Normal Modes

Normal Modes occur as solutions to coupled linear differential equations, and are defined as a solution for the variables $x$ in which all variables have the same dependence $f(t)$:

$$x = f(t) \mathbf{c}.$$  \hspace{1cm} \text{(A)}

Here $\mathbf{c}$ is a constant vector — in other words: all the variables that make up the vector $x$ vary in the same way with time, but with different amplitudes given by the corresponding component of $\mathbf{c}$. Note that box (A) only defines $f(t)$ up to a constant: any constant factor can be moved from the definition of $f(t)$ into the definition of $\mathbf{c}$. In particular we shall assume that the elements of $\mathbf{c}$ are pure dimensionless numbers, and it is the function $f(t)$ which gives the solution its required dimension of length.

We shall be looking at examples of coupled second order differential equations, and in these cases the function $f(t)$ is a sine or cosine function, but similar techniques can be applied to first order systems, where $f(t)$ would be an exponential.

1 Setting up the problem.

We shall use two sample systems as examples. The first involves two equal masses $m$ oscillating horizontally, with equal springs $k$ joining them to fixed side walls, and a coupling spring $k'$ between them (see Figure 1). We measure both displacements from equilibrium in the same direction: from the position of mass 1 towards the position of mass 2. Thus a displacement $x_1$ lengthens the side spring and compresses the coupling spring, whereas a displacement $x_2$ lengthens the coupling spring and compresses the side spring. Then the equations of motion are

$$m \frac{d^2 x_1}{dt^2} = -k x_1 + k' (x_2 - x_1)$$ \hspace{1cm} (1)

$$m \frac{d^2 x_2}{dt^2} = -k x_2 + k' (x_1 - x_2).$$ \hspace{1cm} (2)

If we write $x_1$ and $x_2$ as elements of a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then we can re-write the equations of motion as

$$m \frac{d^2 \mathbf{x}}{dt^2} = -\mathbf{K} \mathbf{x} \hspace{1cm} \text{Normal Mode Problem: Equal Mass Case} \hspace{1cm} (B)$$

\hspace{10cm}

$\begin{array}{cccc}
\hline
k & m & k' & m & k \\
\hline
\end{array}$

Figure 1: First Normal Modes Example
where the matrix $\mathbf{K}$ is given by

$$
\mathbf{K} = \begin{pmatrix}
    k + k' & -k' \\
    -k' & k + k'
\end{pmatrix}.
$$

The second example is two equal masses hanging on two springs in the order (from the top) spring $k_1$, mass $m$, spring $k_2$, mass $m$ (see Figure 2). We have two choices about the origin of displacements; the simplest is to leave the system to settle under gravity, so that the weights of the two masses are balanced by the forces exerted by the springs, and then measure the displacements from that sagged position. The forces that appear are then the additional forces, over and above those necessary to balance the weights. (This is in fact exactly the approach we took with the horizontal spring system, where the three springs may well be under tension in the $x = 0$ position, but in this case the spring forces on each mass cancel out.) However an alternative is to measure the displacements from the unstretched position.

Taking the simpler option of measuring from the sagged position, a displacement $x_1$ extends spring $k_1$ and compresses spring $k_2$, whereas an extension $x_2$ extends spring $k_2$. Thus the equations of motion are

$$
m \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2 (x_2 - x_1), \quad (3)
$$

$$
m \frac{d^2 x_2}{dt^2} = k_2 (x_1 - x_2). \quad (4)
$$

We can write this in the same form as box (B):

$$
m \frac{d^2 \mathbf{x}}{dt^2} = -\mathbf{K} \mathbf{x}
$$

where now the matrix $\mathbf{K}$ is given by

$$
\mathbf{K} = \begin{pmatrix}
    k_1 + k_2 & -k_2 \\
    -k_2 & k_2
\end{pmatrix}. \quad (5)
$$

If instead we measure displacements from the unstretched spring positions then we must include the weights in the equations of motion:

$$
m \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2 (x_2 - x_1) + mg \quad (6)
$$

$$
m \frac{d^2 x_2}{dt^2} = k_2 (x_1 - x_2) + mg. \quad (7)
$$

Figure 2: Second Normal Modes Example
which we can write as

\[
\frac{d^2 x}{dt^2} = -Kx + w \quad \text{Inhomogeneous Normal Mode Problem} \quad (C)
\]

where the weight vector is
\[
w = \begin{pmatrix} mg \\ mg \end{pmatrix}.
\]

We shall return to this formulation in section 5. We have thus cast both of these normal mode problems in the same form as each other, and in the same form as a simple mass on a spring problem

\[
\frac{d^2 x}{dt^2} = -kx
\]

with the single variable \( x \) replaced by a vector \( \mathbf{x} \) and the spring constant \( k \) replaced by a matrix \( \mathbf{K} \). We shall solve this equation in section 3, after introducing the relevant mathematics in the following section.

We first note some features of the matrix \( \mathbf{K} \). In both cases the matrix is symmetric: \( \mathbf{K} = \mathbf{K}^T \). This is generally true: in fact it is always true in any normal mode problem where the forces can be expressed as the gradients of a potential energy function, which covers almost every case except those where magnetic forces are present. We can thus use this feature to structure our solution. However the first matrix \( \mathbf{K} \) has an additional symmetry not possessed by the second: if we write the components of \( \mathbf{K} \) in the standard way
\[
\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}
\]

then in addition to the symmetry \( k_{12} = k_{21} \) noted above, the first \( \mathbf{K} \) matrix has the additional symmetry \( k_{11} = k_{22} \), which the second one lacks. We shall see that this affects what other options are available to us apart from the normal mode method of solution, but does not affect the normal mode method at all.

