Energy in Waves on Strings

One of the defining properties of a wave is that it can transport energy. This handout analyses energy transport and storage in waves on a tensioned string. We shall assume that the string has mass density ρ, tension T, giving a wave speed of $c = \sqrt{T/\rho}$. Positions on the string are labelled by the x co-ordinate, and the purely transverse displacement is y, which satisfies the Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{1}$$

1 Kinetic Energy Density

The total energy of the wave motion is distributed along the string as a certain energy u per unit length, or energy density, at each point. The total energy E can thus be written

$$E = \int u(x,t)\,\mathrm{d}x \tag{2}$$

where the integral runs over the whole string.

The energy density is made up of two contibutions: kinetic energy and potential energy density. The kinetic energy density is simply the kinetic energy per unit length of the transverse motion: a section of length δx has mass $\rho\delta x$ and transverse velocity $\partial y/\partial t$, giving a kinetic energy of

$$\frac{1}{2}\rho\delta x\Big(\frac{\partial y}{\partial t}\Big)^2.$$

The kinetic energy density u_K is thus

$$u_K = \frac{1}{2}\rho\Big(\frac{\partial y}{\partial t}\Big)^2. \tag{3}$$

2 Potential Energy Density.

Simple Approach. Provided we can assume that the transverse motion of the string causes negligible change in the tension then the stretching of the string requires work done against a constant tension. For a section δx the stored energy is just the tension multiplied by the increase of the length of this section. Thus if the stretched length of this section whan the wave passes is δl then the stored energy is given by

$$u_P\,\delta x = T(\delta l - \delta x) \tag{4}$$

Let the two ends of the section be at $(x,0)$ and $(x+\delta x,0)$ in the absence of the wave, but these are displaced to (x,y) and $(x+\delta x, y+\delta y)$ by the wave. The length increases to

$$\delta l = \sqrt{\delta x^2 + \delta y^2} = \delta x\sqrt{1 + \left(\tfrac{\partial y}{\partial x}\right)^2}.$$

The assumption underlying our whole treatment of waves is that $\partial y/\partial x \ll 1$ so we can expand by the binomial theorem, keeping only the first two terms:

$$\delta l = \delta x\Big(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\Big).$$

Substituting into equation (4) we get the potential energy density

$$u_P(x) = \frac{1}{2} T \Big(\frac{\partial y}{\partial x}\Big)^2 \tag{5}$$

Putting these two contributions tgether the energy density is given by:

$$\boxed{u(x) = \frac{1}{2}\Big(\rho\Big(\frac{\partial y}{\partial t}\Big)^2 + T\Big(\frac{\partial y}{\partial x}\Big)^2\Big) \quad \textit{Energy Density} \qquad \text{(A)}}$$

More Detailed Approach. This very simple approach for calculating u_P is clearly rather inadequate because we don't really know under what conditions, if indeed there are any, the assumption of contant T is valid, since *any* change in length leads to *some* change in T. To analyse this in more detail we need to step back a bit, and consider the elastic properties of the stretched string more generally.

Suppose the unstretched length of this section of string is δx_0. When it is stretched to any other length δl, the tension $T(\delta l)$ and stored energy $\delta U_P(\delta l)$ are given by the usual formulae:

$$T(\delta l) = \lambda \frac{\delta l - \delta x_0}{\delta x_0} \quad \text{and} \quad \delta U_P(\delta l) = \frac{\lambda}{2\delta x_0}(\delta l - \delta x_0)^2,$$

where λ is the elastic constant of the string. The change in δU_P between two different stretched states is, using the difference of squares:

$$\delta U_P(\delta l_2) - \delta U_P(\delta l_1) = \frac{\lambda}{2\delta x_0}(\delta l_2 + \delta l_1 - 2\delta x_0)(\delta l_2 - \delta l_1).$$

Substituting from the tension formula, this can be written

$$\delta U_P(\delta l_2) - \delta U_P(\delta l_1) = \frac{T(\delta l_2) + T(\delta l_1)}{2}(\delta l_2 - \delta l_1).$$

