Energy in Normal Modes and Waves on Strings

1 Introduction

This document compares the theory for energy in Normal Modes and in Waves on Strings by developing it in parallel in the two cases. The Normal Mode case is printed in black and the Waves case in blue. In each case the equivalent boxed equations have the same numbering.

2 Equation of Motion

The normal modes equation of motion we write as

$$\boxed{\mathbf{M}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x} \qquad (1)}$$

where $\mathbf{x}$ is the vector of particle positions, $\mathbf{M}$ is the diagonal mass matrix and $\mathbf{K}$ is the symmetric stiffness matrix. This solved by a product solution (function of time multiplied by a "function" of the particle index i, in other words a vector).

$$\boxed{\mathbf{x}(t) = f(t)\mathbf{c}. \qquad (2)}$$

Substitution of this shows that it is a possible solution provided

$$\boxed{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -\omega^2 f \quad \text{and} \quad \mathbf{K}\mathbf{c} = \omega^2\mathbf{M}\mathbf{c}. \qquad (3)}$$

The wave equation is

$$\boxed{\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}. \qquad (1)}$$

where $y(x,t)$ is the string displacement, ρ is the mass per unit length and T is the tension. This is solved by a product solution (function of time multiplied by a function of position):

$$\boxed{y(x,t) = f(t)u(x). \qquad (2)}$$

Substitution of this shows that it is a possible solution provided

$$\boxed{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -\omega^2 f \quad \text{and} \quad -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = k^2 u \quad \text{where} \quad \omega^2 = k^2c^2, \quad c^2 = T/\rho. \qquad (3)}$$

3 The Energy Equation

We have written the equation of motion in box (1) in the form of Newton's Second Law (rate of change of momentum equals force), and if we multiply this by velocity we always get an equation relating rate of change of energy and work done. We define

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

and then we have

$$\boxed{\mathbf{v}^T\mathbf{M}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{v}^T\mathbf{K}\mathbf{x}. \qquad (4)}$$

The left-hand side is the rate of change of kinetic energy:

$$\mathbf{v}^T\mathbf{M}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v}\right)$$

while the right-hand side is the work done by the force, which in this case can be written as minus the rate of change of potential energy:

$$-\mathbf{v}^T\mathbf{K}\mathbf{x} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}\right)$$

so we have

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \text{where} \quad E = \frac{1}{2}\left(\mathbf{v}^T\mathbf{M}\mathbf{v} + \mathbf{x}^T\mathbf{K}\mathbf{x}\right) \qquad (5)}$$

Multiplying the equation of motion by the velocity

$$v(x,t) = \frac{\partial y}{\partial t}$$

we get again an energy equation

$$\boxed{v\rho\frac{\partial^2 y}{\partial t^2} = vT\frac{\partial^2 y}{\partial x^2}. \qquad (4)}$$

However, at this point the parallelism between the two theories breaks down: the left-hand side is, as with the normal modes case, the rate of change of the kinetic energy:

$$\rho v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right),$$

but the right-hand side is *not*, as it was previously, the rate of change of the potential energy. This is because these energy *densities* are a measure of energy around a particular point, and this is *not* a closed system in the way that the normal modes system is. Thus the rate of change of local kinetic energy can come either from the local potential energy density, or by flowing along the string from somewhere else. We therefore have an extra term in the energy conservation equation for the rate at which energy flow delivers energy locally. We still need a potential energy term on the left to complete the time derivative of the total energy density:

$$\frac{\partial}{\partial t}\left(\frac{T}{2}\frac{\partial y}{\partial x}\right)^2 = T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}.$$

. Adding this to the equation in box (4) gives

$$\frac{\partial u}{\partial t} = T\left(\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t} + \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right).$$

The energy flux is

$$\mathcal{F} = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

so the energy conservation equation is now

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0 \qquad (5)}$$

and this equation is *not* structurally equivalent to the previous box (5). The direct equivalent of the previous equation (5) is given in section 5.

