Normal Modes and Waves

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2025 Problem Set

1 Normal Modes

(1.1) Standard Solution Method. Two identical pendula each of length l and with bobs of mass m are free to oscillate in the same plane. The bobs are joined by a massless spring with a small spring constant k, such that the tension in the spring is k times its extension.

(a) Show that the motion of the two bobs is governed by the equations

$$m\frac{\mathrm{d}^2\mathbf{x}_1}{\mathrm{d}t^2} = -\frac{mg}{l}x_1 + k(x_2 - x_1) \tag{1}$$

and

$$m\frac{\mathrm{d}^2\mathbf{x}_2}{\mathrm{d}t^2} = -\frac{mg}{l}x_2 + k(x_1 - x_2) \tag{2}$$

(b) Write these equations in the form

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x}$$

and write down the $\mathbf{K}$ matrix.

(c) Substitute a normal mode solution $\mathbf{x} = \mathbf{a}\,f(t)$ and show that this satisfies the equation of motion provided $\mathbf{a}$ is an eigenvector of $\mathbf{K}$. Find and solve the corresponding equation for $f(t)$.

(d) How many eigenvectors does $\mathbf{K}$ have? Find them and write down a general solution for the problem.

(e) At $t = 0$, both pendula are at rest, with $x_1 = A$ and $x_2 = A$. Describe the subsequent motion of the two pendula.

(1.2) Fitting Initial Conditions. Consider the two coupled pendula of question 1.1.

(a) At $t = 0$, both pendula are at rest, with $x_1 = A$ and $x_2 = 0$. They are then released. Describe the subsequent motion of the system. If $k/m = 0.105g/l$, show that

$$x_1 = A\cos\bar{\omega}t\cos\Delta t$$

and

$$x_2 = A\sin\bar{\omega}t\sin\Delta t$$

where $\bar{\omega} = 1.05\sqrt{g/l}$ and $\Delta = 0.05\sqrt{g/l}$.
Sketch x_1 and x_2, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

(b) State a different set of initial conditions such that the subsequent motion of the pendula corresponds to that of a normal mode.

(c) At $t = 0$, both bobs are at their equilibrium positions: the first is stationary but the second is given an initial velocity v_0. Show that subsequently

$$\mathbf{x} = \frac{v_0}{2\omega_1}\begin{pmatrix}1\\1\end{pmatrix}\sin\omega_1 t + \frac{v_0}{2\omega_2}\begin{pmatrix}-1\\1\end{pmatrix}\sin\omega_2 t$$

(d) For the initial conditions of part (c), and with $k/m = 0.105g/l$, sketch the subsequent positions and velocities of the two bobs.

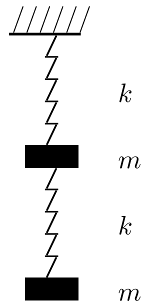

Figure 1: Example for Question 1.3

(1.3) Non-trivial Eigenvectors. Two equal masses m are connected as shown in Figure 1 with two identical massless springs, of spring constant k.

(a) Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by $\omega_i^2 = q_i(k/m)$ where $q_1 = [3 - \sqrt{5}]/2$ and $q_2 = [3 + \sqrt{5}]/2$

(b) Find corresponding eigenvectors.

(c) Why does the acceleration due to gravity not appear in these answers?

(1.4) Zero Eigenvalue. Two particles, each of mass m, are connected by a light spring of stiffness k, and are free to slide along a frictionless horizontal track.

(a) Find the normal frequencies and eigenvectors of this system.

(b) Why does a zero-frequency mode appear in this problem? Write down the general solution.

(1.5) Energy in the equal mass case. The system of question 1.3 with equal springs ($k_1 = k_2 = k$) (shown in Figure 1) is excited by pulling the lower mass down a distance $2a$ and releasing from rest.

(a) Show that the initial condition is $\mathbf{x}(0) = \begin{pmatrix} a \\ 2a \end{pmatrix}$.

(b) Show that the system energy is ka^2.

(c) Show that the energy in normal mode i is given by $\lambda_i[\mathbf{c}_i^T\mathbf{x}(0)]^2/[2\mathbf{c}_i^T\mathbf{c}_i]$. Find the energies in each of the normal modes.(You can give the result numerically, or in the form $\alpha + \beta\sqrt{5}$.))

(1.6) Unequal mass case. In the coupled pendulum example of question 1 the two masses are unequal, m_1 and m_2.