2 Eigenvalues and Eigenvectors

2.1 Definitions

For any square matrix \( \mathbf{V} \) we define the eigenvalue \( \lambda \) and right eigenvector \( \mathbf{c} \) by

\[
\mathbf{Vc} = \lambda \mathbf{c} \quad \text{Right Eigenvector Definition} \quad (D)
\]

The vector \( \mathbf{c} \) is clearly special with respect to \( \mathbf{V} \), because when we multiply it by \( \mathbf{V} \) we get a vector which is parallel to \( \mathbf{c} \), whereas in general the resulting vector would not be. However, note that any multiple of \( \mathbf{c} \), \( \mathbf{c}' = \alpha \mathbf{c} \) is also an eigenvector with the same eigenvalue. Thus the eigenvectors of \( \mathbf{V} \) define directions in \( \mathbf{x} \)-space, but have no natural length.

In similar fashion we define the left eigenvector \( \mathbf{d} \) by

\[
\mathbf{d}^T \mathbf{V} = \lambda \mathbf{d}^T \quad \text{Left Eigenvector Definition} \quad (E)
\]
In each case the eigenvalues are defined by the \textit{characteristic equation}:

\[
\det (V - \lambda I) = 0 \quad \text{Characteristic Equation (F)}
\]

which shows that both sets of eigenvectors have a common set of eigenvalues. All square matrices possess at least one such eigenvalue and left and right eigenvectors.

\subsection{2.2 Orthogonality}

Eigenvalues have certain important orthogonality properties: consider the construct \( d_i^T V c_j \) for vectors belonging to different eigenvalues \( \lambda_i, \lambda_j \). We can use either the left or right eigenvector definition to deduce

\[
d_i^T V c_j = \lambda_i d_i^T c_j = \lambda_j d_i^T c_j
\]

and hence \((\lambda_i - \lambda_j)d_i^T c_j = 0\). Thus left eigenvectors are orthogonal to all right eigenvectors belonging to different eigenvalues:

\[
d_i^T c_j = \begin{cases} 
\gamma_i & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \quad \text{Left-Right Eigenvector orthogonality. (G)}
\]

The two sets of mutually orthogonal vectors are called dual vectors.

We can write some of these results in a more compact form by assembling the eigenvectors into matrices. The matrix \( C \) is formed from all the right eigenvectors \( c_i \):

\[
C = (c_1 \ c_2 \ \ldots)
\]

and the matrix \( D \) from all the left eigenvectors:

\[
D = (d_1 \ d_2 \ \ldots).
\]

(Note that if the number of eigenvectors \( L \) is less than the order of the matrix \( N, L < N \), then these are non-square matrices.) Then the right and left eigenvector definitions in boxes (D) and (E) can be written

\[
VC = \Lambda \quad \text{and} \quad D^T V = \Lambda D^T \quad \text{(8)}
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L) \) is a diagonal matrix with the eigenvalues on the diagonal. (It’s important to understand the order of the matrix factors in equation (8): the point is that post-multiplying by a diagonal matrix multiplies the columns by the diagonal elements, whereas pre-multiplying multiplies the rows. You might like to check with a \( 2 \times 2 \) example!)

Secondly the orthogonality relations of box (G) become

\[
D^T C = \Gamma \quad \text{(9)}
\]

where \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_L) \).
2.3 Special properties of a symmetric matrix

If \( V \) is symmetric matrix, such as \( K \), then we can prove a number of additional properties.

(a) **Identity of left and right eigenvectors.** Taking the transpose of the left eigenvector definition, box (E), we obtain

\[
Kd = \lambda d,
\]

which is the right eigenvector definition. Thus the first special property of a symmetric matrix is that the left and right eigenvectors are the same, \( D = C \). (Strictly speaking we can only say that we can choose them to be the same; we still have the option of choosing left eigenvectors with a different normalisation if we want to.) Both of equations (8) are now equivalent to

\[
KC = CA. \tag{10}
\]

(b) **Mutual orthogonality of eigenvectors.** Substituting \( D = C \) into box (G), or equation (9) we find

\[
c_i^T c_i = \gamma_i \quad \text{or} \quad C^T C = \Gamma. \tag{11}
\]

Thus the second special property of a symmetric matrix is that the eigenvectors are all orthogonal.

(c) **Reality of eigenvalues.** The third special property of a real symmetric matrix is that the eigenvalues are all real. (Note that, for an \( N \times N \) matrix, the characteristic equation, box (F), is an \( N \)’th order polynomial, which in general may have either real or complex roots.) To prove this consider the construct \( c_i^{†}Kc_i \). This can be evaluated by using either the right eigenvector definition, or the complex conjugate of the left eigenvector equation \( c_i^{†}K^{†} = \lambda_i^{†}c_i^{†} = c_i^{†}K \) since \( K \) is real. These alternatives give

\[
c_i^{†}Kc_i = \lambda_i c_i^{†}c_i = \lambda_i^{†}c_i^{†}c_i
\]

so that \( (\lambda_i - \lambda_i^{†})c_i^{†}c_i = 0 \). The product of vectors is the sum of the square moduli of the elements, and thus positive definite, so that \( \lambda_i = \lambda_i^{†} \): \( \lambda_i \) is real.

Since \( K \) is real and \( \lambda \) is real, we are free to choose real eigenvectors \( c \). This entails that the elements of \( \Gamma \), \( \gamma_i = c_i^{†}c_i \) can also be written \( \gamma_i = |c_i|^2 > 0 \). Since the elements of the diagonal matrix \( \Gamma \) are all positive-definite, \( \Gamma \) is non-singular.