The additional elastic energy caused by the wave is specifically the difference caused by the stretching from δx to δl:

$$u_P \delta x = \frac{T(\delta l) + T(\delta x)}{2}(\delta l - \delta x)$$

whereas the simple approach above gave equation (4), which in this notation is

$$u_P\, \delta x = T(\delta x)\,(\delta l - \delta x).$$

Thus we can correct this formula by re-writing our new result as

$$u_P \delta x = T(\delta x)\,(\delta l - \delta x) + \frac{T(\delta l) - T(\delta x)}{2}(\delta l - \delta x) = T(\delta x)\Big((\delta l - \delta x) + \frac{1}{2}\frac{(\delta l - \delta x)^2}{(\delta x - \delta x_0)}\Big) \tag{6}$$

Thus there is indeed a correction to the simple result above, but it is higher order in the additional stretching, and consequently equation (5) is correct to lowest order.

Simplification for one-way waves. In a region where the wave $y(x,t)$ consists simply of a forward-going wave $y = f(x - ct)$ then an important simplification occurs. In this region the one-way wave equation is satisfied:

$$\frac{\partial y}{\partial x} = -\frac{1}{c}\frac{\partial y}{\partial t}.$$

In this case

$$u_P = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}T\left(-\frac{1}{c}\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\frac{T}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 = u_K \tag{7}$$

Thus in a forward-going wave the energy denisty is equally divided between u_P and u_K, and the total is given by

$$u^+ = T\left(\frac{\partial y}{\partial x}\right)^2 = \rho\left(\frac{\partial y}{\partial t}\right)^2.$$

Similarly in a region where $y = f(x + ct)$ the backwards-going one-way wave equation is satisfied:

$$\frac{\partial y}{\partial x} = +\frac{1}{c}\frac{\partial y}{\partial t},$$

and again we have $u_P = u_K$, and the total energy density is given by

$$u^- = T\left(\frac{\partial y}{\partial x}\right)^2 = \rho\left(\frac{\partial y}{\partial t}\right)^2.$$

Note that these two special cases give the same formula for the energy density, $u^+ = u^-$, but this formula is *not* correct in a region where both forward- and backwards-going waves are present. The problem set contains an example of two waves which interfere destructively so that $y = 0$ instantaneously, so that u_P is instantaneously zero, but u is nonetheless non-zero.

3 Energy Flux

The defining characteristic of waves referred to at the outset was that they transport energy. The existence of the energy density does not acheive this: we need to calculate the rate at which energy flows, the flux $\mathcal{F}$.

Suppose waves are travelling along a string; we wish to calculate the rate $\mathcal{F}(x)$ at which the wave transports energy at the point x in the direction of increasing x. That is, the half-string to the negative side is losing energy, and the half-string to the positive side is gaining energy, at a rate $\mathcal{F}$. This is occurring because the negative half-string is doing work at a rate $\mathcal{F}$ on the positive half-string. Suppose the string at this point makes an angle θ to the undisturbed equilibrium direction: the slope of the string is $\tan\theta = \partial y/\partial x$. The force exerted by the negative half-string on the positive half-string is thus

$$\mathbf{F} = \begin{pmatrix} -T\cos\theta \\ -T\sin\theta \end{pmatrix}.$$

The rate of doing work is the scalar product of the force with the velocity of the point at which the force acts

$$\mathbf{v} = \begin{pmatrix} 0 \\ \frac{\partial y}{\partial t} \end{pmatrix}.$$

Thus $\mathcal{F} = \mathbf{F}\cdot\mathbf{v} = -T\sin\theta\,\frac{\partial y}{\partial t}$. We make the usual small-angle approximation $\sin\theta \approx \tan\theta$, $\cos\theta \approx 1$ to arrive at the formula for the energy flux:

$$\boxed{\mathcal{F} = -T\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t}. \quad \textit{Energy Flux Definition} \qquad \text{(B)}}$$