(a) Modify the equations of motion and write in matrix form using the same $\mathbf{K}$ matrix as before and a mass matrix $\mathbf{M}$ which you should define.

(b) Solve the equations by finding the eigenvalues and eigenvectors of $\mathbf{K}$ with respect to the metric $\mathbf{M}$.

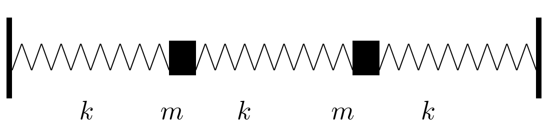

Figure 2: Example for Question 1.7

(1.7) Transverse oscillations; transition to Waves. Consider transverse oscillations of the system shown in Figure 2. The fixed side walls are separated by a distance $3l$, and the two masses divide the distance into three equal spaces of length l. The three springs are all identical with natural length $l_0 \ll l$, so that there is a equilibrium tension T_0 in all three springs.

(*a*) Show that small transverse displacements y_1, y_2 lead to extensions of the springs which are quadratic expressions of the displacements. Hence argue that for suficiently small displacements the dominant restoring force is the transverse component of the tension T_0.

(*b*) Hence show that the linear approximation to the equations of motion is

$$m\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{y} \quad \text{where} \quad \mathbf{K} = \begin{pmatrix} 2T_0/l & -T_0/l \\ -T_0/l & 2T_0/l \end{pmatrix}$$

(*c*) Now consider a much larger system with total length $(N+1)l$ divided into equal sections by N equal masses, with tension T in all springs. Deduce the form of the $\mathbf{K}$ matrix for this case.

(*d*) Hence show that the general row ($i \neq 1$ or N) of the eigenvector equation $\mathbf{Kc} = \lambda\mathbf{c}$ is $c_{i+1} + c_{i-1} = (2 - \lambda l/T)c_i$, where c_i denotes the i'th element of the eigenvector $\mathbf{c}$. Show that $c_i = \sin(i\phi)$ satisfies this equation and show that λ must then be given by $\lambda = (2T/l)(1 - \cos\phi)$. Show that this form also satisfies the $i = 1$ row of the equation, and that to satisfy the $i = N$ row of the equation we need $\phi = n\pi/(N+1)$ for n= 1, 2,... N.

(*e*) Now consider very large N, and $n \ll N$. Define $L = (N+1)l$ as the total length of the string, and $k_n = n\pi/L$ as the rate of change of phase with distance along the string. Show that $\lambda \approx Tk_n^2 l$. Hence deduce that for these low-n modes ω_n and k_n are related by $\omega_n^2/k_n^2 = Tl/m$. Comment on the relation to the wave equation.

2 Waves I: General, Dispersion, Wave equation.

(2.1) Standing and Travelling Waves.

(a) Outline the differences between a travelling wave and a standing wave.

(b) Convince yourself that $y_1 = A\sin(kx - \omega t)$ corresponds to a travelling wave. Which way does it move and with what velocity? What are the amplitude a, wavelength λ, wavenumber $\bar{\nu}$, wavevector, period T, frequency ν, and angular frequency of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y_1}{\partial t^2}$$

provided that ω and k are suitably related.

(2.2) Superposition: Counter-propagating Waves

(a) Write down a wave y_2 of the same form as y_1 above, but travelling in the opposite direction. Show that the superposition $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t).$$

Convince yourself that this superposition of two travelling waves is a standing wave.

(b) Show that the standing wave $y = A\sin(kx)\sin(\omega t)$ can be written as a superposition of two travelling waves.

(2.3) Superposition: Co-propagating Waves

(a) Consider instead a superposition of two co-propagating waves of different angular frequencies ω, $y_1 = A\cos(\omega_1 t + k_1 x)$ and $y_2 = A\cos(\omega_2 t + k_2 x)$. ($\omega_2 > \omega_1$.) Show that the superposition can be written as the product of two travelling waves, and find the velocity of each. Sketch the wave at $t = 0$ and $t = \pi/(\omega_2 - \omega_1)$.