(d) **Completeness.** The fourth and final special property of a symmetric matrix is that the eigenvectors form a complete set; that is, for an \( N \times N \) matrix there are \( N \) of them. This sounds obvious, until we consider that the characteristic equation in box (F) may have repeated roots. In such a case we require multiple eigenvectors to belong to this eigenvalue, and while this is exactly what happens for a symmetric matrix, it cannot be guaranteed in general. Suppose we have a repeated root, and have found only \( N - 1 \) eigenvectors \( c_1 \ldots c_{N-1} \). We then complete the set by adding the unique vector \( f \) which is orthogonal to all the \( c_i \): \( f^T c_i = 0 \). Now consider the construct \( c_i^{†}Kf \). Using the left eigenvector relation we deduce

\[
c_i^{†}Kf = \lambda_i c_i^{†}f = 0.
\]

This shows that the vector \( Kf \) is orthogonal to all the \( c_i \), and hence can only consist of a multiple of \( f \): in other words \( f \) is the missing eigenvector. This argument can generalised to any number of missing eigenvectors. Thus we can always find a complete set of eigenvectors: \( L = N \).
(e) **Inverses.** Taken together theses special properties show that the matrix of eigenvectors \( C \) is non-singular and has an immediately accessible inverse. The fact that the number of eigenvectors \( L \) equals the length of the \( x \)-vector \( N \) means that the matrix \( C \) is square. From equation (11) we see that \( C^T \) is almost the inverse of \( C \) except that we need to divide the \( i \)'th row by \( \gamma_i \). More formally:

\[
\Gamma^{-1} C^T = I \quad \text{or} \quad C^{-1} = \Gamma^{-1} C^T.
\]

### 3 Solution of the Normal Modes Equation

#### 3.1 Methods of Solution

The dynamical equations of box (B) are \( N \) coupled linear homogeneous second order differential equations. We shall restrict ourselves to \( N = 2 \), but the generalisation to arbitrary \( N \) should be obvious. We expect the solution to consist of a sum of \( 2N = 4 \) functions each with an arbitrary multiplicative constant. There are (at least) three solution methods:

(a) Elimination;

(b) Uncoupling, or Normal Coordinates;

(c) Normal Modes;

and we shall discuss the first two briefly before returning to the third, Normal Modes.

**Elimination.** We shall take the second example, equations (3) and (4). Re-arranging equation (4) we get

\[
x_1 = x_2 + \frac{m}{k_2} \frac{d^2 x_2}{dt^2},
\]

which we can differentiate to give

\[
\frac{d^2 x_1}{dt^2} = \frac{d^2 x_2}{dt^2} + \frac{m}{k_2} \frac{d^4 x_2}{dt^4}.
\]

These allow us to eliminate \( x_1 \) in favour of \( x_2 \). Substituting into equation (3) we get

\[
\frac{d^4 x_2}{dt^4} + \left( \frac{2k_2}{m} + \frac{k_1}{m} \right) \frac{d^2 x_2}{dt^2} + \frac{k_1 k_2}{m^2} x_2 = 0.
\]

This is now a fourth-order linear differential equation which we can solve in the usual way to give four functions each with a multiplicative constant. We can then complete the solution by using equation (13) to find \( x_1 \). This approach clearly gives an answer in the expected form, but is fairly lengthy, and gives no insight into the problem whatever.

**Normal Co-ordinates.** These are defined to be linear combinations of the co-ordinates \( x_1 \) and \( x_2 \) which obey uncoupled equations. In the case of a \( K \) matrix with the extra symmetry \( k_{11} = k_{22} \) referred to in section 1 they are easy to find. We therefore consider the first example, equations (1) and (2). If we add these we obtain

\[
m \frac{d^2}{dt^2} (x_1 + x_2) = -k(x_1 + x_2);
\]
while if we subtract them we obtain
\[ m \frac{d^2}{dt^2}(x_1 - x_2) = -(k + 2k')(x_1 - x_2). \]

Thus the linear combinations \( q_1 = x_1 + x_2 \) and \( q_2 = x_1 - x_2 \) are normal co-ordinates, satisfying uncoupled equations that we can solve in the usual way. We can then invert the relations and find \( x_1 \) and \( x_2 \) in terms of \( q_1 \) and \( q_2 \). If the factors (in this case 1, 1 and 1, -1) with which the original equations have to be combined are obvious, then this method is very straightforward. It also gives more insight than the elimination: we learn that certain combinations of the co-ordinates perform simple harmonic motion. However, although the factors for forming the normal co-ordinates always exist, they cannot usually be spotted easily — try spotting them for the second example problem! Thus the method of choice in general is the third alternative, to which we now turn.

### 3.2 Normal Mode Solution.

This method begins by simply substituting the normal mode form, box (A), into the equation of motion, box (B):
\[ m \frac{d^2 f}{dt^2} \mathbf{c} = -f(t) \mathbf{Kc}. \]

Requiring that \( f \) is not always zero (a modest requirement for a non-trivial solution!) we can divide by \( f \) to give
\[ \left( m \frac{d^2 f}{f \frac{dt^2}} \right) \mathbf{c} = -\mathbf{Kc}. \]

We now observe that the right side of this equation is constant, not a function of \( t \). Thus the bracket on the left must also be constant, and we call this constant \(-\lambda\). Thus we have
\[ \frac{d^2 f}{dt^2} = -\frac{\lambda}{m} f \quad \text{and} \quad \mathbf{Kc} = \lambda \mathbf{c}. \]

The right-hand equation tells us that \( \lambda \) is an eigenvalue of \( \mathbf{K} \) and \( \mathbf{c} \) is a corresponding eigenvector. The theory of the last section immediately tells us that there are two eigenvalues \( \lambda_i \), with orthogonal eigenvectors \( \mathbf{c}_i \). Thus we have found not just one but two solutions in the specified form, and the linearity of the equations of motion allows us to superpose them with arbitrary constants.

