

Further Quantum Mechanics

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2012 Problem Set

There are three problem sets, suitable for use at the end of Hilary Term, beginning of Trinity Term and Week 3 or 4 of Trinity Term. The problems are labelled according to their difficulty. So some of the problems have a double dagger †† to indicate that they are a bit more challenging. Some of the problems have a single dagger †. They are straightforward extensions and applications of material in the lectures. Finally there are problems with no daggers; these are closely based on problems we've done in the lectures.

1 Perturbation theory

- (1) **Derivation.** Derive the expressions for the first-order ($E_n^{(1)}$) and second-order ($E_n^{(2)}$) changes in the energy of a non-degenerate level $|n_0\rangle$ with unperturbed eigenvalue $E_n^{(0)}$, induced by a perturbation H_1 , so that the total Hamiltonian is $H_0 + H_1$. Calculate the first-order change in the wavefunction $|n_1\rangle$. Under what conditions would you expect perturbation theory to be reliable for this level?
- (2) **SHO with x^2 perturbation.** A particle of mass m is in a harmonic oscillator potential $V_0 = (1/2)m\omega^2 x^2$. A perturbation is introduced which changes the potential to $V = V_0 + V_1$ with $V_1 = (1/2)\lambda m\omega^2 x^2$ where λ is small, $\lambda \ll 1$. The perturbation is thus $H_1 = V_1$.
- (a) Write H_1 in terms of the operators $a = \sqrt{\frac{1}{2\hbar m\omega}}(m\omega x + ip)$ and $a^\dagger = \sqrt{\frac{1}{2\hbar m\omega}}(m\omega x - ip)$.
- (b) Use perturbation theory to compute the first order change $E_0^{(1)}$ in the ground state energy. Can you extend your calculation to an arbitrary level $|n\rangle$?
- (c) Part (b) just used the $aa^\dagger + a^\dagger a$ part of H_1 . Can you work out how the $aa^\dagger + a^\dagger a$ part gives the second order perturbation? In effect it tells us which states give non-zero contributions in the otherwise infinite sum. Show that the second order shift to the ground state is $-\lambda^2 \hbar \omega / 16$.
- (d) Of course the problem with the full potential V is exactly solvable because it's just a SHO with a shifted frequency ω' ! So write down the exact expression for the energy eigenvalues. Now expand it in powers of λ and check all the results you obtained in earlier parts.
- (e) †Having worked out which states contribute it's now pretty trivial to find the correction to the ket. Find the first-order correction $|0_1\rangle$ to the ground state ket $|0_0\rangle$. Use the exact result, this time in the x -representation, to check your answer. For any ω , $\langle x|0\rangle = (m\omega/\pi\hbar)^{1/4} \exp(-m\omega x^2/2\hbar)$ and $\langle x|2\rangle = \sqrt{2}[(m\omega x^2/\hbar) - 1/2]\langle x|0_0\rangle$. Substitute the expansion of ω' to first order, and expand consistently to first order in λ to find $\langle x|0_0\rangle + \langle x|0_1\rangle$.
- (3) **Particle in box with non-flat bottom.** Consider the one-dimensional box, or infinite square well, of width a , with sides at $x = 0$ and $x = a$. The unperturbed problem is well-known: the eigenvalues are $E_n^{(0)} = n^2 \hbar^2 \pi^2 / 2ma^2 = n^2 E_1^{(0)}$, and the eigenfunctions are

$$\langle x|n_0\rangle = u_n(x) = \sqrt{2/a} \sin(n\pi x/a).$$

- (a) We now add a perturbation

$$H_1 = W \cos(\pi x/a).$$

Sketch the perturbed potential well as a function of x . Show that all the first-order energy shifts are zero. Is there a simple explanation for this?

- (b) Find the first order correction to the ground state wavefunction. Sketch the ground state wavefunction and the correction, showing that the particle is more localised around the minimum of the potential.
- (c) Discuss the constraints on W for perturbation theory to be a suitable approximation method.
- (d) Can you see how to evaluate the second-order shift $E_n^{(2)}$ for $n = 1$ and $n = 2$?