As in the case of the energy density u there is a simplification a region where there is only a positive-going or negative-going wave. If there is only a forward-going wave then the one-way wave equation applies:

$$\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}.$$

Substituting this into $\mathcal{F}$ we find

$$\mathcal{F}^+ = cT\left(\frac{\partial y}{\partial x}\right)^2 = cu^+.$$

Similarly in a region where there is only a backwards-going wave we can substitute

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}.$$

to give

$$\mathcal{F}^- = -cT\left(\frac{\partial y}{\partial x}\right)^2 = -cu^-.$$

Thus, in a region where the wave is travelling in a single direction, the energy flux is the energy density multiplied by the wave speed, $+c$ in the forward-going case and $-c$ in the backward-going case. This gives an indication that we can think of the energy as locked into the waveform with a certain spatially-varying density $u^\pm$ and transported with it at at the wave speed. We find additional support for that concept in the final section.

4 Conservation of Energy

In view of the global conservation of energy, and bearing in mind that we have in our equations no way in which the wave energy can leave the string into its environment, we expect that the total wave energy E is conserved. However we expect more than that. As we have seen E is made up from local contributions u distributed along the string, and we have an expression for the local rate of flow of energy. Thus we expect that energy is *locally conserved.* This means that the energy in any section of the string only changes because it flows into neighbouring sections of the string as a result of a non-zero energy flux at the two ends. This assumption allows us to derive a differential equation connecting u and $\mathcal{F}$.

Consider a section of string extending from $x = a$ to $x = b$. The energy content of this section is

$$E_{a\to b} = \int_a^b u(x,t)\,dx.$$

and the implication of the assumed local conservation of energy is that this only changes because energy flows in at a rate $\mathcal{F}(a)$ at one end and out at a rate $\mathcal{F}(b)$ from the other end:

$$\frac{\mathrm{d}E_{a\to b}}{\mathrm{d}t} = \mathcal{F}(a) - \mathcal{F}(b). \tag{8}$$

The argument applies to any section of string, but we can make up a longer section from two shorter sections. Adding the two equations for the shorter sections we find the rate of change of the sum of the two energies is still given by the flow in and out of the two ends of the combined section, the contributions at the join cancelling. Thus it is sufficient to consider a very short section of length δx centred at x and extending from $x - \delta x/2$ to $x + \delta x/2$. For a sufficiently short section we can neglect the variation in u over the length, and the energy content is just $u\,\delta x$. Applying equation (8) we find

$$\frac{\partial u}{\partial t}\delta x = \mathcal{F}(u - \delta x/2) - \mathcal{F}(x + \delta x/2). \tag{9}$$

Retaining only the first-order term in the Taylor series we substitute

$$\mathcal{F}(x + \delta x/2) = \mathcal{F}(x) + (\delta x/2)\frac{\partial \mathcal{F}}{\partial x}$$

and similarly for $\mathcal{F}(x - \delta x/2)$. Substituting these into equation (9) we find

$$\boxed{\frac{\partial u}{\partial t} = -\frac{\partial \mathcal{F}}{\partial x}. \quad \textit{Equation of Continuity} \qquad \text{(C)}}$$

This is an particular example of what is generally known as the Equation of Continuity, a rather unhelpful name since it doesn't really relate to the content of the equation. The equation expresses local conservation of something, and in this case the something is energy. But that played no rôle in the derivation of it. We had two functions, a local density u of some stuff, and a flux $\mathcal{F}$ of stuff. Assuming the local conservation of stuff then gives the equation. You will meet during the physics course many examples of this equation, in different numbers of dimensions. This is a one-dimensional example; in a three-dimensional example the spatial derivative becomes the divergence $\nabla \cdot \mathcal{F}$. In this form it applies to charge, and also to mass in a fluid. In free space it applies to energy in the elecromagnmetic field, but in a medium there is an additional term representing transfer of energy between the electromagnetic field and the medium, and is known as Poynting's Theorem. In quantum mechanics in the x-representation it applies to the conservation of probability in however many dimensions the wavefunction lives in. Given the importance of conservation laws in physics this is indeed one of the most important equations in mathematical physics.