The type of solution \( f_i \) depends on the value of \( \lambda_i \): the left-hand equation tells us that \( f_i \) is oscillatory (complex-exponential or trigonometric) if \( \lambda_i > 0 \), real-exponential or hyperbolic if \( \lambda_i < 0 \), and linear if \( \lambda_i = 0 \). Of course we do not expect unbounded motion to occur in these problems, so we expect only oscillatory solutions, and we shall therefore assume \( \lambda_i > 0 \). Thus each \( f_i \) is given by
\[ f_i(t) = A_i \cos \left( \sqrt{\frac{\lambda_i}{m}} t \right) + B_i \sin \left( \sqrt{\frac{\lambda_i}{m}} t \right). \]

If we define the angular frequencies \( \omega_i = \sqrt{\lambda_i/m} \) then the full solution can be written
\[ x(t) = (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) \mathbf{c}_1 + (A_2 \cos \omega_2 t + B_2 \sin \omega_2 t) \mathbf{c}_2 \]
containing, as expected, four constants. These can be determined by initial conditions, the positions \( x(0) \) and velocities \( v(0) \) at \( t = 0 \). Evaluating equation (18) at \( t = 0 \) we find

\[
x(0) = A_1 c_1 + A_2 c_2,
\]

which we can easily invert by dotting with the eigenvectors and using the orthogonality. For example if we dot equation (19) with \( c_1 \) we obtain

\[
c_1^T x(0) = A_1 c_1^T c_1 + A_2 c_1^T c_2.
\]

But the second term on the right is zero by orthogonality, so

\[
A_1 = \frac{c_1^T x(0)}{c_1^T c_1}.
\]

Similarly the velocity at \( t = 0 \) is given by

\[
v(0) = B_1 \omega_1 c_1 + B_2 \omega_2 c_2
\]

which we can solve in a similar fashion. For example if we dot with \( c_1 \):

\[
c_1^T v(0) = B_1 \omega_1 c_1^T c_1 \quad \text{and hence} \quad B_1 = \frac{c_1^T v(0)}{\omega_1 c_1^T c_1}.
\]

This gives a complete solution in terms of the initial conditions for all normal mode problems:

\[
x(t) = \sum_i \left[ \frac{c_i^T x(0)}{c_i^T c_i} \cos(\omega_i t) c_i + \frac{c_i^T v(0)}{\omega_i c_i^T c_i} \sin(\omega_i t) c_i \right]. \quad \text{Complete Solution (H)}
\]

Up till this point we have not specified the normalisation of the eigenvectors \( c_i \); indeed we started with the observation that the eigenvectors have no natural length. Now that we have a complete solution we observe, which we should probably have expected, that the normalisation plays no role in the solution at all. Each term of the expression in box (H) contains two factors of \( c_i \) in the numerator and two in the denominator. Any scaling of the eigenvector \( c_i \) simply cancels out of the solution. It would clearly simplify this result slightly if we normalised the eigenvectors to unity, \( c_i^T c_i = 1 \) or \( \Gamma = I \). This allows a simplification of box (H), and also entails that \( C \) is an orthogonal matrix, \( C^T C = I \). The cost of this slight simplification is the necessity for an otherwise redundant step in the solution, that of normalising the eigenvectors.

We shall also find that in the more general case of unequal masses for the particles there is no corresponding simplification, and so we continue to use eigenvectors of unspecified length.

### 3.3 Normal Co-ordinates re-visited.

The method of normal co-ordinates introduced above is, naturally, related to the normal mode method. If we take the original equation of motion from box (B) and dot it with an eigenvector \( c_i \) we obtain

\[
m \frac{d^2}{dt^2} c_i^T x = -c_i^T K x.
\]
Using the left eigenvector relation from box (E) we obtain

\[ \frac{d^2}{dt^2} (c_i^T x) = -\frac{\lambda_i}{m} (c_i^T x) \]  

which tells us that for every \( i \) the linear combination \( c_i^T x \) satisfies an uncoupled equation: that is: \textit{the products} \( q_i = c_i^T x \) \textit{are normal co-ordinates}. We observe that the \( q_i \) have the same ambiguity of scale as the eigenvectors: any multiple of a specific \( q_i \) is also a normal co-ordinate. In order to be a normal co-ordinate the \( q_i \) only has to combine the \( x_j \) in the correct relative proportions. This freedom in the normal co-ordinates is independent of the freedom in \( C \); that is we could use differently normalised eigenvectors to form the \( q_i \) from those used in the direct solution for \( x \). A sensible way to use this freedom is to use the dual vectors to form the normal co-ordinates; that is: vectors with unit scalar product with the \( c_i \), \( q_i = \gamma_i^{-1} c_i^T x \). If we then assemble the \( q_i \) into a vector \( q \) it will be defined by

\[ q = C^{-1} x = \Gamma^{-1} C^T x \quad \text{and hence} \quad x = C q. \quad \text{Normal Co-ordinate Definition} \quad (I) \]

(See equation (12) for the definition of \( C^{-1} \).) This particular choice of \( q \) preserves the role of \( c \) as the vector appearing in the expression for \( x \) as given in the very first equation in Box (A).

Thus spotting the factors which allow the equations of motion to be uncoupled is equivalent to spotting the eigenvectors of \( K \). The reason that \( K \) matrices with the extra symmetry \( k_{11} = k_{22} \) can be easily decoupled is that they all have (unnormalised) eigenvectors \( c_1 = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \) and \( c_2 = \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \). These obviously have \( \gamma_1 = \gamma_2 = 2 \) and the vectors we use to form the normal co-ordinates are \( \left( \begin{array}{c} 1/2 \\ 1/2 \end{array} \right) \) and \( \left( \begin{array}{c} 1/2 \\ -1/2 \end{array} \right) \).

