frac{1}{2}\omega_i^2 q_i^2\right]. \tag{19}$$

There are two things to be said about this expression for the modal energy.

(a) The modal energies E_i, as well as adding up to the constant total energy E, are themselves constant. If we differentiate E_i we obtain

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} = m\dot{q}_i\left[\ddot{q}_i + \omega_i^2 q_i\right].$$

The equation of motion for q_i, equation (17), ensures that the expression in the square brackets is zero, so E_i is conserved.

(b) The expression for E_i has the form of a harmonic oscillator energy, so that we have decomposed the system into a set of non-interacting harmonic oscillators. In order to interpret the two terms in equation (19) as the potential and kinetic energy of a harmonic oscillator we can identify q_i as the displacement, m as the mass, $m\lambda_i$ (the eigenvalue of the **K** matrix) as the spring constant, and consequently ω_i as the angular frequency.

5 Inhomogeneous Problems.

In this section we look briefly at two problems giving rise to inhomogeneous equations, as opposed to the homogeneous linear systems we have solved so far.

5.1 Sagging under gravity.

We briefly return to the inhomogeneous problem of box (C) which arises when the displacements of the masses are measured from the zero extension position of the springs. As we stressed above, the standard problem is a homogeneous linear system

$$\mathcal{L}f = 0$$

where $\mathcal{L}$ is the linear operator $m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \mathbf{K}$, and f is the vector-valued function $\mathbf{x}(t)$. What we have now is the corresponding inhomogeneous problem

$$\mathcal{L}f = h$$

where the inhomogeneous term is the weight vector $\mathbf{w}$. The solution is thus given by any particular integral plus any solution of the homogeneous problem. Thus to the solutions we have already found we have to add the simplest possible solution of the inhomogeneous problem.

However since the inhomogeneous term is a constant, the simplest solution is also a constant, $\mathbf{x}_p$:

$$\mathcal{L}\mathbf{x}_p = \mathbf{w}.$$

The second derivative term vanishes to leave

$$\mathbf{K}\mathbf{x}_p = \mathbf{w} \quad \text{or} \quad \mathbf{x}_p = \mathbf{K}^{-1}\mathbf{w}.$$

This represents the sag of the masses under gravity to reach the equilibrium position, which the simpler homogeneous problem takes as origin in the first place. Thus nothing interesting is added by the choice of zero force position as origin, just an added complication in finding the extensions required to reach to sagged position.

5.2 External Driving force.

Suppose in the second normal modes example the upper boundary, from which the masses are hung, oscillates vertically. We take a fixed reference point O at the centre of the oscillation, so that at time t the boundary is located at $y = h \sin \omega t$ relative to O. The origins for measuring the displacements of the masses are unchanged, at O_1 and O_2. This means that at time t the extension of the upper spring relative to the distance between O and O_1 is changed to $x_1 - y$. This changes the first equation of motion to

$$m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -k_1(x_1 - y(t)) + k_2(x_2 - x_1)$$

to be compared with equation (3). Assembling the two equations into the usual matrix form we find

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + \mathbf{K}\mathbf{x} = \mathbf{F}(t) \quad \text{where} \quad \mathbf{F} = \begin{pmatrix} k_1 y(t) \\ 0 \end{pmatrix}.$$

Thus the external force vector $\mathbf{F}$ appears as an inhomogeneous term in the equation of motion, just as the weight $\mathbf{w}$ did in the earlier example, but $\mathbf{F}$ is a function of t. This means that an appropriate form for the particular integral is now $\mathbf{x}_p = \mathbf{a} \sin \omega t$. (More generally, with an inhomogeneous term varying as a sine or cosine, we would need both sine and cosine terms in the trial form for the particular integral. However, given that the linear operator here only contains even derivatives, that is, a second derivative, we can use just a sine given the sine-dependence of $\mathbf{F}$.) Substituting the trial form into the equation of motion we obtain, after cancelling the sine function throughout,

$$-m\omega^2\mathbf{A} + \mathbf{K}\mathbf{a} = \begin{pmatrix} k_1 h \\ 0 \end{pmatrix}.$$

We can combine the two terms on the left by inserting an identity matrix:

$$(\mathbf{K} - m\omega^2\mathbf{I})\mathbf{a} = \begin{pmatrix} k_1 h \\ 0 \end{pmatrix}$$

