2008 S8 Covariant Electromagnetism: Problems

Questions marked with an asterisk are more difficult.

1. Eliminate **B** instead of **H** from the standard Maxwell equations. Show that the effective source terms are now a magnetic charge density $-\nabla\cdot\mathbf{M}$ and a magnetic current density $\frac{\partial\mathbf{M}}{\partial t}$ in complete analogy with the electric polarization **P**. This illustrates that the choice of **B** instead of **H** for the fundamental magnetic field is governed by the assumption that magnetic effects in atoms are really the result of electric currents. What would be the effect of eliminating **E** instead of **D**?

2. A crude model of a polarized atom is an electron cloud of charge $-Q$ and radius a, with a spherical nucleus of charge $+Q$ and radius b, the centre of which is displaced by **d** from the centre of the cloud. Show that the dipole moment of such an atom is $\mathbf{p} = Q\mathbf{d}$. (Hint: the answer is independent of origin; take the origin at the centre of the cloud.)

If there are N such atoms per unit volume then $\mathbf{P} = N\mathbf{p}$. Show that the spatially averaged current density is $\mathbf{J}_b = \partial\mathbf{P}/\partial t$. For this purpose we can take **P** spatially uniform, and just average over a single unit cell. Note that the cell volume is $1/N$. You should find that the answer is independent of whether you assume the electron cloud moves relative to a fixed nucleus, or *vice versa*, or anything in between.

***3.** Show that the spatially averaged charge density in the atomic model of problem 2 is $\rho_b = -\nabla\cdot\mathbf{P}$. This is more difficult, as it involves spatial variation, but only first order. Take the spatial average by integrating over a smooth averaging function such as a Gaussian $S(\mathbf{R}) = e^{-r^2/R^2}/(\sqrt{\pi}R)^3$ with $R \gg a$:

$$\rho_b(\mathbf{r}) = \int S(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d^3\mathbf{r}'.$$

In the integration over each charged sphere replace the averaging function $S(\mathbf{r}-\mathbf{r}')$ by the value at the centre of the sphere $S(\mathbf{r}-\mathbf{r}'_0)$, which is correct to second-order in a/R. Note that this entails that the positive and negatively charged spheres have different values of S, evaluated at points separated by **d**. To the same degree of accuracy the integration over each unit cell is then $Q\mathbf{d}\cdot\nabla S$ which we can write as a volume integral over a local density $\mathbf{P}\cdot\nabla S$. The result follows on integration by parts.

4. Poynting's Theorem. A long straight wire radius a lies along the z-axis. The wire is subject to a uniform field $\mathbf{E} = E\mathbf{k}$ which induces a current density $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the material. Show that the **B** field in the exterior region is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 a^2\,\mathbf{J}\wedge\mathbf{r}}{2\rho^2}$$

where $\rho^2 = x^2 + y^2$. Find Poynting's vector **S**, and show that it represents an inward energy flow. Evaluate the surface integral of **S** over a cylinder of radius b, $(b > a)$ and length l, and interpret the result in terms of energy dissipation in the wire.

5. Use the equation on page 9 to find expressions for Poynting's vector and the energy density in generalized units. (Note that because of the four equations linking the six constants the answer is not unique. The answer can be expressed in terms of two, for example α and γ.) Hence show, using the table on page 13, that in Gaussian units Poynting's vector is given by $(c/4\pi)\mathbf{E}\wedge\mathbf{B}$, while the energy density is $(E^2+B^2)/8\pi$.

6. Using the B field of a long straight wire (see question 4) find an expression for the force per unit length between two parallel wires carrying the same current. The S.I. definition of the Ampère is 'that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.' Hence show that the numerical value of μ_0 is $4\pi\times10^{-7}$, and assign dimensions to it. In the corresponding definition of the cgs emu of current the wires are 1 cm apart and the force is two cgs force units (dyne) per cm. Hence show that the emu of current is 10 A. (You will need to show that 1 dyne is 10^{-5} N.)