4 Eigenvectors

We can develop the parallel theories further by considering the modes of the two systems. Comparing the black and blue versions of boxes (1) and (3) suggests that apart from obvious replacements — displacement $y(x,t)$ for position $\mathbf{x}(t)$ and density ρ for mass matrix $\mathbf{M}$ — the significant change is $-T\,\mathrm{d}^2/\mathrm{d}t^2$ for stiffness matrix $\mathbf{K}$. How can these play equivalent roles in the two theories? The answer is as linear operators on a vector space.

In the normal modes case this is very obvious. The position vector $\mathbf{x}(t)$ is obviously an element of a vector space $\mathcal{V}$ over the reals $\mathcal{R}$, and so is the vector $\mathbf{c}$ introduced in box (2) — we can add two such vectors, or multiply them by real numbers and the result is still a possible element of $\mathcal{V}$. The real, symmetric matrix $\mathbf{K}$ is then a linear operator on $\mathcal{V}$, taking an element $\mathbf{x}$ to another element of $\mathcal{V}$ in a linear way.

The equation in box (3), $\mathbf{Kc} = \omega^2\mathbf{Mc}$, then defines a generalised eigenvalue problem with $\mathbf{c}$ as the generalized eigenvector of $\mathbf{K}$ and $\mathbf{M}$. The equation in box (2) tells us that for each eigenvector $\mathbf{c}_i$, with eigenvector ω_i^2, there is a possible state of motion of the system $\mathbf{x}(t) = f_i(t)\mathbf{c}_i$ where $f_i(t)$ satisfies the SHM equation with angular frequency ω_i. Thus the normal modes are defined as states where the position vector is always one of the eigenvectors, with time-varying amplitude.

We next give the vector space an inner product. If $|a\rangle \equiv \mathbf{a}$ and $|b\rangle \equiv \mathbf{b}$ are elements of $\mathcal{V}$, we define the inner product as

$$\boxed{\langle b|a\rangle \equiv \mathbf{b}^T\mathbf{a} \quad \text{and hence} \quad \langle b|a\rangle = \langle a|b\rangle. \qquad (6)}$$

This has the required property that the inner product is real and $\langle a|a\rangle$ is positive definite. If we denote the vector $\mathbf{Ka}$ by $|Ka\rangle$, and consider the inner product $\langle b|Ka\rangle$, then we have

$$\langle b|Ka\rangle \equiv \mathbf{b}^T\mathbf{Ka} = \mathbf{a}^T\mathbf{K}^T\mathbf{b} \equiv \langle a|K^Tb\rangle.$$

But since $\mathbf{K}$ is a symmetric matrix, $\mathbf{K}^T = \mathbf{K}$, and so we have proved the *self-adjoint* property of $\mathbf{K}$:

$$\boxed{\langle b|Ka\rangle = \langle a|Kb\rangle. \qquad (7)}$$

(Exactly the same argument applies equally to $\mathbf{M}$.)

If we now take this result and apply it to two different eigenvectors $\mathbf{c}_i$ and $\mathbf{c}_j$, we find

$$\omega_j^2\langle c_i|Mc_j\rangle = \omega_i^2\langle c_j|Mc_i\rangle = \omega_i^2\langle c_i|Mc_j\rangle.$$

Hence we find that eigenvectors belonging to different eigenvalues are orthogonal:

$$\boxed{(\omega_j^2 - \omega_i^2)\langle c_j|Mc_i\rangle = 0. \qquad (8)}$$

In normal modes problems the eigenvalues are usually all distinct, in which case this is sufficient to prove that the eigenvectors form an orthonormal basis in $\mathcal{V}$. This completeness of the eigenvectors ensures that *any* solution of the equation of motion $\mathbf{x}(t)$ can be written as a linear superposition of these modal solutions:

$$\boxed{\mathbf{x}(t) = \sum_i f_i(t)\mathbf{c}_i. \qquad (9)}$$

The parallel development of modes in the waves case requires boundary conditions to give discrete modes — so that we are discussing not an unbounded string but a string of fixed length L: $y(0,t) = y(L,t) = 0$. Displacement functions $y(x,t)$ satifying these requirements form a vector space. (There are technical details regarding the other conditions on displacement functions in order that they form a function space that I am completely ignoring here!) Imposing these boundary conditions on the purely spatial function $u(x)$ introduced in box (2) ensures that $u(x)$ is also a element of this space. Hence, in the equation in box (3) $D = -\mathrm{d}^2/\mathrm{d}x^2$ is is acting as a linear operator on the vector space, and the equation defines an eigenvalue problem (though not in this case a generalised eigenvalue problem). We get a discrete set of eigenfunctions

$$u_j(x) = \sin k_jx \quad \text{with eigenvalues} \quad k_j^2 = j^2\pi^2/L \quad \text{and} \quad \omega_j^2 = k_j^2c^2.$$