The matrix in the bracket is singular when $m\omega^2$ equals one of the eigenvalues of the matrix $\mathbf{K}$, $m\lambda_i$. This of course is when the system is being driven at one of its resonance frequencies $\omega = \omega_i = \sqrt{\lambda_i}$. In these cases the trial solution fails, and we need a different form for the particular integral. In all other cases we can simply invert the matrix to find

$$\mathbf{a} = (\mathbf{K} - m\omega^2\mathbf{I})^{-1}\begin{pmatrix} k_1 h \\ 0 \end{pmatrix} \tag{20}$$

giving the amplitudes of oscillation of the two masses in response to the sinusoidal driving force. As usual we can add to this particular solution any solution of the homogeneous equation.

6 The General Case

6.1 Setting up the problem

In both of the examples so far, the two oscillating masses are identical, which is obviously a special case. If we remove this restriction and label the two masses as m_1 and m_2, then the equations of motion in the second example, equations (3) and (4), become:

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -k_1 x_1 + k_2(x_2 - x_1) \tag{21}$$

$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = k_2(x_1 - x_2). \tag{22}$$

We can still write the right side as $\mathbf{Kx}$, but how do we write the left side as a vector involving the second derivative vector $\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$? We have to write the masses into a diagonal mass matrix:

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{23}$$

The two equations can then be written, replacing box (B):

$$\mathbf{M}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{Kx}. \quad \textit{Normal Mode Problem: General Case} \tag{L}$$

This is still in the general form of the simple mass on a spring problem, but with the scalar m replaced by the matrix $\mathbf{M}$, in addition to the replacements for k and x. However $\mathbf{M}$ is not just any matrix; it is specifically diagonal and positive definite, which are very important constraints. In particular $\mathbf{M}$ is non-singular: its inverse is simply

$$\mathbf{M}^{-1} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix}. \tag{24}$$

We follow the same route as in section 3.2, by moving the mass to the other side of ths equation of motion. But the mass now appears as a mass matrix, so we must multiply by the inverse matrix defined above:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{M}^{-1}\mathbf{Kx}.$$

In the case of equal masses, $\mathbf{M} = m\mathbf{I}$, a multiple of the identity, and multiplying by $\mathbf{M}^{-1}$ is equivalent to multiplying by $(1/m)$. Thus $\mathbf{M}^{-1}\mathbf{K}$ is a natural generalisation of $\mathbf{G}$ defined earlier. There is however, one crucial difference: $\mathbf{G}$ is not symmetric.

With this generalisation of the matrix $\mathbf{G}$ we have the SHM form of the equation of motion in the same form as previously:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{Gx}. \tag{25}$$

The solution now proceeds *identically* to the equal mass case through the complete solution in box (I) and the definition of the normal coordinates q_i as far as the modal decomposition of the position and velocity $\mathbf{x} = \sum q_i \mathbf{c}_i$ in box (J). This is despite the fact that $\mathbf{G}$ is no longer symmetric: the only properties of $\mathbf{G}$ that are required are the right and left eigenvector definitions, together with the duality relations between them.

There is, however, one small detail. Because $\mathbf{G}$ is no longer symmetric, its eigenvectors are no longer *guaranteed* to be complete: there could in principle be fewer than N. This would have the consequence that when we fit the initial conditions, for example $\mathbf{x}(0) = \sum_i A_i \mathbf{c}_i$, there will be possible initial conditions that cannot be expressed in this way. In fact it would mean that we do not have a general solution. The conditions for there being fewer than N eigenvectors may seem difficult to meet — a repeated root in the characteristic equation which does not allow us to find the required multiple eigenvectors belonging to it — but we need to be sure that this can never happen.

Within this mathematical framework this is difficult; however there is an underlying symmetric problem which we have treated in an unsymmetrical way by multiplying the symmetric $\mathbf{K}$ by $\mathbf{M}^{-1}$ on the left to form the unsymmatric $\mathbf{G}$. If we take a different route through the solution which preserves the symmetry, then we can show that the eigenvectors *will* always form a complete set. I have chosen the unsymmetrical route because it allows the two cases of equal and unequal masses to be solved in an identical way. The alternative route is discussed in the Appendix.