### 3.4 A more elegant formulation.

The solutions given above (equation (18) and box (H)) treat the normal modes one at a time, with an explicit \( i \) subscript labelling the mode. A slightly more elegant approach uses the vector notation to remove this. We approach this through the normal co-ordinate vector defined above. We multiply the equation of motion, box (B), by \( C^{-1} \) to give an equation for the normal co-ordinates:

\[ m \frac{d^2q}{dt^2} = -C^{-1} K x = -\Gamma^{-1} C^T K x = -\Gamma^{-1} \Lambda C^T x = -\Lambda q \]

where the third equality uses the matrix form of the right eigenvector equation, (10), and the final equality the fact that the diagonal matrices \( \Gamma^{-1} \) and \( \Lambda \) commute. We now define the diagonal matrix of angular frequencies \( \omega_i = \sqrt{\lambda_i/m} \):

\[ \Omega = \text{diag}(\omega_1, \omega_2, \ldots, \omega_N), \]

so that \( \Lambda = m \Omega^2 \). Dividing by \( m \), the equation of motion for \( q \) becomes

\[ \frac{d^2q}{dt^2} = -\Omega^2 q. \]  

(21)
Each row $i$ of this vector of equations is an equation for the corresponding normal co-ordinate $q_i$, so since $\Omega^2$ is diagonal we have completely uncoupled the equations of motion. If we define the solution matrices

$$F(t) = \begin{pmatrix} \cos \omega_1 t & 0 \\ 0 & \cos \omega_2 t \end{pmatrix} \quad \text{and} \quad G(t) = \begin{pmatrix} \sin \omega_1 t & 0 \\ 0 & \sin \omega_2 t \end{pmatrix}$$

then we can write the solution in terms of constant vectors $a$ and $b$:

$$q(t) = Fa + Gb$$

or, multiplying by $C$ to transform back to $x(t)$ rather than $q(t)$:

$$x(t) = C(Fa + Gb).$$

This is equivalent to the solution in equation (18). The constants in the two equations are related by

$$a = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

We can evaluate the constant vectors in essentially the same way as in the development from equation (18) to box (H). At $t = 0$, $G = 0$ and $F$ is the identity matrix, so $x(0) = Ca$ or $a = C^{-1}x(0)$. Similarly the velocity is given by $v(0) = C\Omega b$. Thus $b = \Omega^{-1}C^{-1}v(0)$ giving the complete solution in the form

$$x(t) = C(Fa + Gb)$$

which is completely equivalent to box (H).


4.1 Kinetic Energy.

We are still dealing with the rather special case when all the oscillating masses are equal. The final topic to be looked at is the energy of the oscillating system. The kinetic energy of the oscillating masses $m$ is obviously

$$T = \sum_i \frac{1}{2} m v_i^2.$$

We can write this as a matrix product using the velocity vector $v$:

$$T = \frac{1}{2} m v^T v.$$  \hspace{1cm} (23)$$

4.2 Potential Energy.

We can derive the forces from a potential as long as the stiffness matrix $K$ is symmetric. We shall find the potential energy at extension $x$ by integrating the work done to get to $x$ from zero extension (also the zero of potential energy) along a path $y = \alpha x$ from $\alpha = 0$ to $\alpha = 1$. The forces exerted by the mechanical system at extension $y$ are given by $f = -Ky$; thus the external source of energy must supply forces $f_{\text{ext}} = +Ky$. The work done is found by integrating $f_{\text{ext}} \cdot dy$:

$$V(x) = \int_0^x (Ky) \cdot dy = x^T Kx \int_0^1 \alpha \, d\alpha = \frac{1}{2} x^T Kx.$$
If we now differentiate to find the forces (force on mass \( i \) given by \( \partial V / \partial x_i \)) we find
\[
f = \frac{1}{2} (Kx + K^T x)
\]
which demonstrates that this approach only works for symmetric \( K \).

### 4.3 Conservation of Energy.

Putting these two results together we find the energy is given by
\[
E = \frac{1}{2} \left( m\dot{v}^T v + x^T Kx \right).
\]

Differentiating with respect to \( t \):
\[
\frac{dE}{dt} = m\dot{v}^T \dot{v} + \dot{x}^T Kx = \dot{v}^T \left( m\frac{dv}{dt} + Kx \right).
\]

However the equation of motion of box (L) ensures that the bracket is zero and so energy is conserved
\[
\frac{dE}{dt} = 0.
\]

### 4.4 Division of the Energy between Normal Modes

If we substitute the decomposition of \( x \) into normal coordinates from box (I) into box (K) we can find how the energy is divided between the normal modes:
\[
E = \frac{1}{2} \left( m\dot{q}^T C^T C\dot{q} + q^T C^T K C q \right)
\]

where we have used the dot notation for compactness to denote \( dq/dt \). In the first term we use \( C^T C = \Gamma \). In the second term we use the right eigenvector equation \( 8 \) to simplify:
\[
C^T K C = C^T C \Lambda = \Gamma \Lambda.
\]

With these two simplifications we obtain
\[
E = \frac{1}{2} \left( m\dot{q}^T \Gamma \ddot{q} + q^T \Gamma \Lambda q \right).
\]

The significant thing about this expression for the total energy is that the matrices in the middle of the two terms, \( \Gamma \) and \( \Gamma \Lambda \), are both diagonal. This means that the expression contains no cross-terms linking one mode to another, so that the energy is expressed as a sum over terms relating to each mode. The two terms for each mode are
\[
E_i = \frac{1}{2} m\gamma_i (\ddot{q}_i)^2 + \frac{1}{2} \gamma_i \lambda_i q_i^2 \quad \text{where} \quad q_i = c_i^T x / \gamma_i \quad \text{and} \quad \gamma_i = c_i^T c_i.
\]

There are three things to be said about this expression for the modal energy.
(a) The explicit dependence on the arbitrary eigenvector normalisation $\gamma_i$ is at first sight surprising. However its effect is to cancel the implicit dependence on normalisation introduced by the normal co-ordinate $q_i$. If we double the length of the eigenvector $c_i$ then $q_i$ will be halved. The two factors of $q_i$ (or $\dot{q}_i$) therefore contribute a factor of a quarter, in this scenario, which is cancelled by the explicit factor of four from the $\gamma_i$.