(b) Consider now a more general superposition of a range of waves with different amplitudes and phases given by the complex amplitude a_i, all with wavevectors around $\bar{k}$, and angular frequencies around $\bar{\omega}$. (As usual, the real part of this complex expression is assumed.)

$$y(x,t) = \sum_i a_i \exp[\mathrm{i}(\omega_i t - k_i x)].$$

Show that if, over this narrow frequency range, we can approximate $\omega(k)$ by a linear dependence $\omega_i = \bar{\omega} + v_g(k_i - \bar{k})$ we can still factor the summation into a wave of angular frequency $\bar{\omega}$ and an envelope, and give the speed of each.

(2.4) Dispersion

(a) What is meant by a dispersive medium and what is the dispersion relation? Define the phase velocity $v_p = \omega/k$ and the group velocity $v_g = \partial\omega/\partial k$. Explain carefully what travels at each velocity.

(b) Show that an alternative expression for v_g is

$$v_g = v_p + k\frac{\partial v_p}{\partial k}.$$

(c) Evaluate v_p and v_g as a functions of k for the following cases:

i. Long wavelength surface waves on water $\omega = \sqrt{gk}$ (where g is the acceleration due to gravity).

ii. Short wavelength ripples on water $\omega = \sqrt{\sigma k^3/\rho}$ (where σ is the surface tension and ρ the density).

iii. In the crossover region where both effects are important $\omega^2 = gk + \sigma k^3/\rho$.

iv. Guided electromagnetic waves in a waveguide (with a non-zero longitudinal component of either E or B) $\omega^2 = \omega_0^2 + c^2k^2$ (where c is the speed of light).

(*d*) In the first two cases but not the other two you should have found $v_g = \alpha v_p$, where the constant α is different in the two cases. What type of dispersion relation leads to this result?

(*e*) In the fourth case you should have found $v_p v_g = c^2$, so that either v_p or v_g is greater than c. Which is it, and why does this *not* allow signalling faster than the speed of light?

(2.5) Waves on Water: cross-over region In the long wavelength limit of question 2.4(c)i., v_p and v_g are decreasing functions of k, while in the short-wavelength limit of 2.4(c)ii. they increase with k. Thus in the cross-over region of question 2.4(c)iii. both pass through minima.

(*a*) Many physical problems can be simplified with the use of dimensionless variables. In the dispersion relation

$$\omega^2 = gk + (\sigma/\rho)k^3$$

the physical constants g and σ/ρ clutter up the equation. g has dimensions $[LT^{-2}]$, and σ/ρ has dimensions $[L^3T^{-2}]$. Find combinations of these with dimensions of length, time and velocity. Hence find dimensionless variables proportional to ω and k, which we denote $\tilde{\omega}$ and $\tilde{k}$. Show that if we multiply the dispersion relation by $\sqrt{\sigma/(g^3\rho)}$ the result can be written

$$\tilde{\omega}^2 = \tilde{k} + \tilde{k}^3.$$

This form brings out the essential mathematical structure by removing the physical constants.

(*b*) Sketch the dispersion relation over the range $\tilde{k}$ between 0 and 2. At any point on the curve the slope of the tangent is the group velocity, and the slope of the line connecting $(\tilde{\omega}, \tilde{k})$ to the origin is the phase velocity. Use your sketch to demonstrate that the smallest possible phase velocity is at the point where $v_p = v_g$. Show that this occurs at $\tilde{k} = 1$, $\tilde{\omega} = \sqrt{2}$, $v_p = v_g = \sqrt{2}$.

(*c*) Your sketch should show an obvious point of inflexion at a lower value of $\tilde{k}$. By differentiating the dispersion equation show that this occurs at

$$\tilde{k} = \sqrt{\frac{2}{\sqrt{3}} - 1}.$$

(2.6) Derivation of The Wave Equation for a string.

(*a*) A string of uniform linear density ρ is stretched to a tension T. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c^2 = T/\rho$.

(*b*) Show that the equation is linear and homogeneous, of the form $\mathcal{L}y = 0$ where $\mathcal{L}$ is a linear differential operator.

(*c*) What does this imply for solutions of the equation?

(2.7) General Solution of Wave Equation The general solution to the one-dimensional wave equation is due to d'Alembert:

$$y(x,t) = f(x - ct) + g(x + ct).$$

(*a*) Give an outline derivation of this.

(*b*) A semi-infinite string is fixed at $x = 0$ so that $y(0,t) = 0$. Find g in terms of f.

(*c*) If the string is now additionally fixed at $x = L$ show that f must be a periodic function, and give its period in x and t.

