QM2

General Principles of Quantum Mechanics

a) Define a Hermitian operator (you will probably need to know what an adjoint operator is; Hermitian operators are self-adjoint). Show that if the adjoint of operator A is $A^\dagger$ then the adjoint of cA is $c^\star A^\dagger$ where c is a complex number. Derive the important properties of eigenfunctions and eigenvalues of Hermitian operators.

b) We normalise kets such that $\langle\psi|\psi\rangle = 1$, and this follows from our interpretation of the amplitudes (such as $\langle x|\psi\rangle$) as probability amplitudes whose square modulus gives a probability which must sum to 1. However, the time evolution of the ket is governed by Schroedinger's time-dependent equation:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

Thus the time-evolution must preserve the normalisation, or else a wavefunction normalised at time t would not be normalised at subsequent times. Show that the normalisation is preserved provided the Hamiltonian H is Hermitian.

c) How do you expand an arbitrary ket $|\psi\rangle$ in a complete orthonormal basis $|n\rangle$?

d) If the Hamiltonian H does not depend on time (as is usually the case) then the Schrodinger equation looks very simple — it's a first order linear ODE with a trivial integrating factor:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad \rightarrow \qquad |\psi(t)\rangle = \exp(-iHt/\hbar)|\psi(0)\rangle.$$

But of course it's not so simple, because $|\psi\rangle$ is a ket vector not a simple function of t. Or is it so simple? What's the *real* difficulty with this solution, and how is it overcome? Show, using the result of c), how this general solution produces the usual time-dependent solution in terms of the eigenkets which satisfy $H|n\rangle = E_n|n\rangle$.

e) What is the quantum-mechanical prediction for the probability distribution of the results of a measurement of a property with operator A on a state $|\psi\rangle$? What is the state after the measurement? How does this relate to the definition of the expectation value?

f) What is meant by the correspondence principle? How does the correspondence work; *i.e.* what corresponds with the classical $x(t)$?

g) A particle is in a one-dimensional box of width a between $x = 0$ and $x = a$. The ket $|A(0)\rangle$ at time $t = 0$ given by

$$|A(0)\rangle = N(|n\rangle + |n+1\rangle)$$

where $|n\rangle$ and $|n+1\rangle$ are two of the eigenkets for the infinite square well. Find the normalizing constant N, and write down $|A(t)\rangle$. Find the expectation value of x at time t. (You may quote the following results, although you may also be able to prove them: $\langle n|x|n\rangle = a/2$ for all n, $E_n = n^2\pi^2\hbar^2/2ma^2$, and

$$\langle n|x|n'\rangle = \begin{cases} \frac{2a}{[(n+n')\pi]^2} - \frac{2a}{[(n-n')\pi]^2} & \text{if } n+n' = \text{odd;} \\ 0 & \text{otherwise.} \end{cases}$$

h) Find the classical motion of a particle in the infinite square well and sketch $x(t)$. In the limit of large n, show that the frequency that appears in g) is that given by the correspondence principle.

Blundell Questions 2.4, 2.7, 2.9, 2.10, 2.11