You will find the evaluation of the integrals much simplified if you start by proving for the perturbation a relationship of the form

$$H_1 u_n = \alpha(u_{n-1} + u_{n+1}).$$

This relationship turns the integrals into orthogonality integrals. You will need to think about the meaning of this equation for $n = 1$ since $n - 1$ is then zero, while u_n is only defined for $n > 0$.

- (4) **SHO with linear perturbation.** A particle of charge q and mass m is in a harmonic oscillator potential $V_0 = (1/2)m\omega^2 x^2$. A weak external electric field of magnitude f is applied along the x direction. This gives an electrostatic potential $-fx$ and potential energy $V_1 = -qfx$. Treating this as a small perturbation, calculate the shift in energy of all the states, as follows.

- (a) Write down the energy shift to be calculated, $E_n^{(1)}$, in the form of a matrix element by quoting the standard result of first-order perturbation theory. (Call the energy eigenstates of the unperturbed Hamiltonian $|n_0\rangle$ as usual.)
- (b) Deduce that the first order shift is zero for every n . This is an easy calculation using the properties of raising and lowering operators. Is there an even simpler argument?
- (c) We will next work out the first order change in the wavefunctions (this will turn out to be non-zero). First write down the standard result, derived in question 1, involving a sum of matrix elements divided by energy differences. How many terms contribute to the sum?
- (d) Hence show that the first order correction to the state is

$$|n_1\rangle = \frac{qf}{\sqrt{2m\hbar\omega^3}} \left((n+1)^{1/2} |(n+1)_0\rangle - n^{1/2} |(n-1)_0\rangle \right).$$

Does this formula give the correct result when $n = 0$?

- (e) Write down the formula for the 2nd order energy shift, and use the result of part (d) to show it is $E_n^{(2)} = -q^2 f^2 / 2m\omega^2$.
- (f) †This is a nice test of perturbation theory, because the problem can be solved exactly by a simple insight. To get the general idea, plot a graph of $V_0(x)$ and also of $V_0(x) + V_1(x)$. (For the purpose of this graph alone, choose $m = \omega = q = f = 1$). The point is, the new potential energy is still exactly a quadratic, it is merely shifted over and down a bit. Therefore write $V_0(x) + V_1(x) = A(x - x_0)^2 + B$ and find A and B . Use this to deduce the exact energy eigenvalues for this problem, and hence confirm your answers from parts (b) and (f).

- (5) ††**WKB Approximation and interpretation of the perturbed wavefunction.** This question introduces the WKB approximation, a completely different approach to obtaining approximate wavefunctions, based on the de Broglie relation $p = \hbar k$. Here k is the rate of change of phase with distance:

$$\phi = \int_0^x k dx = \int_0^x \frac{p}{\hbar} dx.$$

The momentum is given by $p^2/2m = E - V$ so k is given by

$$k = \sqrt{\frac{2m(E_n - V(x))}{\hbar^2}}.$$

For a 1-dimensional bound state problem we must have both forward e^{ikx} and backward e^{-ikx} waves, so with a boundary condition of $u_n(0) = 0$, like the particle in a box, the wavefunction must look something like

$$u_n(x) \propto \sin \phi.$$

An even better approximation is to adjust the amplitude of the wavefunction so that it is smaller where the particle moves faster (think about traffic on a road when you enter or leave a speed limit!). We need probability density $\propto 1/\text{velocity}$, and hence wavefunction $\propto 1/\sqrt{\text{velocity}}$:

$$u_n(x) \propto \frac{1}{\sqrt{k}} \sin \phi.$$

This is a very good approximation to the exact wavefunction where the wavelength is small compared to the distance over which the potential changes: that is at large n . We shall use it to look at the box with a non-flat bottom of question (3).