Again we define an inner product: for vectors $|u\rangle \equiv u(x)$ and $|v\rangle \equiv v(x)$ we have

$$\boxed{\langle u|v\rangle = \int_0^L u(x)v(x)\,\mathrm{d}x. \qquad (6)}$$

This is real, symmetric and has a positive definite norm as required.

Finally we enquire whether the linear operator D is self-adjoint. Consider the following inner product;

$$\langle v|Du\rangle \equiv -\int_0^L v(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\,\mathrm{d}x.$$

We can integrate by parts:

$$-\int_0^L v(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\,\mathrm{d}x = -\left[v(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L + \int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

and then by parts again:

$$-\int_0^L v(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x = -\left[v(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L + \left[\frac{\mathrm{d}v}{\mathrm{d}x}u(x)\right]_0^L - \int_0^L \frac{\mathrm{d}^2v}{\mathrm{d}x^2}u\,\mathrm{d}x.$$

The boundary conditions on u and v imply that both integrated parts vanish and so

$$-\int_0^L v(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x = \int_0^L \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = -\int_0^L \frac{\mathrm{d}^2v}{\mathrm{d}x^2}u\,\mathrm{d}x.$$

The first and last of these terms suffice to show that D is self adjoint:

$$\boxed{\langle v|Du\rangle = \langle u|Dv\rangle. \qquad (7)}$$

At this point the extra effort required to set this discussion in a linear algebra context finally pays a dividend: the formal argument that links the definition of the inner product (box (6)) and the self-adjoint property of the operator (box (7)) to the orthogonality of the eigenfunctions is identical, apart from the fact that there is no equivalent of M in this version:

$$\boxed{(k_j^2 - k_i^2)\langle u_j|u_i\rangle = 0. \qquad (8)}$$

The completeness of the eigenfuctions is a much more delicate subject since the vector space has infinite dimension, but the following superposition of modes is indeed a general solution

$$\boxed{y(x,t) = \sum_i f_i(t)u_i(x). \qquad (9)}$$

5 Modal Energy

We can now put together the definition of energy (box (5)) and the modal decomposition of the displacement (box(9)) to identify the energy in each mode. Substituting the expansion in modes for both of the displacement vectors in the energy we find

$$E = \sum_i \sum_j \frac{1}{2} \left[\mathbf{c}_i \mathbf{M} \mathbf{c}_j \frac{\mathrm{d}f_i}{\mathrm{d}t} \frac{\mathrm{d}f_j}{\mathrm{d}t} + \mathbf{c}_i^T \mathbf{K} \mathbf{c}_j f_i(t) f_j(t) \right].$$

Using the eigenvector property in the second term we can factor out $\mathbf{c}_i \mathbf{M} \mathbf{c}_j$:

$$E = \sum_i \sum_j \frac{1}{2} \left[\frac{\mathrm{d}f_i}{\mathrm{d}t} \frac{\mathrm{d}f_j}{\mathrm{d}t} + \omega_j^2 f_i(t) f_j(t) \right] \mathbf{c}_i \mathbf{M} \mathbf{c}_j.$$

The orthogonality of eigenvectors results in all terms with $i \neq j$ being zero:

$$\mathbf{c}_i \mathbf{M} \mathbf{c}_j = \mathbf{c}_i^T \mathbf{M} \mathbf{c}_i \delta_{ij}.$$

Thus *all* the terms that couple different modes are zero, and the double sum reduces to a single summation over modal energies:

$$E = \sum_i \frac{\mathbf{c}_i^T \mathbf{M} \mathbf{c}_i}{2} \left[\left(\frac{\mathrm{d}f_i}{\mathrm{d}t} \right)^2 + \omega_i^2 f_i(t)^2 \right].$$