(d) Show that if a string is released *from rest* at $t = 0$, $f(x) = g(x)$. If the string is also fixed at $x = 0$ and $x = L$ show that $f(-x) = -f(x)$, tht is f is an odd function.

(2.8) Fitting Initial Conditions A tensioned string of length L is plucked at $x = L/4$ a distance h and released from rest.

(a) Using the results of the previous question deduce the form of the function $f(x)$, and sketch it in the interval $[-L, 2L]$.

(b) By superposing two versions of the this function displaced appropriately forwards and backwards, sketch $y(x, t)$ in the (physical) interval $[0, L]$ at times $t = L/8c$, $L/4c$, $3L/8c$, $L/2c$, L/c.

3 Waves II: Separation of Variables, Boundary Problems, Energy

(3.1) Solution by separation of variables. Transverse waves are excited on a string stretched between two fixed points at $x = 0$ and $x = L$.

(a) Outline the solution of the wave equation using the method of separation of variables. Explain carefully how the boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$ determine the sign of the separation constant.

(b) Show that there exist two classes of separated solutions which satisfy the wave equation and the boundary conditions: $y = \sin(n\pi x/L)\sin(n\pi ct/L)$ and $y = \sin(n\pi x/L)\cos(n\pi ct/L)$ for positive integer n. Hence write down a general solution for y.

(c) Show that this general solution is periodic in time and find the period.

(d) Suppose the string is plucked at its midpoint and released from rest at $t = 0$:

$$y(x,0) = \begin{cases} 2ax/L & \text{for } 0 \le x < L/2, \\ a & \text{for } x = L/2 \\ 2a(L-x)/L & \text{for } L/2 < x \le L. \end{cases}$$

Show that the solution must now be of the form

$$y(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

where the coefficients A_n satisfy

$$y(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right).$$

The A_n can be found by the method of Fourier Series, which is not on the first year maths course.

(e) Suppose instead that the initial conditions are $y(x,0) = 0$ and $y_t(x,0) = V(x)$, where y_t denotes $\partial y/\partial t$. Write down the form of the solution in this case, and the equation from which the coefficients can be determined.

(3.2) Fitting Initial Conditions. Suppose instead that the initial conditions for question 3.1 (c) are $y(x,0) = \sin(\pi x/L) + 2\sin(2\pi x/L)$ and $y_t(x,0) = 0$.

(a) Find an explicit expression for $y(x,t)$.

(b) Make rough sketches of $y(x,t)$ at the following times: $t = 0$, $t = L/4c$, $t = L/2c$, $t = 3L/4c$, $t = L/c$.

Note that this solution is neither a standing wave (no fixed nodes) nor a travelling wave (no net progression).

(3.3) Energy Density.

(a) Show that the kinetic energy density u_K and the potential energy density u_P for a transverse wave on a string of linear density ρ and at tension T are given by

$$u_K = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

and

$$u_P = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

(b) Evaluate these for the wave $y = A\sin(kx - \omega t)$ where k and ω are such that y satisfies the wave equation.

(*c*) Show that $u_k = u_P$.

(*d*) Define the energy flux $\mathcal{F}$, and give a formula by which $\mathcal{F}$ can be calculated from $y(x,t)$.

(*e*) Calculate $\mathcal{F}$ for the above y at $t = 0$ and comment on the result.

(*f*) Show that in general

$$\frac{\partial \mathcal{F}}{\partial x} = -\frac{\partial (u_K + u_P)}{\partial t}.$$

and comment on the significance of this result.

(3.4) d'Alembert Solution, Superposition, Energy. At tensioned string has a wave with displacement $y(x,t) = f(x - ct) + g(x + ct)$ where

$$f(u) = \begin{cases} A \sin ku & \text{if } -2\pi \le ku \le -\pi; \\ 0 & \text{otherwise.} \end{cases} \qquad g(u) = \begin{cases} A \sin ku & \text{if } \pi \le ku \le 2\pi; \\ 0 & \text{otherwise.} \end{cases}$$

(*a*) Sketch $y(x,0)$ and $v(x,0)$ where $v = \partial y/\partial t$.

(*b*) Calculate $y(x, 3\pi/2kc)$ by two methods:

i. Using the explicit form in terms of f, and g;

ii. *Harder!* Using the d'Alembert solution in terms of initial conditions on $y(x,0) = Y(x)$ and $v(x,0) = V(x)$:

$$y(x,t) = \frac{1}{2}\left[Y(x - ct) + Y(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x')\,\mathrm{d}x'.$$

(Evaluate y at $x = 0$ and $x = 3\pi/k$ only).

