Transition from Normal Modes to waves

We consider transverse oscillations of a very large line of N masses m and $N+1$ springs connecting them, stretched to a tension T and fixed at two points separated by $L = (N+1)l$ where l is the equilibrium distance between the masses. I have in mind an atomic model of a tensioned string, so the masses are atomic masses and the separations are of atomic size, and $N \sim 10^{10}$. We shall call the transverse displacements $\mathbf{y}$. In this case the equations of motion become

$$m\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{y}$$

where $\mathbf{K}$ takes the form

$$\mathbf{K} = \frac{T}{l}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

in the case $N = 5$, that is, a principal diagonal of $+2$ and a super-diagonal and sub-diagonal of -1. We write this as

$$\mathbf{K} = \frac{T}{l}\mathbf{D}$$

where $\mathbf{D}$ is the matrix with 2's and 1's in it. We're now going to assume that on an atomic scale the displacement $y_i(t)$ varies very slowly with i, since this represents the macroscopic wave and the distances between masses are multiples of the microscopic l. So we can in fact introduce a second variable x on which y_i depends, the distance from one of the fixed points. Thus the i'th mass is at $x_i = il$, and

$$y_i(t) = \tilde{y}(x_i, t).$$

Thus we have introduced a smooth function $\tilde{y}$ such that if we evaluate it at a array of points x_i at time t we obtain $y_i(t)$.

The i'th row of the equations of motion is then

$$-\tilde{y}(x_{i+1}, t) + 2\tilde{y}(x_i, t) - \tilde{y}(x_{i+1}, t) = \frac{ml}{T}\frac{\partial \tilde{y}(x_i, t)}{\partial t^2}, \tag{1}$$

where the time derivative has become a partial derivative since $\tilde{y}$ depends on two variables. Bearing in mind the very small separation of the points x_i and x_{i+1} we can use a Taylor series to evaluate $\tilde{y}(x_{i-1}, t)$ and $\tilde{y}(x_{i+1}, t)$.

If we take just the first term in the series we get

$$\tilde{y}(x_{i-1}, t) = \tilde{y}(x_i, t) \quad \text{and} \quad \tilde{y}(x_{i+1}, t) = \tilde{y}(x_i, t).$$

If we substitute this into equation (1) we get

$$\frac{\partial \tilde{y}(x_i, t)}{\partial t^2} = 0.$$

This is not generally true, so this approximation is obviously inadequate. The next approximation is

$$\tilde{y}(x_{i-1}, t) = \tilde{y}(x_i, t) - l\frac{\partial \tilde{y}}{\partial x} \quad \text{and} \quad \tilde{y}(x_{i+1}, t) = \tilde{y}(x_i, t) + l\frac{\partial \tilde{y}}{\partial x}.$$

Again this gives zero on the left side of equation (1), so the lowest order approximation we can use is the second order:

$$\tilde{y}(x_{i-1},t) = \tilde{y}(x_i,t) - l\frac{\partial \tilde{y}}{\partial x} + \frac{l^2}{2}\frac{\partial^2 \tilde{y}}{\partial x^2} \quad \text{and} \quad \tilde{y}(x_{i+1},t) = \tilde{y}(x_i,t) + l\frac{\partial \tilde{y}}{\partial x} + \frac{l^2}{2}\frac{\partial^2 \tilde{y}}{\partial x^2}.$$

This gives us exactly $\partial^2\tilde{y}/\partial x^2$ in the left-hand side of equation (1): in fact this is a finite difference approximation to l^2 times the second derivative, in exactly the same way that $\tilde{y}(x+l,t)-\tilde{y}(x-l,t)$ is a finite difference approximation to $2l$ times $\partial\tilde{y}/\partial x$.

We also note that successive approximations in the Taylor series bring in the small quantity l to one higher power, so the lowest non-zero approximation is also a very good approximation, since neglected terms are much smaller.

Thus we have derived the general equation satisfied by the smooth function that passes through all the y_i:

$$\frac{\partial^2 \tilde{y}}{\partial x^2} = \frac{m}{lT}\frac{\partial^2 \tilde{y}}{\partial t^2}.$$

Finally we note that m/l is the mass per unit length on the string, which we defne to be ρ. We have derived the Wave Equation

$$\boxed{\frac{\partial^2 \tilde{y}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \tilde{y}}{\partial t^2} \quad \text{where} \quad c^2 = \frac{T}{\rho}. \qquad \text{(A)}}$$