6.2 Energy in the General Case

By contrast, the earlier treatment of energy does require small changes to accomodate the general case. The formula for the potential energy is unchanged, but the kinetic energy becomes

$$(1/2)\sum_i m_i v_i^2 = (1/2)\mathbf{v}^T\mathbf{M}\mathbf{v}.$$

This gives the total energy as

$$\boxed{E = \frac{1}{2}\Big(\mathbf{v}^T\mathbf{M}\mathbf{v} + \mathbf{x}^T\mathbf{K}\mathbf{x}\Big). \qquad \text{(M)}}$$

which replaces box (K). If we differentiate E with respect to t we get an expression which includes the equation of motion and is hence zero, just as we did in the equal mass case.

If we substitute for $\mathbf{x}$ and $\mathbf{v}$ in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$ we get

$$E = \frac{1}{2} = \sum_{\text{modes } i}\sum_{\text{modes } j}\Big(\dot{q}_i\mathbf{c}_i^T\mathbf{M}\mathbf{c}_j\dot{q}_j + q_i\mathbf{c}_i^T\mathbf{K}\mathbf{c}_j q_j\Big). \tag{26}$$

The final term in this equation is the point at which we need to change our approach. We consider the product $\mathbf{c}_i^T\mathbf{K}\mathbf{c}_j$ which occurs in that term. We can substitute for $\mathbf{K}$ either $\mathbf{M}\mathbf{G}$ or $\mathbf{G}^T\mathbf{M}$. Thus

$$\mathbf{c}_i^T\mathbf{M}\mathbf{G}\mathbf{c}_j - \mathbf{c}_i^T\mathbf{G}^T\mathbf{M}\mathbf{c}_j = 0$$

On the left we can use a right eigenvector equation, and on the right its transpose, to yield

$$\lambda_i\mathbf{c}_i^T\mathbf{M}\mathbf{c}_j - \lambda_j\mathbf{c}_i^T\mathbf{M}\mathbf{c}_j = (\lambda_i - \lambda_j)\mathbf{c}_i^T\mathbf{M}\mathbf{c}_j = 0$$

Thus $\mathbf{c}_i^T\mathbf{K}\mathbf{c}_j$ is zero unless $\lambda_i = \lambda_j$, in which case it equals $\mathbf{c}_i^T\mathbf{K}\mathbf{c}_i$ which we define to be $\mathcal{M}_i$:

$$\mathbf{c}_i^T\mathbf{M}\mathbf{c}_j = \mathcal{M}_i\delta_{ij}. \tag{27}$$

This combination of a right eigenvector and the mass matrix right turns out to have physical significance and we need to understand what it represents. If we denote the j'th component of the vector $\mathbf{c}_i$ by $(\mathbf{c}_i)_j$, then the normalisation condition is $\sum_j [(\mathbf{c}_i)_j]^2 = 1$, and $\mathcal{M}_i$ is given by

$$\mathcal{M}_i = \sum_j m_j (\mathbf{c}_i)_j^2. \tag{28}$$

Thus $\mathcal{M}_i$ is a weighted sum of the particle masses m_j, and the weights add up to one: in other words $\mathcal{M}_i$ is a *weighted average.* But the components of $\mathbf{c}_i$ give the relative amplitudes of oscillation of the different masses in the i'th mode, so that $\mathcal{M}_i$ is an average of the various masses weighted by their squared amplitude in the i'th mode. $\mathcal{M}_i$ is therefore called the modal mass or moving mass in this mode.

Returning to equation (26), both terms now contain the factor $\mathbf{c}_i^T \mathbf{M} \mathbf{c}_j$, which we found above to be $\mathcal{M}_i \delta_{ij}$, where $\mathcal{M}_i$ is the modal mass. The Kronecker delta ensures that all terms linking different modes are zero, and we find

$$E = \frac{1}{2} \sum_{\text{modes } i} \mathcal{M}_i \Big(\dot{q}_i^2 + \omega^2 q_i^2 \Big). \tag{29}$$

This expresses the energy as a sum of terms associated with each normal mode, each of harmonic oscillator form. $\mathcal{M}_i$ is the modal mass, q_i the displacement and ω_i the angular frequency. Thus in this final result there is a change in the formulation as compared to the equal mass case, replacing m with $\mathcal{M}_i$. But, as with the replacement of $\mathbf{G} = m^{-1}\mathbf{K}$ by $\mathbf{G} = \mathbf{M}^{-1}\mathbf{K}$, it is a natural generalisation in the sense that it is still applicable in the equal mass case. If $\mathbf{M}$ is of the form $m\mathbf{I}$, then $\mathcal{M}_i$ becomes m. So the modal mass is m for every mode in the equal mass case, as we found earlier.