(*c*) Calculate the energy at $t = 0$ and at $t = 3\pi/2kc$.

(3.5) Reflections and Transmission at an Interface. Two semi-infinite strings are connected at $x = 0$ and stretched to a tension T. They have linear density ρ_1 and ρ_2 respectively. A harmonic travelling wave

$$y = \cos[\omega(t - x/c_1)]$$

travels along string 1 towards the boundary at $x = 0$.

(*a*) Determine the amplitudes $\mathbf{r}$ and $\mathbf{t}$ of the reflected and transmitted waves.

(*b*) Calculate the energy flux in both regions and show that energy is conserved at the boundary.

(*c*) Write down the reflection and transmission coefficients for a wave incident from the other side of the boundary. Verify that they satisfy the Stokes relations $\mathbf{r}' = -\mathbf{r}$ and $\mathbf{r}^2 + \mathbf{t}\mathbf{t}' = 1$.

(3.6) The same in the time domain

(*a*) Consider now the same boundary with a wave incident on string 1 $y = f(t - x/c_1)$. Write down the reflected ($g(x,t)$) and transmitted ($h(x,t)$)waves. (*Hint: does the cosine function play an essential role in the calculation in question 3.5?*)

(*b*) Calculate c_2/c_1, $\mathbf{r}$,$\mathbf{r}'$,$\mathbf{t}$ and $\mathbf{t}'$ for the case $\rho_2 = 4\rho_1$.

(*c*) String 2 is now terminated by fixing it to a wall at $x = L$: $y(L,t) = 0$. Show that any incident wave at the wall is reflected with reflection coefficient -1.

(*d*) A single short pulse amplitude A, width $W \ll L$ is now sent along the string 1 towards the boundary, from negative x, arriving at $x = 0$ at $t = 0$. Sketch the displacement of the string at times $t = L/c_1$, $3L/c_1$, $5L/c_1$, $9L/c_1$, with showing the height, width and velocity of any pulses.

(*e*) Justify that the energy E of the initial pulse is proportional to A^2/W. Hence show how E is divided between the reflected pulses.

(3.7) Other types of wave; Wave Impedances. In in many cases waves can be thought of as a dynamic interaction between a force-like quantity which drives the wave and a velocity-like quantity which is the response. Consider, for example, signals propagating on a two-conductor transmission line, such as an ethernet cable or a television aerial cable. The force-like quantity is the voltage between the two conductors $V(x,t)$ and the velocity-like quantity is the current flowing along the conductors (in the positive x-direction) $I(x,t)$. These are related by two equations:

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \quad \text{and} \quad \frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}.$$

The constants L and C are material properties of the specific cable. (Physicists may like to work out what they represent: they are both 'per unit length' quantities. P&P need not concern themselves with physical interpretation for the the purposes of this paper: it's explicitly excluded from the syllabus since you don't know the relevant electromagnetism.)

(a) Show that both I and V satisfy the wave equation, and find the wave speed c in terms of the material constants. Show that the solution for a forwards-going signal has V and I proportional to each other:

$$V(x,t) = V_+(t - x/c) \qquad I(x,t) = \frac{V_+(t - x/c)}{Z}$$

and find the wave impedance Z in terms of the material constants.

(b) Derive the corresponding current/voltage relation for a backwards-going signal $V(x,t) = V_-(t + x/c)$. Why can we immediately write down a general solution $V = V_+ + V_-$ and $I = I_+ + I_-$?

(c) The energy density and energy flux associated with this signal are $U = \frac{1}{2}(LI^2 + CV^2)$ and $\mathcal{F} = IV$. Show that energy is locally conserved.

(d) Repeat for plane electromagnetic waves in vacuum for which the force-like quantity is the y-component of the electric field E and the velocity-like quantity is the z-component of the magnetic field H. These are related by

$$\frac{\partial H}{\partial x} = -\epsilon_0\frac{\partial E}{\partial t} \quad \text{and} \quad \frac{\partial E}{\partial x} = -\mu_0\frac{\partial H}{\partial t}.$$

(e) On a tensioned string there is only one variable, the displacement y. However we can still apply this approach if we take as the velocity-like quantity the string velocity $\partial y/\partial t$, and for the force-like quantity the force in the +ve direction $-T(\partial y/\partial x)$. Show that these choices lead to the same standard form for the equations connecting I and V, and hence identify c, Z, u and $\mathcal{F}$.