7. Show that in suffix notation Maxwell's equations become

$$\frac{\partial E_i}{\partial x_i}=\rho/\epsilon_0 \qquad \epsilon_{ijk}\frac{\partial E_k}{\partial x_j}+\frac{\partial B_i}{\partial t}=0$$
$$\frac{\partial B_i}{\partial x_i}=0 \qquad \epsilon_{ijk}\frac{\partial B_k}{\partial x_j}-\frac{1}{c^2}\frac{\partial E_i}{\partial t}=\mu_0 J_i.$$

Hence show that if we define $\mathcal{B}_{ij}=\epsilon_{ijk}B_k$, and correspondingly $B_i=\frac{1}{2}\epsilon_{ijk}\mathcal{B}_{jk}$, then the three equations containing $\mathbf{B}$ can be written

$$\epsilon_{ijk}\frac{\partial \mathcal{B}_{jk}}{\partial x_i}=0 \qquad \begin{aligned}\epsilon_{ijk}\left(\frac{\partial E_k}{\partial x_j}+\frac{1}{2}\frac{\partial \mathcal{B}_{jk}}{\partial t}\right)&=0\\ \frac{\partial \mathcal{B}_{ij}}{\partial x_j}-\frac{1}{c^2}\frac{\partial E_i}{\partial t}&=\mu_0 J_i.\end{aligned}$$

Note that another contraction of the curl $\mathbf{E}$ equation with ϵ_{ilm} gives

$$\frac{\partial E_m}{\partial x_l}-\frac{\partial E_l}{\partial x_m}+\frac{\partial \mathcal{B}_{lm}}{\partial t}=0.$$

***8.** Use Maxwell's equations to derive

$$\begin{aligned}\epsilon_0\Big(\mathbf{E}\,\nabla\cdot\mathbf{E}-\mathbf{E}\wedge(\nabla\wedge\mathbf{E})\Big)+\frac{1}{\mu_0}\Big(\mathbf{B}\,\nabla\cdot\mathbf{B}+(\nabla\wedge\mathbf{B})\wedge\mathbf{B}\Big)&\\ -\frac{1}{\mu_0c^2}\left(\frac{\partial\mathbf{E}}{\partial t}\wedge\mathbf{B}+\mathbf{E}\wedge\frac{\partial\mathbf{B}}{\partial t}\right)&=\rho\mathbf{E}+\mathbf{J}\wedge\mathbf{B}.\end{aligned}$$

(This involves all four equations, each combined with either $\mathbf{E}$ or $\mathbf{B}$). Combine terms to derive

$$\frac{\partial T_{ij}}{\partial x_j}+\frac{\partial g_i}{\partial t}+f_i=0$$

where $T_{ij}=\epsilon_0\big(\frac{1}{2}E^2\delta_{ij}-E_iE_j\big)+(1/\mu_0)\big(\frac{1}{2}B^2\delta_{ij}-B_iB_j\big)$, $\mathbf{g}=\mathbf{E}\wedge\mathbf{B}/\mu_0c^2$ and $\mathbf{f}$ is the Lorentz force density. Interpret this result in terms of conservation of momentum.

9. Plane Waves. The equations satisfied by the potentials in the absence of sources are

$$-\nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla^2 \Phi = 0 \qquad \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial \nabla \Phi}{\partial t} = 0$$

Show that in the Lorentz gauge these uncouple to give homogeneous wave equations. Consider plane wave solutions in the Lorentz gauge

$$\Phi = \varphi \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \qquad \mathbf{A} = \mathcal{A} \, \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

with constant amplitudes φ (scalar) and $\mathcal{A}$ (vector). Show that these are solutions of the wave equation provided $\omega^2/c^2 = k^2$, and that they satisfy the Lorentz gauge condition provided $\mathbf{k} \cdot \mathcal{A} = \omega\varphi/c^2$. This defines φ in terms of $\mathcal{A}$ but leaves $\mathcal{A}$ arbitrary. There are thus *three* independent solutions: for example,

$$\mathcal{A}_0 = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \qquad \mathcal{A}_1 = \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} \qquad \mathcal{A}_2 = \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix}$$

Show that when $\mathbf{k} = \hat{\mathbf{z}}\omega/c$, $\varphi_1$and φ_2 are zero but φ_0 is not. Show that the fields derived from the solution with $\mathcal{A}_0$ and φ_0 vanish, and find the fields derived from the potentials with $\mathcal{A}_1$ and $\mathcal{A}_2$, representing the *two* transverse polarization states. Find a gauge transformation function χ which eliminates the solution involving $\mathcal{A}_0$ and φ_0.