7 Lagrangian and Hamiltonian formulations.

7.1 In terms of Position Coordinates x.

Having explicit expressions for the kinetic and potential energies we can readily define the Lagrangian v

$$\mathcal{L}(\mathbf{x}, \mathbf{v}) = T - V = \frac{1}{2}\Big(\mathbf{v}^T \mathbf{M} \mathbf{v} - \mathbf{x}^T \mathbf{K} \mathbf{x} \Big). \tag{30}$$

The usual Lagrangian equation of motion then gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{M}\mathbf{v} + \mathbf{K}\mathbf{x} = 0$$

which is, of course, the same equation of motion we found above, box (L).

If we find the corresponding momenta $\mathbf{p} = \nabla_{\mathbf{v}} \mathcal{L} = \mathbf{M}\mathbf{v}$ we can find the Hamiltonian:

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) = \dot{\mathbf{x}}^T \mathbf{p} - \mathcal{L} = \frac{1}{2}\Big(\mathbf{p}^T \mathbf{M}^{-1} \mathbf{p} + \mathbf{x}^T \mathbf{K} \mathbf{x} \Big).$$

which is just E from box (M) expressed in terms of $\mathbf{x}$ and $\mathbf{p}$. The Hamiltonian equations of motion are then

$$\dot{\mathbf{x}} = \mathbf{M}^{-1}\mathbf{p} \qquad \dot{\mathbf{p}} = -\mathbf{K}\mathbf{x}$$

which reduce, unsurprisingly, to the equation of motion of box (L) .

7.2 In terms of Normal Coordinates q_i.

If instead we set up the Lagrangian in normal co-ordinates $q_i = \mathbf{d}_i^T \mathbf{x}$

$$\mathcal{L}'(q_i, \dot{q}_i) = \sum_{\text{modes } i} \frac{\mathcal{M}_i}{2} \left(\dot{q}_i^2 - \omega_i^2 q_i^2 \right) = \sum_{\text{modes } i} \mathcal{L}_i.$$

Since this is a sum of Lagrangians, one for each co-ordinate q_i, it yields uncoupled equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = -\omega_i^2 q_i$$

as in equation (17).

The momenta are simply $P_i = \mathcal{M}_i \dot{q}_i$ so this time the Hamiltonian is given by

$$\mathcal{H}'(q_i, P_i) = \sum_{\text{modes } i} \left(\frac{P_i^2}{\mathcal{M}_i} - \mathcal{L}_i \right) = \sum_{\text{modes } i} \frac{1}{2} \left(\frac{P_i^2}{\mathcal{M}_i} + \mathcal{M}_i \omega_i^2 q_i^2 \right),$$

which is just the normal co-ordinate form of E, equation (29) expressed in terms of momenta. It is also a sum of Hamiltonians, and again this yields uncoupled equations:

$$\dot{q}_i = \frac{P_i}{\mathcal{M}_i} \qquad \dot{P}_i = -\mathcal{M}_i \omega_i^2 q_i.$$

8 Summary.

The following table collects all the significant equations we have looked at. The special case of equal masses m is obtainable from the general case by replacing the mass matrix $\mathbf{M}$ with $m\mathbf{I}$ (and consequently $\mathcal{M}_i$ with m).

Equation of Motion	$\mathbf{M}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x}$
Eq of Motion: SHM form	$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{G}\mathbf{x} \quad \mathbf{G} = \mathbf{M}^{-1}\mathbf{K}$
Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}\right) = 0$
Normal Node Ansatz	$\mathbf{x} = f(t)\mathbf{c} \quad \text{where} \quad \mathbf{c}^T\mathbf{c} = 1$
Right Eigenvectors	$\mathbf{G}\mathbf{c}_i = \lambda_i \mathbf{c}_i$
Eigenvalue Ansatz	$\lambda_i = \omega_i^2$
Characteristic Equation	$\det \lvert \mathbf{G} - \omega^2 \mathbf{I} \rvert = 0$
Left Eigenvectors	$\mathbf{d}_i^T \mathbf{G} = \omega_i^2 \mathbf{d}_i^T$
Dual Vectors	$\mathbf{d}_i^T \mathbf{c}_j = \delta_{ij}$
General Solution	$\mathbf{x}(t) = \sum_i \left[[\mathbf{d}_i^T \mathbf{x}(0)] \cos(\omega_i t) + [\mathbf{d}_i^T \mathbf{v}(0)/\omega_i] \sin(\omega_i t) \right] \mathbf{c}_i$
Normal Coordinates	$q_i = \mathbf{d}_i^T \mathbf{x}$
Modal Mass	$\mathcal{M}_i = \mathbf{c}_i^T \mathbf{M} \mathbf{c}_i$
Modal Decomposition: $\mathbf{x}$	$\mathbf{x} = \sum_i q_i \mathbf{c}_i$
Modal Decomposition: E	$E_i = \frac{1}{2}\mathcal{M}_i \left(\dot{q}_i^2 + \omega_i^2 q_i^2\right)$