4 Normal Modes and Waves: Harder Problems

(4.1) Symmetric Three-mass Normal Modes (CO_2 molecule). Three masses are joined in a straight line by two equal springs, in an arrangement similar to question 1.4. The central mass is M and the other two are both m, and the spring constant is k. We consider longitudinal oscillations of the masses.

(*a*) Show that the **K**-matrix and **M**-matrix for this problem are

$$\mathbf{K} = \begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

and calculate the matrix $\mathbf{G} = M^{-1}\mathbf{K}$.

(*b*) Find the eigenvalues and eigenvectors of G. Interpret the zero eigenvalue physically and explain why the eigenvector has the form that it does.

Explain the behaviour of the centre of mass of the molecule in the three modes, in terms of Newton's first law of motion.

What is the ratio of the squared frequencies of the other two modes, in the case of CO_2, where the masses C:O are in the ratio 12:16?

(*c*) The observed ratio of these squared frequencies in CO_2 is 3, which is lower than the value found above. This is because of direct interaction between the two oxygen atoms not included in this model. We can add this to the model by including extra restoring forces $\pm k(x_1 - x_3)$ on the two oxygen atoms. Show that this changes only one of the two normal mode frequencies, and none of the eigenvectors. (Hint: if this is true you can just multiply the given eigenvectors by **G** to find the new eigenvalues.) What value of k/K will give the observed squared-frequency ratio?

(*d*) The system has an obvious reflection symmetry about the middle. Show that for a given set of displacements **x** the matrix **S** generates the reflected displacements, where

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Obviously, since it is a reflection, $\mathbf{S}^2 = 1$. Show that **S** commutes with **G** (either version).

(*e*) Hence show that if **c** is an eigenvector satisfying $\mathbf{Gc} = \lambda\mathbf{c}$, then so is **Sc**. In this problem there is no degeneracy, and so **Sc** must be a multiple of **c**: in other words **c** is an eigenvector of **S**. But since $\mathbf{S}^2 - 1 = 0$, the eigenvalues of **S** are ± 1. Show that these correspond to symmetric and antisymmetric vibrations. Thus we have shown that the eigenvectors all have definite symmetry under **S**. Classify the eigenvectors found in part (b).

(4.2) Hanging mass on spring system. We return to the second example from the Normal Modes handout, two hanging masses on springs, but this time with the masses unequal, as shown in figure 3. The **K**-matrix was given in the handout:

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}.$$

Show that the **G**-matrix is

$$\mathbf{G} = \begin{pmatrix} a + c & -c \\ -b & b \end{pmatrix}$$

where $a = k_1/m_1$, $b = k_2/m_2$, and $c = k_2/m_1$. We define $\alpha = a/(a+b+c)$, $\beta = b/(a+b+c)$ and $\gamma = c/(a+b+c)$; note that $\alpha + \beta + \gamma = 1$. (The equal mass and equal spring case, which we solved in question 1.3, has $\alpha = \beta = \gamma = 1/3$.) Show that the eigenvalues of $\mathbf{G}$ are given by

$$\lambda = (a + b + c)\left(\frac{1 \pm \sqrt{1 - 4\alpha\beta}}{2}\right).$$

By choosing different values of α, β and γ (subject to them adding up to one) we can explore several different limits of this generic system. We shall look only at one case, $k_1 \gg k_2$, and also $m_1 \gg m_2$, so that we have a large mass m_1 quasi-rigidly connected to the fixed point, while m_2 is loosely connected. The specific choice we make (chosen to make the numbers easy!) is $m_2 = m, \quad m_1 = 3m$, and $k_2 = k, \quad k_1 = 4k$.

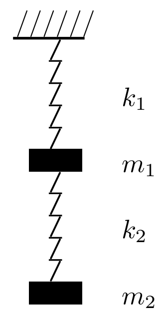

Figure 3: Hanging mass system

(a) Find the $\mathbf{G}$-matrix for this problem.

(b) Find the eigenvalues of the $\mathbf{G}$-matrix.

(c) Find the right eigenvectors of the $\mathbf{G}$-matrix, and use them to find the moving mass in each mode.