10. Show that the metric tensor for the two-dimensional oblique co-ordinates on page 42 is

$$(\mathbf{L}^{-1})^T \mathbf{L}^{-1} = \begin{pmatrix} \alpha^{-2} & -\gamma/\alpha^2\beta \\ -\gamma/\alpha^2\beta & (\alpha^2+\gamma^2)/\alpha^2\beta^2 \end{pmatrix}.$$

Confirm that the squared distance from the origin to the point $(x_1', x_2') = (13, 9)$ is 200.

11. Unifom Field A uniform (constant both spatially and in time) electromagnetic field F is represented by a four-potential A. Since F is derived from A by a single differentiation, A must depend linearly on the co-ordinates: $A^\mu = K^{\mu\nu} x_\nu$. Find the relationship between K and F. What constraint on K does the Lorentz gauge condition represent?

12. Show that the $\mathbf{E}$ and $\mathbf{B}$ fields transform under a Lorentz transformation as

$$\begin{aligned} \mathbf{E}'_\parallel &= \mathbf{E}_\parallel & \mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{v} \wedge \mathbf{B}) \\ \mathbf{B}'_\parallel &= \mathbf{B}_\parallel & \mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \mathbf{v} \wedge \mathbf{E}/c^2) \end{aligned}$$

where $\mathbf{E}_\parallel$ and $\mathbf{E}_\perp$ denote the vector components parallel and perpendicular to $\mathbf{v}$.

13. $D = E^2/c^2 - B^2$ and $G = \mathbf{E}\cdot\mathbf{B}$ are both Lorentz invariants. Show this

a) by writing them in terms of manifestly invariant constructions;

b) by explicitly evaluating them in two reference frames using the results of problem 12. (Remember $\mathbf{E}_{||} = (\mathbf{E}\cdot\mathbf{v}/v^2)\mathbf{v}$.)

Discuss the possibility of finding reference frames in which either $\mathbf{E}$ or $\mathbf{B}$ are zero.

14. Non-relativistic motion in uniform field. The non-relativistic equation of motion of a charged particle in a uniform fields $\mathbf{E}$ and $\mathbf{B}$ is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}).$$

The solution divides into four classes:

1) $\mathbf{B} = 0$;
2) $\mathbf{E} = 0$;
3) $\mathbf{E}\cdot\mathbf{B} = 0$;
4) $\mathbf{E}\cdot\mathbf{B} \neq 0$.

Case 1) is simply motion under a constant force, and the solution is immediate: $\mathbf{v} = \mathbf{v}_0 + (q/m)\mathbf{E}t$. Solve for $\mathbf{v}$ in cases 2) and 3), and show that the solution in case 4) is essentially the sum of solutions in cases 1) and 3). You may find it easier to go into components, and take the z-axis along $\mathbf{B}$, and $\mathbf{E}$ in the x-direction in case 3).

***15.** Write out the spatial components of the equation of motion of a charged particle:

$$\frac{dp^\mu}{d\tau} = eF^\mu{}_\nu u^\nu \qquad \text{or} \qquad \frac{du^\mu}{d\tau} = \frac{e}{m}F^\mu{}_\nu u^\nu$$

and show that they represent the usual definition of the Lorentz force. (Remember $dt/d\tau = \gamma$.) What does the zeroth component represent?

One way of solving these coupled equations is to use eigenvectors to find uncoupled combinations. The matrix $F^\mu{}_\nu$ is not symmetric but it still has a complete set of right eigenvectors $d^\mu_{(p)}$ satisfying $F^\mu{}_\nu d^\nu_{(p)} = \lambda_{(p)} d^\mu_{(p)}$, and left eigenvectors satisfying $c_{\mu(p)}F^\mu{}_\nu = \lambda_{(p)} c_{\nu(p)}$. Show that the eigenvalues $\lambda_{(p)}$ are

$$\lambda^2 = \left(\frac{D}{2} \pm \sqrt{(\frac{D^2}{4} - G^2)}\right)$$

in terms of the field invariants of problem 13. (Without loss of generality we can take the z-axis along $\mathbf{B}$, and $\mathbf{E}$ in the xz-plane.) Show that there are now six classes of solution: $D > 0$, $D = 0$ and $D < 0$ with either $G = 0$ or $G \neq 0$.

How do these correspond with the classes in the non-relativistic limit (problem 14)?