Appendix: Roads Not Taken.

A number of specific choices have been made in presenting the material above. There are two alternative choices that I could have made (and indeed did make in earlier editions of this handout) that are sufficiently important to be made available here.

A.1 Maximally Matrix Formulation

In the mathematics there are two types of indices, and consequently two types of summation over them: There is the index labelling the different masses in the system, and the index labelling the different normal modes of the system. At a very early stage the position coordinates of the masses are arranged into the vector $\mathbf{x}$, and thereafter all summations over their index are handled by matrix multiplication. However the modes have been treated differently, and there are repeated instances of sums over modes. This choice was made to keep the mathematics as simple as possible, but at some sacrifice of unity and elegance.

We can use matrix multiplication to carry out these sums as well, as follows. The first step is to arrange the right eigenvectors $\mathbf{c}_i$ into the columns of a matrix:

$$\mathbf{C} = (\mathbf{c}_1\mathbf{c}_2 \dots \mathbf{c}_N)$$

and the eigenvalues into a diagonal matrix Λ. The right eigenvector equation can then be written

$$\mathbf{GC} = \mathbf{C}\Lambda.$$

Similarly we arrange the left eigenvectors $\mathbf{d}_i$ into the columns of a matrix $\mathbf{D}$, and the left eigenvector equation becomes

$$\mathbf{D}^T\mathbf{G} = \Lambda\mathbf{D}^T.$$

The left-right duality relations are $\mathbf{D}^T\mathbf{C} = \mathbf{I}$. The matrices $\mathbf{C}$ and $\mathbf{D}^T$ are thus inverses of each other. We can then define a vector of normal coordinates

$$\mathbf{q} = \mathbf{D}^T\mathbf{x} \quad \text{and consequently} \quad \mathbf{x} = \mathbf{Cq},\ \mathbf{v} = \mathbf{C}\dot{\mathbf{q}}.$$

We shall apply these definitions to formulate three specific points in the theory.

(a) **General Solution:** The general solution has the form

$$\mathbf{x}(t) = \mathbf{Cq}(t)$$

where the components of $\mathbf{q}$ are

$$q_i = A_i \cos\omega_i t + B_i \sin\omega_i t.$$

The double occurrence of the index i on the right and the single occurrence on the left tells us that either the constants or the sine and cosine functions have to be placed in a diagonal matrix, and the other in a vector. Since $A_i = q_i(0)$ we choose to put the constants into vectors $\mathbf{A}$ and $\mathbf{B}$, and the functions into diagonal matrices:

$$\mathbf{F} = \begin{pmatrix} \cos\omega_1 t & 0 & \dots & 0 \\ 0 & \cos\omega_2 t & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \cos\omega_N t \end{pmatrix} \quad \mathbf{G} = \begin{pmatrix} \sin\omega_1 t & 0 & \dots & 0 \\ 0 & \sin\omega_2 t & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \sin\omega_N t \end{pmatrix}.$$

We can then write

$$\mathbf{q} = \mathbf{FA} + \mathbf{GB}.$$

Finally the initial conditions can be written

$$\mathbf{A} = \mathbf{q}(0) = \mathbf{D}^T\mathbf{x}(0) \quad \text{and} \quad \Omega\mathbf{B} = \dot{\mathbf{q}}(0) = \mathbf{D}^T\mathbf{v}(0)$$

where Ω is a diagonal matrix of frequencies ω_i. Thus our final result is

$$\mathbf{x}(t) = \mathbf{C}\left[\mathbf{FD}^T\mathbf{x}(0) + \mathbf{G}\Omega^{-1}\mathbf{D}^T\mathbf{v}(0)\right].$$