In order to show that these modal energies are themselves independent of time we need to look at the general solution of the SHM equation satisfied by f. We normally write this solution as a sum of sine and cosine terms, but in this instance it is simpler to write it in the amplitude and phase version:

$$f_i = a_i \sin(\omega_i t + \phi_i).$$

The apparently time-dependent bracket in the modal energy reduces to

$$\left[\left(\frac{\mathrm{d}f_i}{\mathrm{d}t} \right)^2 + \omega_i^2 f_i(t)^2 \right] = a_i^2 \omega_i^2 \left[\cos^2(\omega_i t) + \sin^2(\omega_i t) \right] = a_i^2 \omega_i^2.$$

Thus the modal energy is independent of time:

$$\boxed{E = \sum_i E_i \quad \text{where} \quad E_i = \frac{1}{2} \mathbf{c}_i^T \mathbf{M} \mathbf{c}_i \omega_i^2 a_i^2.} \qquad (10)$$

We will follow an almost identical path, but first we must take account of the fixed length of the string so that we can discuss the total energy E:

$$E(t) = \int_0^L u(x,t)\,\mathrm{d}x = \frac{1}{2} \int_0^L \left[\rho \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right] \mathrm{d}x.$$

The time derivative of this is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L \frac{\partial u}{\partial t}\,\mathrm{d}x = -\int_0^L \frac{\partial \mathcal{F}}{\partial x}\,\mathrm{d}x = \mathcal{F}(0) - \mathcal{F}(L),$$

where we have used the conservation of energy equation from blue box (5). So the total energy changes only if energy flows in or out of the ends of the string. But the boundary conditions $y(0,t) = y(L,t) = 0$ imply that $\partial y/\partial t = 0$ at both ends, and hence $\mathcal{F} = 0$ at both ends also. We therefore arrive at the real equivalent of black box (5):

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \text{where} \quad E = \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x. \qquad (5)}$$

We now substitute in E the modal representation of the displacement y from box (9):

$$E = \sum_i \sum_j \frac{1}{2}\left[\rho \frac{\mathrm{d}f_i}{\mathrm{d}t}\frac{\mathrm{d}f_j}{\mathrm{d}t}\int_0^L u_i u_j\,\mathrm{d}x + T f_i f_j \int_0^L \frac{\mathrm{d}u_i}{\mathrm{d}x}\frac{\mathrm{d}u_j}{\mathrm{d}x}\,\mathrm{d}x\right].$$

Referring to the equation before box (7) we see that the integral over the spatial derivatives of u is linked, by an integration by parts, to the expressions from which we derived the orthogonality of the u eigenfunctions:

$$\int_0^L \frac{\mathrm{d}u_i}{\mathrm{d}x}\frac{\mathrm{d}u_j}{\mathrm{d}x}\,\mathrm{d}x = -\int_0^L \frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} u_j\,\mathrm{d}x = k_i^2 \int_0^L u_i u_j\,\mathrm{d}x.$$

Thus we can factor out the integrals from the square bracket, and all of them with $i \neq j$ are zero by orthogonality. The terms with $i = j$ are given by

$$\int_0^Y \sin^2(k_i x)\,\mathrm{d}x = \frac{L}{2}.$$

Thus the expression for E reduces to

$$E = \frac{L}{4}\sum_i \left[\rho\left(\frac{\mathrm{d}f_i}{\mathrm{d}t}\right)^2 + T k_i^2 f_i^2\right].$$

Finally we note that $T = c^2\rho$ to obtain

$$E = \sum_i \frac{\rho L}{4}\left[\left(\frac{\mathrm{d}f_i}{\mathrm{d}t}\right)^2 + \omega_i^2 f_i^2\right].$$

The terms in the square bracket combine exactly as before to give E as a sum of constant modal energies E_i:

$$\boxed{E = \sum_i E_i \quad \text{where} \quad E_i = \frac{\rho L}{4}\omega_i^2 a_i^2. \qquad (10)}$$