(b) The expression has the general form of a harmonic oscillator energy, so that we have in a manner decomposed the system into a set of non-interacting harmonic oscillators.

(c) However this interpretation is not straightforward because of the involvement of the arbitrary $\gamma_i$ factors. If we identify $q_i$ as the displacement of the harmonic oscillator, then we must identify $\gamma_i \lambda_i$ as its spring constant, and $\gamma_i m$ as its mass, in order to interpret the two terms in equation (24) as potential and kinetic energy. Thus the oscillator has a well-defined energy $E_i$ and frequency $\omega_i = \sqrt{(\gamma_i \lambda_i)/(\gamma_i m)}$, but its other physical parameters are arbitrary and do not have unique values! These are inevitable consequences of the indeterminacy in the normalisation of the $q_i$, which we are using as the displacement of the oscillator.

5 Inhomogeneous Problems.

In this section we look briefly at two problems giving rise to inhomogeneous equations, as opposed to the homogeneous linear systems we have solved so far.

5.1 Sagging under gravity.

We briefly return to the inhomogeneous problem of box (C) which arises when the displacements of the masses are measured from the zero extension position of the springs. As we stressed above, the standard problem is a homogeneous linear system

$$\mathcal{L} f = 0$$

where $\mathcal{L}$ is the linear operator $m \frac{d^2}{dt^2} + K$, and $f$ is the vector-valued function $x(t)$. What we have now is the corresponding inhomogeneous problem

$$\mathcal{L} f = h$$

where the inhomogeneous term is the weight vector $w$. The solution is thus given by any particular integral plus any solution of the homogeneous problem. Thus to the solutions we have already found we have to add the simplest possible solution of the inhomogeneous problem. However since the inhomogeneous term is a constant, the simplest solution is also a constant, $x_p$:

$$\mathcal{L} x_p = w.$$

The second derivative term vanishes to leave

$$K x_p = w \quad \text{or} \quad x_p = K^{-1} w.$$

This represents the sag of the masses under gravity to reach the equilibrium position, which is the simpler homogeneous problem takes as origin in the first place. Thus nothing interesting is added by the choice of zero force position as origin, just an added complication in finding the extensions required to reach to sagged position.
5.2 External Driving force.

Suppose in the second normal modes example the upper boundary, from which the masses are hung, oscillates vertically. We take a fixed reference point $O$ at the centre of the oscillation, so that at time $t$ the boundary is located at $y = h \sin \omega t$ relative to $O$. The origins for measuring the displacements of the masses are unchanged, at $O_1$ and $O_2$. This means that at time $t$ the extension of the upper spring relative to the distance between $O$ and $O_1$ is changed to $x_1 - y$.

This changes the first equation of motion to

$$m \frac{d^2 x_1}{dt^2} = -k_1(x_1 - y(t)) + k_2(x_2 - x_1)$$

to be compared with equation (3). Assembling the two equations into the usual matrix form we find

$$m \frac{d^2 \mathbf{x}}{dt^2} + \mathbf{K} \mathbf{x} = \mathbf{F}(t) \quad \text{where} \quad \mathbf{F} = \begin{pmatrix} k_1 y(t) \\ 0 \end{pmatrix}.$$  

Thus the external force vector $\mathbf{F}$ appears as an inhomogeneous term in the equation of motion, just as the weight $\mathbf{w}$ did in the earlier example, but $\mathbf{F}$ is a function of $t$. This means that an appropriate form for the particular integral is now $\mathbf{x}_p = a \sin \omega t$. (More generally, with an inhomogeneous term varying as a sine or cosine, we would need both sine and cosine terms in the trial form for the particular integral. However, given that the linear operator here only contains even derivatives, that is, a second derivative, we can use just a sine given the sine-dependence of $\mathbf{F}$.) Substituting the trial form into the equation of motion we obtain, after cancelling the sine function throughout,

$$-m\omega^2 \mathbf{A} + \mathbf{K} \mathbf{a} = \begin{pmatrix} k_1 h \\ 0 \end{pmatrix}.$$  

We can combine the two terms on the left by inserting an identity matrix:

$$(\mathbf{K} - m\omega^2 \mathbf{I}) \mathbf{a} = \begin{pmatrix} k_1 h \\ 0 \end{pmatrix}.$$  

The matrix in the bracket is singular when $m\omega^2$ equals one of the eigenvalues of the matrix $\mathbf{K}$, $\lambda_i$. This of course is when the system is being driven at one of its resonance frequencies $\omega = \omega_i = \sqrt{\lambda_i / m}$. In these cases the trial solution fails, and we need a different form for the particular integral. In all other cases we can simply invert the matrix to find

$$\mathbf{a} = (\mathbf{K} - m\omega^2 \mathbf{I})^{-1} \begin{pmatrix} k_1 h \\ 0 \end{pmatrix}.$$  

(25)

giving the amplitudes of oscillation of the two masses in response to the sinusoidal driving force. As usual we can add to this particular solution any solution of the homogeneous equation.