So the equation of continuity expresses local conservation of energy. We simply have to show that it is satisfied by the specific forms for u and $\mathcal{F}$ given in boxes (A) and (B). The time derivative of u generates two terms:

$$\frac{\partial u}{\partial t} = \rho\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}.$$

The spatial derivative of $\mathcal{F}$ also generates two terms:

$$\frac{\partial \mathcal{F}}{\partial x} = -T\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t} - T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}.$$

If we substitute into the equation of continuity, the mixed second derivatives cancel to leave

$$\frac{\partial u}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = \frac{\partial y}{\partial t}\left[\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 x}{\partial x^2}\right]$$

The two terms in the bracket cancel provided y satisfies the wave equation, so these forms of u and $\mathcal{F}$ are consistent with the equation of continuity specifically for waves on strings.

Conservation of Energy at Boundaries. The energy flux $\mathcal{F}$ provides a very quick answer to a frequently asked question: is energy conserved at a boundary between two strings? We first suppose that two strings with different values of ρ are joined together. The tension T is obviously continuous across the boundary. The boundary conditions that we use for the displacement y are that y is also continuous across the boundary, as is $\frac{\partial y}{\partial x}$. We use subscripts + and − to denote values just on the positive and negative side of the boundary. The boundary conditions are thus

$$T_+ = T_- \quad y_+ = y_- \quad \left(\frac{\partial y}{\partial x}\right)_+ = \left(\frac{\partial y}{\partial x}\right)_- .$$

The continuity of y implies that

$$\left(\frac{\partial y}{\partial t}\right)_+ = \left(\frac{\partial y}{\partial t}\right)_- \quad \text{and so taking all these together} \quad \mathcal{F}_+ = \mathcal{F}_-.$$

Thus the boundary conditions we impose ensure that $\mathcal{F}$ is continuous across the boundary, and hence all the energy that arrives at the boundary on the negative side leaves it on the positive side, at all times.

If there is a non-zero mass M located on the boundary then this changes the boundary conditions. There is now a change of slope at the boundary leading to a net force on the the mass:

$$M\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_+ - \left(\frac{\partial y}{\partial x}\right)_-\right].$$

If we multiply this by the (common) value of $\frac{\partial y}{\partial t}$ we find

$$M\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial t} = \mathcal{F}_- - \mathcal{F}_+.$$

The left side of this is a time derivative:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}M\left(\frac{\partial y}{\partial t}\right)^2\right] = \mathcal{F}_- - \mathcal{F}_+.$$

Again this expresses conservation of energy: the rate of change of the kinetic energy of the mass located on the boundary equals the rate at which the wave delivers energy to the boundary.

5 Decompositions of the Energy

Just as in the case of normal modes, the total energy can be decomposed into independent parts, and in fact in several different ways in different situations.

5.1 Decomposition into forwards and backwards Waves.

If we substitute into the expressions for energy density u and flux $\mathcal{F}$ the general solution of thw wave equation as a superposition of arbitrary forwards and backwards waves, we find an interesting simplification. We substitute

$$y(x,t) = f(x - ct) + g(x + ct)$$

where the forwards-going and backwards-going waves defined by f and g are arbitrary continuous functions.