16. The relativistic case corresponding to case 1) of problem 15 is $G = 0$ and $D > 0$. Note that the $F^\mu{}_\nu$ matrix is now symmetric so $c_{\mu(p)} = d^\mu_{(p)}$. Verify that the eigenvalues and eigenvectors of $F^\mu{}_\nu$ in this case are (taking the x-axis along $\mathbf{E}$):

$$\lambda_{(1)} = 0 \qquad d^\mu_{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_{(2)} = 0 \qquad d^\mu_{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_{(3)} = \frac{E}{c} \qquad d^\mu_{(3)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_{(4)} = -\frac{E}{c} \qquad d^\mu_{(4)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

Hence show that the combinations $W_p(\tau) = c_{\mu(p)} u^\mu$ satisfy uncoupled equations, and that the solution is

$$u^\mu = \sum_p W_p(0)\, \exp(\omega_{(p)}\tau) d^\mu_{(p)} \qquad \text{where} \quad W_p(0) = c_{\mu(p)} u^\mu(0)$$

and $\omega_p = q\lambda_{(p)}/m$. Show that this reduces to

$$u^\mu(\tau) = \begin{pmatrix} u^0(0)\cosh(\omega\tau) + u^1(0)\sinh(\omega\tau) \\ u^0(0)\sinh(\omega\tau) + u^1(0)\cosh(\omega\tau) \\ u^2(0) \\ u^3(0) \end{pmatrix}.$$

17. In a reference frame $\mathcal{S}$ a wire straight wire of radius a carries a current I. The current four-vector is thus

$$j^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I/\pi a^2 \end{pmatrix} \qquad \text{for } x^2 + y^2 < a^2.$$

A reference frame $\mathcal{S}'$ moves with velocity $\mathbf{v} = v\mathbf{k}$ relative to $\mathcal{S}$. Find the components of the current four-vector in $\mathcal{S}'$. Comment on the emergence of a non-zero charge density.

18. In a reference frame $\mathcal{S}$ there is a uniformly charged sphere with radius a. The current four-vector is

$$j^\mu = \frac{3Qc}{4\pi a^3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for } x^2 + y^2 + z^2 < a^2.$$

and the potential four-vector is

$$A^\mu = \begin{cases} \dfrac{\mu_0 Qc}{4\pi r} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \text{for } x^2 + y^2 + z^2 > a^2 \\ \dfrac{\mu_0 Qc}{4\pi a} \left(\dfrac{3}{2} - \dfrac{r^2}{2a^2} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \text{for } x^2 + y^2 + z^2 < a^2. \end{cases}$$

Find these four-vectors in a reference frame $\mathcal{S}'$ moving with speed v along the x-axis. Find the fields in $\mathcal{S}'$ in the external region either by transforming the fields in $\mathcal{S}$, or by differentiating the potential in $\mathbf{S}'$.

19. Spinless Hydrogen in a Magnetic Field. Spinless hydrogen is a model system in which the electron in hydrogen is replaced by a spinless particle. The Schrodinger equation is

$$\frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \psi_{nlm}(\mathbf{r}) - e\Phi\, \psi_{nlm}(\mathbf{r}) = E\, \psi_{nlm}(\mathbf{r})$$

where the electron has charge $-e$, $\Phi = e/4\pi\epsilon_0 r$ is the scalar potential of the proton (assumed infinitely heavy) and $\mathbf{A} = \frac{1}{2}\mathbf{B} \wedge \mathbf{r}$ is the vector potential of an external uniform magnetic field. The $n = 2$, $l = 1$ and $m = \pm 1$ states have eigenfunctions and eigenvalue (in zero B-field)

$$\psi_{2p\pm} = \frac{1}{8\sqrt{\pi a^3}} \frac{x \pm iy}{a} e^{-r/2a} \qquad E_{2p\pm} = \frac{\hbar^2}{8ma^2}$$

where a is the Bohr radius.

Show that the B-dependent terms in the Hamiltonian are

$$\frac{e}{2m}\mathbf{L} \cdot \mathbf{B} \qquad \text{and} \qquad \frac{e^2}{8m}(B^2 r^2 - (\mathbf{B} \cdot \mathbf{r})^2).$$

(You may find it helpful to start by proving that $\nabla \cdot \mathbf{A} = 0$, and hence that $\mathbf{p} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{p}$.)

Show that if the term quadratic in B is ignored these functions are still eigenfunctions, but the eigenvalues change to $E_{2p\pm} \pm e\hbar B/2m$. Find the gauge-invariant current, and show that it circulates around the z-axis, and that, although the wavefunction is unchanged, the current increases/decreases linearly with B in the $+/-$ state respectively.