- (a) We now look at the WKB solution when we can treat V perturbatively, so that we treat the whole of V inside the box to first order. We will assume that $E_n = n^2 E_1$ is unchanged — the method can be used to deduce this, but we have already proved it to first order, which is good enough. Then we can write our WKB wavefunction as

$$u_n(x) = \sqrt{\frac{2k_0}{ak}} \sin \phi \quad \text{where} \quad k_0 = \sqrt{\frac{2mE_n}{\hbar^2}} = \frac{n\pi}{a}.$$

Show that to first order $k \approx k_0 \left[1 - \frac{W \cos(\pi x/a)}{2n^2 E_1} \right]$ and

$$\phi \approx k_0 x - \frac{W}{2nE_1} \sin(\pi x/a).$$

- (b) By expanding consistently to first order in W show that

$$u_n(x) \approx \sqrt{\frac{2}{a}} \sin(k_0 x) \left(1 + \frac{W \cos(\pi x/a)}{4n^2 E_1} \right) - \sqrt{\frac{2}{a}} \cos(k_0 x) \frac{W \sin(\pi x/a)}{2nE_1}.$$

Here the first correction term changes the amplitude of the existing wavefunction, and the second term corrects its phase, making the wavelength longer where the particle moves slower.

- (c) Show that the first order correction to the wavefunction in perturbation theory is

$$|n_1\rangle = \frac{W}{2(2n-1)E_1} |(n-1)_0\rangle - \frac{W}{2(2n+1)E_1} |(n+1)_0\rangle.$$

Compare this with the WKB wavefunction correction. You should find agreement when n is sufficiently large that we can neglect 1 in comparison with $4n^2$.

2 Degenerate and Time Dependent Perturbation Theory

- (1) **Degenerate Perturbation Theory: 2-Dimensional Harmonic Oscillator** The two-dimensional HO has Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) = H_x + H_y$$

so that it is the sum of two one-dimensional SHO.

- (a) Show that $[H, H_x] = 0$. Hence we can choose eigenstates of H to be eigenstates of H_x as well. (And hence H_y — explain.) The normalized wavefunctions for the ground state and 1st excited state of the *one-dimensional* SHO are $\phi_0(x) = N_0 e^{-\alpha x^2/2}$ and $\phi_1(x) = \sqrt{2\alpha} x \phi_0(x)$ respectively. Let $\Phi_{nm}(x, y) = \phi_n(x)\phi_m(y)$.
- (b) The angular momentum in the x - y plane is $L = xp_y - yp_x$. Show that $[L, H_x] \neq 0$, but $[L, H] = 0$. What do you conclude about possible bases of eigenstates of H ?
- (c) Explain why the ground state wavefunction for the 2D SHO is $\Phi_{00}(x, y)$ and the first excited state is doubly degenerate with wavefunctions $\Phi_{10}(x, y)$ and $\Phi_{01}(x, y)$.
- (d) A small perturbation $H_1 = \lambda xy$ is now added to the Hamiltonian. Show that to 1st order in λ the ground state energy does not change.
- (e) Using degenerate perturbation theory show that the degeneracy of the 1st excited state is lifted and that the wavefunctions of the two resulting states are $(\Phi_{01}(x, y) \pm \Phi_{10}(x, y))/\sqrt{2}$. What are the corresponding energies?
- (f) Show that $[H_x, H_1] \neq 0$, and also $[L, H_1] \neq 0$. So neither the H_x basis used here, nor the angular momentum basis, will avoid the diagonalization. Consider the operation S of reflection about the line $x = y$, so that $Sx = yS$ and $Sy = xS$. Show that $[S, H_1] = 0$. Consider your eigenfunctions from part (e) in terms of behaviour under S .