(d) At $t = 0$ mass m_2 is pulled down by $2a$, and released from rest (as in question 1.5). Find the intial condition on x_1. Decompose $\mathbf{x}(0)$ into normal modes.

(e) Find the total energy of the system and how it is divided between the modes.

(4.3) Mass on boundary: Boundary conditions. On a string, with tension T and mass per unit length ρ, a point mass m is attached at $x = 0$. There are waves on the string. We shall denote the displacement for $x < 0$ by $y_1(x,t)$ and for $x > 0$ by $y_2(x,t)$, and the displacement on the boundary by $Y(t) = y_1(0,t) = y_2(0,t)$.

Show that the boundary conditions on the displacement $y(x,t)$ are as follows:

(a) The displacement $y(x,t)$ is continuous at $x = 0$: $y_1(0,t) = y_2(0,t)$.

(b) The spatial gradient $y'(x,t)$ is discontinuous at $x = 0$, such that

$$T[y_2'(0,t) - y_1'(0,t)] = m\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2}.$$

(4.4) Solution using Complex Waves. A wave approaches the mass from negative x:
$y_i = \quad \exp[\mathrm{i}\omega(t - x/c)]$. There is a reflected wave :
$y_r = \tilde{r}\exp[\mathrm{i}\omega(t + x/c)]$, and a transmitted wave:
$y_t = \tilde{t}\exp[\mathrm{i}\omega(t - x/c)]$.

Show using the boundary conditions that

$$\tilde{r} = \frac{-\mathrm{i}m\omega c/(2T)}{1 + \mathrm{i}m\omega c/(2T)}, \quad \text{and} \quad \tilde{t} = \frac{1}{1 + im\omega c/(2T)}.$$

Hence write down the real solution corresponding to this complex solution.

(4.5) Solution using trignometric functions. The use of complex functions here is obviously not necessary, since thay do not appear in the final answer. Instead we write the input wave as $y_i = \cos[\omega(t - x/c)]$, and the reflected and transmitted waves as
$y_r = \mathsf{r}\cos[\omega(t + x/c)] + \mathsf{r}'\sin[\omega(t + x/c)]$ and $y_t = \mathsf{t}\cos[\omega(t - x/c)] + \mathsf{t}'\sin[\omega(t - x/c)]$.

(*a*) Show that the the boundary conditions imply

$$1 + \mathsf{r} = \mathsf{t} \quad \mathsf{r}' = \mathsf{t}' \quad (T\omega/c)(1 - \mathsf{r} - \mathsf{t}) = m\omega^2\mathsf{t}' \quad (T\omega/c)(\mathsf{r}' + \mathsf{t}') = m\omega^2\mathsf{t}$$

(*b*) Hence find y_r and y_t, and reconcile with the result from 4.2(c)

(4.6) Energy Conservation on the boundary. The discontinuity in y' entails that the Flux $\mathcal{F}$ is not continuous on the boundary either. Show how energy is conserved on the boundary taking account of the kinetic energy of the mass. [Hint: take the gradient boundary condition and multiply by $\mathrm{d}Y/\mathrm{d}t$.]

(4.7) Energy in Plucked String. A string is fixed at $x = 0$ and $x = L$ and plucked at the centre a distance a. Show that the energy in this initial triangle wave is $E_{\text{total}} = 2Ta^2/L$. The resulting wave is described by

$$y = \sum_{n=\text{odd}} A_n \sin k_n x \cos \omega_n t \quad \text{where} \quad A_n = (-1)^{(n-1)/2}\frac{8a}{n^2\pi^2}; \quad k_n = n\pi/L; \quad \omega_n = k_n c.$$

Show that the energy is a sum over modal energies E_n where

$$E_n = \frac{1}{4}MA_n^2\omega_n^2$$

and that these modal energies sum to E_{total}. You may find the following sum helpful:

$$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and so, dividing by 4:} \sum_{n=\text{even}} \frac{1}{n^2} = \frac{\pi^2}{24.}$$

What fraction of the energy is in the $n = 1$ mode?

(4.8) And finally something completely different: Ptolomy was right (to second order)! Before Copernicus, the best description of the solar system was that of Ptolomy, and Copernicus continued to use the same model for the orbit in his heliocentric system. This was based on the perfection of uniform circular motion, and represented the planetary orbit as a superposition of a large uniform circular motion and a small additional circular motion centred on the circumference of the first circle. epicycle. This is in fact the non-trivial second approximation to the Newtonian solution.