For the energy density we find

$$u = \frac{\rho}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)^2.$$

We can easily extract from these products the energy densities of the f and g waves:

$$u = \left[\frac{\rho}{2}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial f}{\partial x}\right)^2\right] + \left[\frac{\rho}{2}\left(\frac{\partial g}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial g}{\partial x}\right)^2\right] + \left[\rho\frac{\partial f}{\partial t}\frac{\partial g}{\partial t} + T\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right]$$

As we noted in section 2, for the one-way waves f and g the two terms in the bracket are equal and both add up to what we defined as u^+ or u^-:

$$u = u^+ + u^- + \left[\rho\frac{\partial f}{\partial t}\frac{\partial g}{\partial t} + T\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right]$$

But we can also use the one-way wave equations in the final term:

$$\frac{\partial f}{\partial x} = -\frac{1}{c}\frac{\partial f}{\partial t} \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{1}{c}\frac{\partial g}{\partial t}.$$

These transform the final term into

$$\left[\rho - \frac{T}{c^2}\right]\frac{\partial f}{\partial t}\frac{\partial g}{\partial t}$$

which vanishes since $c^2 = T/\rho$. Hence the internal energy decomposes as

$$\boxed{u = u^+ + u^-. \quad \textit{Forwards/Backwards Decomposition of } u \qquad \text{(D)}}$$

This is an interesting result: we know that we can superpose the displacement of two waves, because of the linearity of the wave equation, so we can write, for example, $u = f + g$ as we did above. However the energy density is*not* expected to superpose in this way because it is quadratic in y, and so there will be cross-terms, as indeed there were when we made the substition. However these cross-terms cancel and the forwards and backwards energy densities do in fact simply add in the same way that the displacements do.

The same cancellation of cross terms occurs when we calculate the energy flux $\mathcal{F}$:

$$\mathcal{F} = -T\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}\right)$$

Again we can extract the terms involving only f or only g:

$$\mathcal{F} = \left[-T\frac{\partial f}{\partial x}\frac{\partial f}{\partial t}\right] + \left[-T\frac{\partial g}{\partial x}\frac{\partial g}{\partial t}\right] + \left[-T\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial t}\right)\right]$$

Here the first term is the energy flux of the forwards-going wave $\mathcal{F}^+$, and the second term is $\mathcal{F}^-$. The third term cancels using the one-way wave equations. Thus the energy flux also decomposes as

$$\boxed{\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-. \quad \textit{Forwards/Backwards Decomposition of } \mathcal{F} \qquad \text{(E)}}$$

Taken together these two decompositions in boxes (D) and (E) give a very striking picture of the energy locked into the forwards- and backwards-going waves, travelling with it at speed c or $-c$, and passing through each other without disturbance. The displacements of the two waves superpose, which distorts the waveforms of each wave as they pass through each other, but the energy is *not* re-distributed by this effect. We can usefully compare this with a superposition of waves on a string travelling in the *same* direction: the resulting energy distribution is not just the sum of the energy distributions of the two component waves. However in a forwards/backwards superposition the displacement y, the energy density u and the energy flux $\mathcal{F}$ are *all* simply the sum of the contributions from each wave.

5.2 Modal Energy Decomposition.

Our second example of a time-independent decomposition is of the energy in the modes of a fixed string of length L. We first review the derivation of the relevant solution.

Solution in terms of modes. We look for a solution of the wave equation for a string with tension T and mass density ρ which is fixed at $x = 0$ and $x = L$, and hence $y(0,t) = y(L,t) = 0$. The existence of these fixed nodes encourages us to look for a solution in the form of standing waves:

$$y(x,t) = R_n(x)Q_n(t).$$

Substitution into the wave equation yields the separated equations

$$\frac{\mathrm{d}^2 R_n}{\mathrm{d}x^2} = k_n^2 R_n \qquad \frac{\mathrm{d}^2 Q_n}{\mathrm{d}t^2} = \omega_n^2 Q_n \tag{10}$$

where k_n^2 is the separation constant and $\omega_n^2 = c^2 k_n^2$. Applying the boundary conditions at $x = 0$ and L gives

$$R_n(x) = \sin(k_n x) \quad \text{and} \quad k_n = \frac{n\pi}{L}.$$

The solution for Q_n is then

$$Q_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t).$$

Superposing these solutions gives the general solution

$$y(x,t) = \sum_{n=1}^{\infty} R_n(x)Q_n(t). \tag{11}$$

The constants A_n and B_n are determined by the initial conditions. The A_n are determined by the initial displacement:

$$y(x,0) = \sum_n A_n \sin(k_n x)$$

and the B_n by the initial velocity $V(x,0)$ where $V = \partial y/\partial t$:

$$V(x,0) = \sum_n B_n \omega_n \sin(k_n x).$$

The Conserved Energy. The decomposition we shall exhibit is for the total energy of the string, E:

$$E = \int_0^L u(x,t)\,\mathrm{d}x. \tag{12}$$

The calculation of u involves two differentials of y from equation (11). The kinetic energy density requires

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} R_n(x) \frac{\mathrm{d}Q_n}{\mathrm{d}t}.$$

Substituting this into u we find the kinetic energy contribution to E to be

$$E_K = \frac{\rho}{2} \sum_{m,n} \frac{\mathrm{d}Q_m}{\mathrm{d}t} \frac{\mathrm{d}Q_n}{\mathrm{d}t} \int_0^L R_m(x) R_n(x)\,dx. \tag{13}$$

Similarly the potential energy constribution requires

$$\frac{\partial y}{\partial x} = \sum_{n=1}^{\infty} \frac{\mathrm{d}R_n}{\mathrm{d}x} Q_n(t)$$

Substituting this into u we find the potential energy contribution

$$E_P = \frac{T}{2} \sum_{m,n} Q_m(t) Q_n(t) \int_0^L \frac{\mathrm{d}R_m}{\mathrm{d}x} \frac{\mathrm{d}R_n}{\mathrm{d}x}\,dx.$$

We see that the whole expression for $E = E_K + E_P$ involves two different integrals over the spatial functions R:

$$\mathcal{I}_{mn} = \int_0^L R_m(x) R_n(x)\,dx \quad \text{and} \quad \mathcal{J}_{mn} = \int_0^L \frac{\mathrm{d}R_m}{\mathrm{d}x} \frac{\mathrm{d}R_n}{\mathrm{d}x}\,dx. \tag{14}$$

In terms of these E is given by

$$E = \frac{1}{2} \sum_{n=1}^{\infty} \Big[\rho \mathcal{I}_{mn} \frac{\mathrm{d}Q_m}{\mathrm{d}t} \frac{\mathrm{d}Q_n}{\mathrm{d}t} + T \mathcal{J}_{mn} Q_m Q_n \Big]. \tag{15}$$

E is both constant in time and can be written as a sum of time-independent modal energies E_n. These claims are by no means obvious from equation (15), which is composed of explicitly time-dependent terms, most of which depend on two different modes.

The time-independent modal energy. If we substitute the explicit expression for R_n into equation (14) we find

$$\mathcal{I}_{mn} = \int_0^L \sin(k_m x) \sin(k_n x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{J}_{mn} = k_m k_n \int_0^L \cos(k_m x) \cos(k_n x)\,\mathrm{d}x.$$

These are both obviously symmetric in m and n: $\mathcal{I}_{mn} = \mathcal{I}_{nm}$ and similarly for $\mathcal{J}_{mn}$. They can both be calculated by substituting a trignometric addition formula which re-expresses the integrand as a sum or difference of cosines. The result is then

$$\mathcal{I}_{mn} = \begin{cases} \frac{L}{2} & m = n \\ 0, & m \neq n; \end{cases} \quad \text{and} \quad \mathcal{J}_{mn} = \begin{cases} \frac{k_n^2 L}{2} & m = n \\ 0 & m \neq n. \end{cases}$$