Hydrogen-like kets and wavefunctions for the next three questions. Kets are labelled $|n, l, m\rangle$, and the wavefunctions are in terms of $a = \frac{\hbar^2}{mZe^2/4\pi\epsilon_0}$ where $Z =$ atomic number:

$$\psi_{1s} = \langle \mathbf{r} | 100 \rangle = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \quad \psi_{2s} = \langle \mathbf{r} | 200 \rangle = \frac{1}{\sqrt{8\pi a^3}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

$$\psi_{2p0} = \langle \mathbf{r} | 210 \rangle = \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} \cos\theta e^{-r/2a}$$

All the integrals involve the factorial integral

$$\int_0^\infty x^n e^{-kx} dx = \frac{n!}{k^{n+1}}.$$

- (2) **Linear Stark Effect in Hydrogen $n = 2$.** The usual argument for the vanishing of the Stark effect in first order is based on parity: $[H_0, P] = 0$ but $\{H_1, P\} \neq 0$, where H_0 is the unperturbed atomic Hamiltonian and H_1 is the interaction with the applied field $H_1 = e\mathcal{E}z$. The argument fails when states of opposite parity are degenerate, as in all excited states in Hydrogen. We shall consider the $n = 2$ states.
- (a) Show that $[Lz, H_1] = 0$. The states in $n = 2$ are $|200\rangle, |210\rangle, |211\rangle, |21-1\rangle$. Argue that the only states we need consider are $|200\rangle$ and $|210\rangle$. We therefore have a 2×2 matrix to evaluate. Argue that the diagonal elements are zero (remember Parity). The off-diagonal elements are complex conjugates of each other. Hence we have just one integral to evaluate.

- (b) Show that the perturbation matrix is

$$\begin{pmatrix} 0 & -3e\mathcal{E}a_0 \\ -3e\mathcal{E}a_0 & 0 \end{pmatrix}.$$

Hence find the linear Stark shifts in $n = 2$, and the eigenstates for each level.

In fact there are small energy shifts between the s and p levels in $n = 2$ caused by effects we have not studied yet, but this treatment is still valid in the limit of large electric fields such that this extra splitting is small compared with the Stark effect.

- (3) **Quadratic Stark effect in Hydrogen $n = 1$.** As discussed in the lecture the second-order Stark effect in the ground state of hydrogen is given by

$$E_{1(2)} = \sum_{n'\ell' \neq 10} \frac{|\langle n'\ell'0 | e\mathcal{E}z | 100 \rangle|^2}{E_1^{(0)} - E_n^{(0)}}$$

The restriction to $m = 0$ follows from the L_z commutator in the previous question. In fact only $\ell = 1$ states contribute.

- (a) Calculate the energy denominator for the first three terms in the sum. In addition to the denominator getting bigger, the numerator gets smaller as the excited state wavefunction looks less like the ground state.
- (b) Obviously we can find a lower bound on the right answer by calculating just the first, $n = 2$ term. Do so.
- (c) †† The full second-order result can be found by solving the equation for the first-order perturbation $|(100)_1\rangle$ directly as an inhomogeneous differential equation. The result of this is

$$|(100)_1\rangle = -\frac{4\pi\epsilon_0\mathcal{E}a}{e} [r + (r^2/2a)] \cos\theta |100\rangle.$$

Show that the full result for the second-order energy is $-\frac{9}{4}\mathcal{E}^2(4\pi\epsilon_0a^3)$ and compare with the lower bound found in part (b).

- (4) **Sudden approximation** In the β decay H^3 (1 proton + 2 neutrons in the nucleus) \rightarrow $(\text{He}^3)^+$ (2 protons + 1 neutron in the nucleus), the emitted electron has a kinetic energy of 16 keV. We will consider the effects on the motion of the atomic electron, i.e. the one orbiting the nucleus, which we assume is initially in the ground state of H^3 .

- (a) Show by a brief justification that the perturbation is sudden, by considering the location of the emitted electron at a time around $\tau = 5 \times 10^{-17}$ s after emission. How does τ compare with the time-scale on which the wavefunction changes?
- (b) Show that the probability for the electron to be left in the ground state of $(\text{He}^3)^+$ is $2^3(2/3)^6 \simeq 0.7$.