6 The General Case

6.1 Setting up the problem

In both of the examples so far, the two oscillating masses are identical, which is obviously a special case. If we remove this restriction and label the two masses as $m_1$ and $m_2$, then the equations of motion in the second example, equations (3) and (4), become:

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_2(x_1 - x_2).$$  

(26)

(27)
We can still write the right side as $Kx$, but how do we write the left side as a vector involving the second derivative vector $\frac{d^2x}{dt^2}$? We have to write the masses into a diagonal mass matrix:

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{28}$$

The two equations can then be written, replacing box (B):

$$M \frac{d^2x}{dt^2} = -Kx. \tag{L}$$

This is still in the general form of the simple mass on a spring problem, but with the scalar $m$ replaced by the matrix $M$, in addition to the replacements for $k$ and $x$. However $M$ is not just any matrix; it is specifically diagonal and positive definite, which are very important constraints. In particular $M$ is non-singular: its inverse is simply

$$M^{-1} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix}. \tag{29}$$

We follow the same route as in section 3.2, by substituting the normal mode form $x = f(t)c$ into box (L) and dividing by $f$:

$$\begin{pmatrix} 1 & \frac{d^2f}{dt^2} \end{pmatrix} Mc = -Kc, \tag{30}$$

instead of equation (16). We again observe that the bracket on the left-hand side is constant. However, there is no factor of $m$ in this bracket, as there was in equation (16), so the bracket has dimensions of frequency-squared not spring constant. As before we assume that the motion, is bounded so that the separation constant must be negative, so we define it to be $-\omega^2$. Our equation separates into two:

$$\frac{d^2f}{dt^2} = -\omega^2 f \quad \text{and} \quad Ka = \omega^2 Ma \tag{31}$$

as in equation (17). The left-hand equation gives us sine and cosine solutions, but the right-hand equation is no longer a simple eigenvalue equation. It is, instead, a *generalise eigenvalue equation*, or an eigenvalue equation with a metric: $\omega^2$ is the eigenvalue, and the symmetric positive-definite matrix $M$ is the metric.

### 6.2 Eigenvalues and Eigenvectors with respect to a metric.

We can develop the theory for eigenvalue equations in the case of a symmetric matrix $K$ in exactly the same way as without the metric. We define eigenvalues and left or right eigenvectors with respect to $M$ as in boxes (E) and (D):

$$Kc = \omega^2 Mc \quad \text{and} \quad c^T K = \omega^2 c^T M. \tag{M}$$

(Note that, although we are using the same symbol for them, the eigenvectors with respect to $M$ will not be the same as the eigenvectors of $K$.) Here the second equation is the transpose of the first, so the symmetry of $K$ and $M$ ensure that the same vector $c$ is both a left and right eigenvector with respect to $M$. The characteristic equation, as in box (F), is given by
The orthogonality of the eigenvectors is demonstrated from the construct $c_i^T K c_j$ which can be expanded using either the left or right eigenvector definition to deduce $(\omega_i^2 - \omega_j^2) c_i^T M c_j = 0$. This shows that eigenvectors belonging to different eigenvalues are orthogonal with respect to the metric $M$:

$$c_i^T M c_j = \mu_i \delta_{ij}. \quad (32)$$

The presence of $M$ in this equation precludes the choice of $\mu_i = 1$, since this would entail that the eigenvectors, instead of being pure numbers, would acquire dimensions of $[\text{Mass}]^{1/2}$. There is no obvious choice for the elements of this diagonal matrix, so we leave the normalisation of the $c_i$ arbitrary for the moment.

The argument for the eigenvalues being real is almost unchanged, and the argument for completeness of the eigenvectors can also be re-worked in this context. Thus all of the special properties of the eigenvectors of a symmetric matrix of section 2.3 are also true for eigenvectors with respect to $M$ with appropriate changes.

We can then summarize these properties in matrix equations by assembling the eigenvectors into a matrix:

$$C = (c_1 \ c_2 \ \ldots).$$

The equations above can be summarized as

$$KC = MC \Omega^2 \quad C^T MC = M \quad (33)$$

where $M = \text{diag}(\mu_1, \mu_2 \ldots)$. The second equation tells us that $C^{-1} = M^{-1} C^T M$.

6.3 Solution of the problem.

We now recognise that equation (31) defines $c$ to be one of the eigenvectors $c_i$ of $K$ with respect to metric $M$, allowing us to write the complete solution as

$$x(t) = \sum_i c_i \left( A_i \cos \omega_i t + B_i \sin \omega_i t \right) \quad (34)$$

in complete analogy to equation (18). The solution for the parameters goes through almost the same:

$$x(0) = \sum_i A_i c_i$$

as before, but the $c_i$ are no longer orthogonal we cannot just dot with $c_1$ as before. Instead, following equation (32), we multiply by $M$ and then dot with $c_1$. The orthogonality then picks up the term $i = 1$:

$$c_1^T M x(0) = A_1 \ c_1^T c_1 = A_1 \mu_1.$$

Similarly the initial velocity is given by

$$v(0) = \sum_i A_i \omega_i c_i$$

and the same procedure gives

$$c_1^T M v(0) = A_1 \omega_1 \ c_1^T c_1 = A_1 \omega_1 \mu_1.$$

This the complete solution becomes
\[ x(t) = \sum_i \left[ c_i^T Mx(0) \right] \cos(\omega_i t) c_i + \left[ c_i^T Mv(0) / \omega_i \right] \sin(\omega_i t) c_i \]

Complete Solution \((O)\)

which replaces box (H).

### 6.4 Normal Co-ordinates of the Problem.

Following the same route as section 3.3 we left-multiply the equation of motion by \(C^T\):

\[
\frac{d^2}{dt^2} (C^T M x) = -C^T K x = -\omega^2 (C^T M x)
\]

where the second step uses the right eigenvector equation from (33). Thus \(q_i = C^T M x\) is a normal co-ordinate, and we have the same freedom to multiply by a convenient constant. Unlike the equal mass case we now need this freedom to eliminate the extra dimension of mass introduced by the matrix \(M\), so that this \(q_i\) has a different dimension from \(x\). We therefore use the normal co-ordinates \(q_i = (\mu_i^{-1})(C^T M x)\). The vector of normal co-ordinates is then

\[
q = M^{-1} C^T M x = C^{-1} x \quad \text{and hence} \quad x = Cq. \quad (35)
\]

This is identical with box (I) apart from change in the definition of \(C^{-1}\).