(*a*) The first order solution is a uniform circular orbit $\mathbf{r} = a\hat{\mathbf{u}}(t)$ where

$$\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and } \hat{\mathbf{u}} \text{ is a rotating unit vector:} \quad \hat{\mathbf{u}} = \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.$$

Show that this satisfies the Newtonian equation of motion provided $\omega^2 a^3 = GM$ (treating the Sun as so massive that it is essentially fixed).

(*b*) In the second approximation, any nearly circular orbit can be written as $\mathbf{r} = a\hat{\mathbf{u}} + \mathbf{e}$, where $\mathbf{e}$ is the error in the first approximation. (Now that the circle with radius a is no longer the same as the orbit it is referred to as the deferent.) Substitute this into the Newtonian equation and expand in powers of $(|\mathbf{e}|/a)$, keeping only the linear term. You should find that

$$\frac{\mathrm{d}^2\mathbf{e}}{\mathrm{d}t^2} = \omega^2[-\mathbf{e} + 3(\mathbf{e}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}]$$

(c) At this point we could substitute the explicit form for $\hat{\mathbf{u}}$ and solve these two coupled differential equations for e_x and e_y. The equations are linear (by construction, since we arrived at them by linearizing the Newtonian equation) but the coefficients are time-dependent. An alternative is to solve instead for the radial and tangential components of $\mathbf{e}$. To this end we introduce a second rotating unit vector orthogonal to $\hat{\mathbf{u}}$, directed along the circumference of the deferent:

$$\hat{\mathbf{c}} = \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}. \qquad \text{Note that} \quad \frac{\mathrm{d}\hat{\mathbf{u}}}{\mathrm{d}t} = \omega\hat{\mathbf{c}} \quad \frac{\mathrm{d}\hat{\mathbf{c}}}{\mathrm{d}t} = -\omega\hat{\mathbf{u}}.$$

We expand the non-circular displacement $\mathbf{e}$ in the $\hat{\mathbf{u}}$, $\hat{\mathbf{c}}$ basis: $\mathbf{e} = d\hat{\mathbf{u}} + c\hat{\mathbf{c}}$. Substitute this into the linearised equation for $\mathbf{e}$ to find coupled differential equations for d and c:

$$\frac{\mathrm{d}^2 d}{\mathrm{d}t^2} - 2\omega\frac{\mathrm{d}c}{\mathrm{d}t} - 3\omega^2 d = 0 \quad \frac{\mathrm{d}^2 c}{\mathrm{d}t^2} + 2\omega\frac{\mathrm{d}d}{\mathrm{d}t} = 0.$$

(d) Integrate the second equation and substitute into the first equation to obtain an uncoupled equation for d. Solve to find d and hence c. You should find

$$\begin{aligned} d &= A(\ \cos\omega t) + B(\ \sin\omega t) + C \qquad \text{and} \\ c &= -A(2\sin\omega t) + B(2\cos\omega t) - C(3\omega t/2) + D. \end{aligned}$$

(e) Now substitute back for the rotating unit vectors to find the Cartesian components of $\mathbf{e}$. Interpret the four terms. Two of them represent nearby circular orbits, either of a slightly different radius or a slightly different orbital phase, whereas the other two are the epicycles which I promised you at the outset, but with different phases so that by superposing them we can have any relative phase of the epicycle with respect to co-ordinate system we are using. Pick one of the epicyles to write the full solution as

$$\mathbf{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} + \frac{3A}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{A}{2}\begin{pmatrix} \cos 2\omega t \\ \sin 2\omega t \end{pmatrix}.$$

Here the first term is the original uniform motion around the deferent. The second term translates the entire orbit with respect to the deferent, or equivalently it shifts the central body (the Earth in Ptolomy's system, the Sun in the Copernican system) away from the centre of the deferent. The third term is an epicycle added to the deferent to arrive at the orbiting body's position. This is the full Keplerian/Newtonian solution in the limit of suffiently small eccentricity. It's relationship to Ptolomy is not straightforward because his geocentric system creates more a more complicated situation where the eccentricity of the Earth's orbit has to be taken into account in the supposed orbit of another planet around the Earth, but it is striking that it contains the elements of the full Ptolomeic system: the eccentric location of the central body and the epicycle.