QM5

(1.) Parity

We define the parity operator in one dimension P:

$$P\,\psi(x) \;=\; \psi(-x).$$

The operator 'flips over' a function about the origin, or reflects it in the origin. Show that P is Hermitian. Show that $P^2 \;=\; 1$, in other words operating twice is equal to no operation at all. Hence show that the eigenvalues are ± 1, and find the eigenfunctions. (Note that the eigenfunctions are very far from being unique; each is a large class of functions.)

Now return to the square well, this time a well with finite depth. We shift the origin so that the edges are at $\pm a/2$:

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq a/2; \\ V_0 & \text{if } |x| > a/2. \end{cases}$$

Write down the Hamiltonian H, and show that P and H commute. Hence all eigenstates can be chosen to be eigenstates of P. Solve separately to find the $+$ and $-$ eigenstates and eigenvalues for the bound states (that is, states localised around the well). Show that the $\pm$ parity eigenvalues are given by $E = \hbar^2 k^2/2m$ where k satisfies

$$\left\{ \begin{array}{c} \tan \\ -\cot \end{array} \right\} \left(\frac{ka}{2} \right) = \sqrt{\frac{K^2}{k^2} - 1} \qquad \text{where} \quad V_0 = \frac{\hbar^2 K^2}{2m}$$

Sketch the left and right sides of this equation, and hence prove that the lowest state is even parity, there is always at least one bound state, and if there is more than one bound state then even and odd states alternate. If $V_0 a^2 m/\hbar^2 = 18$ how many bound states are there?

There are also unbound states in this potential — what range of eigenvalues do thay have? Find unnormalised eigenfunctions for the even parity states, and show that they cannot be normalised.

(2.) 1996 A3 6. Particles of mass m moving, in one dimension, in a potential $V(x)$ have wavefunction $\psi(x,t)$ satisfying the time-dependent Schrödinger equation. Derive an expression for the probability current density $J(x,t)$.

Particles of a particular energy E moving in the direction of the positive x-axis suffer no reflection when incident on a double potential step defined by

$$V(x) = \begin{cases} 0 & x < 0 \\ V_1, & 0 \leq x < a \\ V_2, & x \geq a\ , \end{cases}$$

where $V_{1,2}$ are constants satisfying the inequalities $E > V_2 > V_1 > 0$. Write down the form of the wavefunction in the three regions and, from the boundary conditions at $x = 0$ and $x = a$, find the relation between E, V_1 and V_2 which is necessary for zero reflection given that

$$\left(\frac{2m(E - V_1)}{\hbar^2} \right)^{\frac{1}{2}} a = \frac{\pi}{2}.$$

Obtain expressions for the magnitude and phase of the complex amplitude of the transmitted wave. Verify that the incident and transmitted waves have equal probability current densities.

(3.) Angular Momentum

a) We define the angular momentum $\mathbf{L}$ to be $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. (This part is algebraically difficult, but not essential for part (b)). Show that

$$L^2 \;=\; r^2 p^2 \;-\; (\mathbf{r}\cdot\mathbf{p})^2 \;+\; i\hbar\,\mathbf{r}\cdot\mathbf{p}.$$

(There are several similar forms for this expression.) We can identify

$$\mathbf{r}\cdot\mathbf{p} \;=\; -i\hbar\,\mathbf{r}\cdot\nabla \;=\; -i\hbar\, r\frac{\partial}{\partial r}.$$

Hence identify the connection between the operator L^2 and the angular part of the operator for p^2 expressed in spherical polar co-ordinates.

N.B. This derivation is very obvious in classical mechanics, because $\mathbf{r}$ and $\mathbf{p}$ commute: it just involves

$$L^2 \;=\; r^2 p^2 \;-\; (\mathbf{r}\cdot\mathbf{p})^2$$

and hence

$$p^2 \;=\; \frac{1}{r^2}\,(\mathbf{r}\cdot\mathbf{p})^2 \;+\; \frac{L^2}{r^2}.$$

b) A particle moves in a central potential $V(r)$. Write down the Schrödinger equation for this system, and separate the variables in spherical polar co-ordinates. As implied by part (b), the angular equation is simply the eigenvalue equation for L^2, and the angular parts of the eigenfunctions are the eigenfunctions of L^2 (spherical harmonics, Y_{lm}). Solve to find the eigenvalues of L_z and L^2, and sketch some of the simpler spherical harmonics.