We then recover the same normal mode equation as in the equal mass case (21):

\[
\frac{d^2 q}{dt^2} = -\Omega^2 q \quad (36)
\]

The solution goes through in the same way:

\[
q(t) = Fa + Gb
\]

where the initial conditions give \(a = q(0)\) and \(b = \Omega^{-1} \frac{dq(0)}{dt}\). We can then invert to give

\[
x(t) = C \left( FC^{-1} x(0) + G \Omega^{-1} C^{-1} v(0) \right) \quad (P)
\]

in complete analogy to box (J), apart from the same replacement of \(C^T\) by \(C^{-1}\).

### 6.5 Energy in the General Case

The earlier treatment of energy requires very small changes to accommodate the general case. The formula for the potential energy is unchanged, but the kinetic energy becomes \((1/2) \sum_i m_i v_i^2 = (1/2) v^T M v\): \(\text{E} = \frac{1}{2} (v^T M v + x^T K x) \quad (Q)\)

which replaces box (K).

If we substitute for \(x\) and \(v\) in terms of \(q\) and \(\dot{q}\): \(x = Cq\) and \(v = C\dot{q}\), we get

\[
E = \frac{1}{2} \left( \dot{q}^T C^T M C \dot{q} + q^T C^T K C q \right).
\]
In the second term \(KC = MC\Omega^2\), and then in both terms \(C^TMC = M\):

\[
E = \frac{1}{2} \left( q^T M \dot{q} + q^T M \Omega^2 q \right).
\]

This expresses the energy as a sum of terms associated with each normal mode, each of harmonic oscillator form with a modal mass given by \(\mu_i\). The same caveat as in the equal mass case applies here: the division of the total energy \(E\) into modal energies \(E_i\) is a rigorous result, as is the modal frequency \(\omega_i\), but the modal mass is indeterminate since it depends on the arbitrary normalisation of the eigenvectors used to define the normal co-ordinates.

### 7 Lagrangian and Hamiltonian formulations

Having explicit expressions for the kinetic and potential energies we can readily define the Lagrangian

\[
\mathcal{L}(x, v) = T - V = \frac{1}{2} \left( v^T M v - x^T K x \right).
\]

The usual Lagrangian equation of motion then gives

\[
\frac{d}{dt} M v + K x = 0
\]

which is, of course, the same equation of motion we found above, box (L).

If we find the corresponding momenta \(p = \nabla_\mathcal{L} = M v\) we can find the Hamiltonian:

\[
\mathcal{H}(x, p) = x^T p - \mathcal{L} = \frac{1}{2} \left( p^T M^{-1} p + x^T K x \right),
\]

which is just \(E\) from box (Q) in terms of \(x\) and \(p\). The Hamiltonian equations of motion are then

\[
\dot{x} = M^{-1} p \quad \dot{p} = -K x
\]

which reduce, unsurprisingly, to the equation of motion of box (L).

If instead we set up the Lagrangian in normal co-ordinates, using the eigenvectors of \(K\) with respect to metric \(M\) we find

\[
\mathcal{L'}(q, \dot{q}) = \frac{1}{2} \left( \dot{q}^T M \dot{q} - q^T M \Omega^2 q \right).
\]

This gives the uncoupled equations of motion

\[
\frac{d\dot{q}}{dt} = -\Omega^2 q
\]

as in equation (36).

The momenta are simply \(P = M \dot{q}\) so this time the Hamiltonian is given by

\[
\mathcal{H'}(q, P) = P^T M^{-1} P - \mathcal{L} = \frac{1}{2} \left( P^T M^{-1} P + q^T M \Omega^2 q \right),
\]

which is just the normal co-ordinate form of \(E\), equation (37). This gives the Hamiltonian equations of motion in essentially the same form as the Lagrangian.

\[
\dot{q} = M^{-1} P \quad \dot{P} = -M \Omega^2 q.
\]
We can of course transform directly from $\mathcal{H}(x, p)$ to $\mathcal{H}'(q, P)$, and this forms an instructive example of a canonical transformation, using a generating function $F_2(x, P)$, according to the classification of Goldstein (*Classical Mechanics*, 3rd ed., section 9.1). The transformation function takes the form

$$F_2(x, P) = P^T C^{-1} x.$$  

The equations of transformation are then

$$q = \nabla_P F_2 = C^{-1} x \quad \text{and hence} \quad x = Cq$$

as required for the co-ordinates, and for the momenta

$$p = \nabla_x F_2 = (C^T)^{-1} P \quad \text{and hence} \quad P = C^T p.$$  

This is absolutely typical of any linear transformation of the co-ordinates: if the new co-ordinates are given by $q = C^{-1} x$ then the new momenta are given by $P = C^T p$; in the special case of a rotation, when $C$ is an orthogonal matrix, then both momenta and co-ordinates transform in the same way since $C^T = C^{-1}$. However in this case the momenta do *not* transform in the same way. The new Hamiltonian is given by

$$\mathcal{H}'(q, P) = \mathcal{H}(x(q), p(P)) = \frac{1}{2} \left( P^T C^{-1} M^{-1} (C^T)^{-1} P + q^T C^T K C q \right).$$

The matrix products simplify greatly: $C^T K C = M \Omega^2$ as above, and $C^{-1} M^{-1} (C^T)^{-1} = (C^T M C)^{-1} = M^{-1}$ so that

$$\mathcal{H}'(q, P) = \frac{1}{2} \left( P^T M^{-1} P + q^T M \Omega^2 q \right),$$

as we found above.