- (5) **Fermi Golden Rule.**

- (a) A particle of mass m is bound by a very short range attractive potential that can be modelled as a delta-function: $V(x) = -W\delta(x)$ where W is a positive constant with dimensions energy \times length. Show that this system has a single bound state $|0\rangle$ with wavefunction

$$\langle x|0\rangle = \frac{1}{\sqrt{a}} \exp(-|x|/a) \quad \text{where} \quad a = \frac{\hbar^2}{mW} \quad \text{and eigenvalue} \quad E_0 = -\frac{mW^2}{2\hbar^2}.$$

In addition there are positive energy states of positive and negative parity $|p, +\rangle$ and $|p, -\rangle$ with energies $E(p) = p^2/2m$ and wavefunctions

$$\langle x|p, +\rangle = \frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{p(|x| + \phi)}{\hbar}\right) \quad \langle x|p, -\rangle = \frac{1}{\sqrt{\pi\hbar}} \sin(px/\hbar)$$

where $\cot\phi = \hbar p/(mW)$. These states have the usual continuum normalisation $\langle p-|p'-\rangle = \delta(p-p')$. Taken together these form a complete set of states.

- (b) This system is initially in the ground state at $t = 0$, when it is perturbed by an odd-parity perturbation $V(x, t) = -Fx e^{-i\omega t}$. We can expand the system ket $|\psi(t)\rangle$ in terms of the complete set of unperturbed states:

$$|\psi(t)\rangle = a_0(t) e^{-iE_0 t/\hbar} |0\rangle + \int_0^\infty [b(p, t) |p, +\rangle + c(p, t) |p, -\rangle] e^{-ip^2 t/2m\hbar} dp$$

where $b(p, t)$ and $c(p, t)$ are the amplitudes for the positive and negative parity continuum states: $c(p, t) e^{-ip^2 t/2m\hbar} = \langle p, - | \psi \rangle$. Use the Schrodinger equation to derive equations of motion for $b(p, t)$ and $c(p, t)$, and show that to first order in F the perturbation creates non-zero amplitudes only in the negative-parity excited states.

- (c) Hence show that to first order

$$c(p, t) = Fa \sqrt{\frac{a}{\pi\hbar}} \frac{4pa/\hbar}{[1 + (pa/\hbar)^2]^2} \frac{\exp[i(E(p) - E_0 - \hbar\omega)t/\hbar] - 1}{E(p) - E_0 - \hbar\omega}.$$

You can assume the integral

$$\int_0^\infty x \exp(-x/a) \exp(ipx/a) dx = \frac{a^2}{[1 - i(pa/\hbar)]^2}$$

- (d) Within this approximation the total excitation probability is

$$P(t) = \int_0^\infty |c(p, t)|^2 dp$$

which cannot be evaluated in closed form. Change the variable of integration to $q = (E(p) - E_0 - \hbar\omega)/2\hbar$ and justify the Fermi golden rule approximation at large times. Hence show that when $\hbar\omega = \frac{mW^2}{\hbar^2}$ the only excited states are those around $p = \hbar/a$, and

$$P(t) \approx \frac{2F^2 \hbar^5 t^2}{\pi m^3 W^4} \int_{-\infty}^\infty \frac{\sin^2 qt}{q^2 t^2} dq = \frac{2F^2 \hbar^5 t}{m^3 W^4}.$$

- (6) **A Gaussian kick.** A harmonic oscillator starts in its ground state ($n = 0$) at $t = -\infty$. A perturbation $H_1 = -xF(t)$ is applied between $t = -\infty$ and $t = T$.

- (a) By considering the corresponding classical interaction, explain why this represents the application of a time-dependent force $F(t)$ to the oscillator.
 (b) Calculate the ket at time T correct to first order in the perturbation. Why does the first-order correction only involve the state $n = 1$?
 (c) Hence write down the expectation value of x at time T .
 (d) †Compare this with the corresponding classical result for the application of a force $F(t)$ to an oscillator starting from rest:

$$x(t) = \int_{-\infty}^t G(t-t') F(t') dt' \quad \text{where} \quad G(t-t') = \frac{1}{m\omega} \sin[\omega(t-t')].$$

(The function G is known as the Green's function for the problem.)