If we substitute these into equation (15), all the terms with $m \neq n$ vanish and we are left with a sum over modes n:

$$E = \sum_{n=1}^{\infty} E_m \quad \text{where} \quad E_m = \frac{\rho L}{4}\Big[\Big(\frac{\mathrm{d}Q_n}{\mathrm{d}t}\Big)^2 + \omega_n^2 Q_n^2\Big], \tag{16}$$

where we have used $T = \rho c^2$ to combine the two terms, and $\omega_n^2 = k_n^2 c^2$ to simplify the second one. This establishes the claim that E is a sum of modal energies; to show that it is constant we simply substitute the explicit expression for Q_n. The two terms in the bracket become

$$\Big[\Big(\frac{\mathrm{d}Q_n}{\mathrm{d}t}\Big)^2 + \omega_n^2 Q_n^2\Big] = \omega_n^2\big[(-A_n \sin(\omega t) + B_n \cos(\omega t))^2 + (A_n \cos(\omega t) + B_n \sin(\omega t))^2\big] = \omega_n^2(A_n^2 + B_n^2).$$

Hence the energy in the n'th mode is

$$E_n = \frac{\rho L}{4}\omega_n^2(A_n^2 + B_n^2). \tag{17}$$

Discussion. Two very remarkable things happened in the previous paragraph on substituting explicit expressions for R_n and Q_n: terms in E with $m \neq n$ vanished, and the time-dependence of E_n in equation (16) cancelled to give equation (17). The reasons for both of these can be traced to the equations satisfied by R_n and Q_n, (10) above. If we integrate $\mathcal{J}_{mn}$ by parts, integrating the first factor and differentiating the second, we find

$$\mathcal{J}_{mn} = \Big[R_m \frac{\mathrm{d}R_n}{\mathrm{d}x}\Big]_0^L - \int_0^L R_m \frac{\mathrm{d}^2 R_n}{\mathrm{d}x^2}\, dx.$$

The integrated part vanishes because R_m is zero at both limits and we can substitute for the second derivative from equation (10):

$$\mathcal{J}_{mn} = k_n^2 \mathcal{I}_{mn}. \tag{18}$$

However if instead we differentiate the first factor and integrate the second we obtain

$$\mathcal{J}_{mn} = k_m^2 \mathcal{I}_{mn}. \tag{19}$$

Subtracting equation (19) from (18) we find

$$(k_n^2 - k_m^2)\mathcal{I}_{mn} = 0$$

which shows that the $\mathcal{I}_{mn}$ is zero unless $m = n$. Substituting this result into equation (18) then shows that the same is true of $\mathcal{J}_{mn}$. Thus this simple calculation has shown that all the integrals are zero except $\mathcal{I}_{nn}$ and $\mathcal{J}_{nn}$, which are related by $\mathcal{J}_{nn} = k_n^2 \mathcal{I}_{nn}$. The actual values of these integrals cannot be found from this general approach, since the solution for R_n includes an arbitrary constant.

The second remarkable fact is that the two time-dependent terms in E_n add to give a constant. The reason for this becomes clear if we differentiate the bracket in equation (16):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\Big(\frac{\mathrm{d}Q_n}{\mathrm{d}t}\Big)^2 + \omega_n^2 Q_n^2\Big] = 2\frac{\mathrm{d}Q_n}{\mathrm{d}t}\Big[\frac{\mathrm{d}^2 Q_n}{\mathrm{d}t^2} + \omega_n^2 Q_n\Big] = 0$$

where the final zero comes from the equation satisfied by Q_n, (10).

The alert reader may also have noticed a strong similarity between the above and the way that the energy of a mass-and-spring system can be expressed in terms of normal mode energies. The mathematics for both can be viewed from the perspective of linear algebra: we can consider the functions R_n to be vectors in a linear vector space, and the operator $\mathrm{d}^2/\mathrm{d}x^2$ to be a linear operator. Integrals over the interval $[0, L]$ are the inner product in this space. The equality of (18) and (19) then establishes that the differential operator is self-adjoint. The differential equation (10) is then an eigenvalue equation, and the orthogonality of eigenvectors for solutions of different eigenvalue then follows. All of this takes us well beyond the first year syllabus, but this approach will be central when studying quantum mechanics in the second year.