(4.) Angular Momentum — Operator method

In this question I am using angular momentum operators with dimensions of angular momentum. Use is often made of dimensionless operators, by dividing by $\hbar$. You can use either as long as you know what you are doing!

a) We define an angular momentum operator $\hat{\mathbf{J}}$. Show that, in the case of the orbital angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, the commutation relations satisfied by the components of $\hat{\mathbf{L}}$ are $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ and cyclically. We take these as defining properties of the more general angular momentum operator $\hat{\mathbf{J}}$. (Do you know why?) Show that $\hat{J}^2$, defined by $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, commutes with $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$. Discuss the possibility of finding a basis of simultaneous eigenstates of more than one of the operators $\hat{J}^2, \hat{J}_x, \hat{J}_y$ and $\hat{J}_z$.

b) We define the operators $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ and $\hat{J}_- = \hat{J}_x - i\hat{J}_y$. Find their commutators with $\hat{J}^2$ and $\hat{J}_z$ and show that $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$. Hence show that if $|a,b\rangle$ is an eigenstate of $\hat{J}^2$ and $\hat{J}_z$:

$$\hat{J}^2|a,b\rangle = a|a,b\rangle$$
$$\hat{J}_z|a,b\rangle = b|a,b\rangle$$

then the two states $\hat{J}_+|a,b\rangle$ and $\hat{J}_-|a,b\rangle$ are either zero or are eigenstates of $\hat{J}^2$ and $\hat{J}_z$ with eigenvalues $a' = a$, $b' = b \pm \hbar$.

c) The eigenvalue b cannot be arbitrarily large: $a \geq b^2$. (Why?) Deduce that there must exist states $|a, b_+\rangle$, $|a, b_-\rangle$, with $b_+ - b_- = \text{integer} \times \hbar$, such that

$$\hat{J}_+|a,b_+\rangle = 0$$
$$\hat{J}_-|a,b_-\rangle = 0.$$

By operating on these equations with $\hat{J}_-$ or $\hat{J}_+$ show that

$$a = b_+(b_+ + \hbar) = b_-(b_- - \hbar).$$

Hence deduce that

$$a = J(J+1)\hbar^2, \quad J = \text{integer or half-integer}$$
$$b = M\hbar, \quad M = -J, (-J+1), \ldots + J.$$

Thus there are $2J+1$ states with different values of M for each J.

Summary of deductions from the commutation relations:

- States can be found which are simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$.
- When this is done, the eigenvalues are found to be $J(J+1)\hbar^2$ and $M\hbar$ as above, which include the $l(l+1)\hbar^2$ eigenvalues found for orbital angular momentum, but also half-integer angular momenta
- $\hat{J}_x \pm i\hat{J}_y$ are shift operators, increasing or decreasing the value of M at fixed J.

d) In exactly the same way as for the harmonic oscillator creation and destruction operators we introduce the factors $N_\pm$:

$$\hat{J}_+|J,M\rangle = N_+\hbar|J,M+1\rangle$$
$$\hat{J}_-|J,M\rangle = N_-\hbar|J,M-1\rangle.$$

Show that $|N_+|^2 = J(J+1) - M(M+1)$, $|N_-|^2 = J(J+1) - M(M-1)$. The conventional phase choice for the angular momentum states makes N_+ and N_- both positive.

e) For the case $J = 1/2$ evaluate all the matrix elements of $\hat{J}_z$, $\hat{J}_+$ and $\hat{J}_-$. Show that the matrix for $\hat{J}_z$ is given by

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Construct the matrices for the other components of $\hat{\mathbf{J}}$, and show that they are given by the corresponding Pauli spin matrices multiplied by $\hbar/2$.

N.B. The use of the caret to distinguish between the operators and the eigenvalues, while useful in this question, is not standard. No typographical distinction is usually made between them!