(b) **Modal Decomposition of the Energy:** In the expression for the energy

$$E = \frac{1}{2}\Big(\mathbf{v}^T\mathbf{Mv} + \mathbf{x}^T\mathbf{Kx}\Big),$$

we substitute the modal decompositions for $\mathbf{x}$ and $\mathbf{v}$:

$$E = \frac{1}{2}\Big(\dot{\mathbf{q}}^T\mathbf{C}^T\mathbf{MC}\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{C}^T\mathbf{KCq}\Big).$$

In the second term we consider the product $\mathbf{C}^T\mathbf{KC}$. It is manifestly symmetric. We can substitute for $\mathbf{K} = \mathbf{MG}$, and equate it to its transpose:

$$\mathbf{C}^T\mathbf{MGC} = \mathbf{C}^T\mathbf{G}^T\mathbf{MC}.$$

On the left we can use the right eigenvector equation, and on the right we can use its transpose:

$$\mathbf{C}^T\mathbf{MC}\Lambda = \Lambda\mathbf{C}^T\mathbf{MC}.$$

The matrix $\mathbf{C}^T\mathbf{MC}$ commutes with the diagonal matrix Λ, and is thus itself diagonal. Since its diagonal elements are $\mathcal{M}_i$ we call the matrix $\mathcal{M}$:

$$\mathbf{C}^T\mathbf{MC} = \mathcal{M} \quad \text{and} \quad \mathbf{C}^T\mathbf{KC} = \mathcal{M}\Lambda.$$

The expression for E then simplifies to

$$E = \frac{1}{2}\Big(\dot{\mathbf{q}}^T\mathcal{M}\dot{\mathbf{q}} + \mathbf{q}^T\mathcal{M}\Lambda\mathbf{q}\Big).$$

Since both $\mathcal{M}$ and Λ are diagonal, this is a sum of modal energies.

(c) **Canonical Transformation between x and q Hamiltonians.** In section 7.2 we calculated each Hamiltonian from the corresponding Lagrangian, but we can of course transform directly from $\mathcal{H}(\mathbf{x}, \mathbf{p})$ to $\mathcal{H}'(\mathbf{q}, \mathbf{P})$. This forms an instructive example of a canonical tranformation, using a generating function $\mathcal{F}_2(\mathbf{x}, \mathbf{P})$, according to the classification of Goldstein (*Classical Mechanics*, 3rd ed., section 9.1). The transformation function takes the form of a dot product of the new momenta $\mathbf{P}$ multiplied by the expressions for the normal co-ordinates $\mathbf{q}$ in terms of the position coordinates $\mathbf{x}$, $\mathbf{q} = \mathbf{D}^T\mathbf{x}$. However, recalling that $\mathbf{D} = \mathbf{C}^{-1}$ we write this is terms of $\mathbf{C}$:

$$\mathcal{F}_2(\mathbf{x}, \mathbf{P}) = \mathbf{P}^T(\mathbf{C}^{-1})^T\mathbf{x}.$$

The equations of transformation are then

$$\mathbf{q} = \nabla_{\mathbf{P}}\mathcal{F}_2 = (\mathbf{C}^{-1})^T\mathbf{x}$$

as required for the co-ordinates, and for the momenta

$$\mathbf{p} = \nabla_{\mathbf{x}}\mathcal{F}_2 = \mathbf{C}^{-1}\mathbf{P}.$$

The canonical transformation is thus, new co-ordinates in terms of old coordinates,

$$\mathbf{q} = (\mathbf{C}^{-1})^T\mathbf{x} \quad \text{and} \quad \mathbf{P} = \mathbf{C}\mathbf{p}. \tag{31}$$

This is absolutely typical of any linear transformation $\mathbf{L}$ of the co-ordinates: if the new co-ordinates are given by $\mathbf{q} = \mathbf{L}\mathbf{x}$ then the new momenta are given by $\mathbf{P} = (\mathbf{L}^{-1})^T\mathbf{p}$; in the special case of a rotation, when $\mathbf{L}$ is an orthogonal matrix, then both momenta and co-ordinates transform in the same way. This is exactly what happens in the equal mass case, where the eigenvectors are orthogonal, but not in the general case.

The new Hamiltonian is given by

$$\mathcal{H}'(\mathbf{q},\mathbf{P}) = \mathcal{H}(\mathbf{x}(\mathbf{q}),\mathbf{p}(\mathbf{P})) = \frac{1}{2}\Big(\mathbf{P}^T\mathbf{C}^{-1}\mathbf{M}^{-1}(\mathbf{C}^T)^{-1}\mathbf{P} + \mathbf{q}^T\mathbf{C}^T\mathbf{K}\mathbf{C}\mathbf{q}\Big).$$

The matrix products simplify greatly: $\mathbf{C}^T\mathbf{K}\mathbf{C} = \mathcal{M}\Lambda$ as above, and $\mathbf{C}^{-1}\mathbf{M}^{-1}(\mathbf{C}^T)^{-1} = (\mathbf{C}^T\mathbf{M}\mathbf{C})^{-1} = \mathcal{M}^{-1}$ so that

$$\mathcal{H}'(\mathbf{q},\mathbf{P}) = \frac{1}{2}\Big(\mathbf{P}^T\mathcal{M}^{-1}\mathbf{P} + \mathbf{q}^T\mathcal{M}\Lambda\mathbf{q}\Big),$$

which is our earlier expression for the energy , but expressed in terms of momenta instead of velocities.

A.2 The Symmetric Approach.

If we substitute the normal mode *ansatz* directly into the general equation of motion in box (L), without first transforming it into the SHM form, then we obtain

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\mathbf{M}\mathbf{c} = -f\mathbf{K}\mathbf{c}.$$

As before, we divide by f:

$$\left(\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right)\mathbf{M}\mathbf{c} = -\mathbf{K}\mathbf{c},$$

and note that the right side is constant, and therefore so is the bracket on the left, which we define to be $-\omega^2$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -\omega^2 f \quad \mathbf{K}\mathbf{c} = \omega^2\mathbf{M}\mathbf{c}.$$

The first equation is the same SHM equation for f that we had previously, but the second equation is a different type of eigenvalue equation. This equation defines a *generalised eigenvalue problem*, or an *eigenvalue problem with a metric.* The diagonal, positive definite matrix $\mathbf{M}$ is the metric.

We can develop the theory for eigenvalue equations of this kind, in the case of a symmetric matrix $\mathbf{K}$, in exactly the same way as without the metric. We define eigenvalues and left or right eigenvectors with respect to $\mathbf{M}$ as in boxes (E) and (D):

$$\boxed{\mathbf{Kc} = \omega^2 \mathbf{Mc} \quad \mathbf{d}^T\mathbf{K} = \omega^2 \mathbf{d}^T\mathbf{M}. \quad \textit{Left and Right eigenvector definitions} \qquad \text{(N)}}$$

(Note that, although we are using the same symbol for them, the eigenvectors of **K** with respect to **M** will not be the same as the ordinary eigenvectors of **K**.) The characteristic equation, instead of box (F), is given by

$$\boxed{\det|\mathbf{K} - \omega^2 \mathbf{M}| = 0. \qquad \text{(O)}}$$

The orthogonality of the eigenvectors is demonstrated from the construct $\mathbf{d}_i^T\mathbf{K}\mathbf{c}_j$ which can be expanded using either the left or right eigenvector definition to deduce $(\omega_i^2 - \omega_j^2)\mathbf{d}_i^T\mathbf{M}\mathbf{c}_j = 0$. This shows that eigenvectors belonging to different eigenvalues are orthogonal with respect to the metric **M**:

$$\mathbf{d}_i^T\mathbf{M}\mathbf{c}_j = 0 \quad \text{for} \quad i \neq j.$$

If we take the transpose of the left eigenvector definition (remembering that both **K** and **M** are symmetric) we obtain

$$\mathbf{K}\mathbf{d}_i = \omega_i^2 \mathbf{M}\mathbf{d}_i$$

which is the *right* eigenvector definition. Thus again we find that the left and right eigenvectors are interchangeable.

The argument for the eigenvalues being real is almost unchanged, and the argument for completeness of the eigenvectors can also be re-worked in this context. Thus all of the special properties of the eigenvectors of a symmetric matrix of section 2.3 are also true for eigenvectors with respect to **M** with appropriate changes.

We could go on to complete the solution within this framework, and to discuss the relationship of it to the solution presented previously. However we simply note that we have demonstrated the earlier claim that there will always be a complete set of eigenvectors for this problem.