- (e) In the case of a gaussian force $F(t) = F_0 \exp(-t^2/\tau^2)$ show that the probability the oscillator makes a transition to the state $n = 1$ in the limit $T \rightarrow \infty$ is

$$P_{0 \rightarrow 1} = \frac{F_0^2 \pi \tau^2}{2m\omega\hbar} e^{-\omega^2 \tau^2/2}$$

- (f) Plot the dependence of $P_{0 \rightarrow 1}$ on the timescale τ of the perturbation, and comment on the behaviour in the limits $\tau \rightarrow 0$ and $\omega\tau \gg 1$, using either quantum or classical arguments.

3 Selection Rules and Helium

- (1) **Summary of selection rules.** Write down the selection rules for electric dipole radiation, distinguishing exact and approximate rules. (Note that for hydrogen the approximate rules are exact as well, because there is no ambiguity about the fact that single electron has spin half.) Draw an energy level diagram for hydrogen (use the vertical direction for energy, and separate the states horizontally by angular momentum ℓ). How do the rules apply to hydrogen? (To answer this you will need to find out the parity of the hydrogen states. Remember that the wavefunction takes the form $R_{nl}(r)Y_{\ell m}(\theta, \phi)$. What happens to the coordinates r , θ and ϕ under the parity operation?)

(2) **Scaling: hydrogen-like ions**

- (a) How do the energy levels of hydrogen-like ions scale as a function of the nuclear charge Z ? Show that the wavelength of $n = 2 \rightarrow n = 1$ transition in hydrogen occurs at a wavelength of 121.57 nm. (You can check the values of physical constants at:

<http://physics.nist.gov/cuu/Constants/index.html>.

If you are getting 121.51 nm you have forgotten the reduced mass correction! But it's obviously a small effect and it you can probably continue to ignore it.) The following wavelengths (as well as many others) are observed in emission from a plasma: 3.375 nm, 2.848 nm, 2.700 nm, 18.226 nm, 13.501 nm, 12.054 nm. Show that this spectrum is consistent with emission from a hydrogen-like ion, and identify the element. [Method: first make a reasonably accurate sketch of this spectrum on a frequency scale, then guess the identity of one or more of the transitions, then confirm your guess using your knowledge of the pattern to be expected, and hence deduce Z].

- (b) How does the mean radius for an electron in the ground state of hydrogen-like ions scale as a function of the nuclear charge Z ? What is the mean radius for an electron in the ground state of this ion?

- (3) **Scaling: muonic ions** A muon has mass 206 times the electronic mass and the same charge as an electron. The particle may be captured by an atom and the radiation which is emitted as the muon cascades through the energy levels can be measured.

- (a) Assuming the muon-nucleus system can be treated as hydrogen-like, find the energy in MeV of the photon emitted as the muon goes from a state with principal quantum number 2 to the ground state in an atom of iron and lead.

- (b) †Is it reasonable to neglect the effect of the electrons in this calculation as a first approximation? (Consider the mean radius of the muon orbits). Do you think the influence of the nuclear charge has been accurately accounted for? (The observed energies are 1.255 MeV for iron and about 5.88 MeV for lead).

- (4) **He: ionisation energy.** The ionization energy is the energy required to remove one electron from an atom or ion in its ground state, leaving it in the ground state of the next higher ionization stage. (The energy is often quoted in eV and referred to as an ionization potential.)

- (a) If we make the (poor) approximation of ignoring the electron-electron repulsion altogether, what value (in eV) is obtained for the ionization potential of the ground state in helium? How much additional energy would then be required to remove the second electron? Assuming these estimates have been made as accurately as reasonably possible within their respective assumptions, state the degree of accuracy of each of these two results (i.e. how close they may be expected to be to the true first and second ionization energies for helium.)

- (b) To do better, use the variational method. Using hydrogen-like wavefunctions for both electrons: $\psi_{1s} = \sqrt{Z^3/\pi a_0^3} \exp(-Zr/a_0)$, show that

$$\left\langle \frac{1}{r_1} + \frac{1}{r_2} \right\rangle = \frac{2Z}{a_0}.$$

Using also $\langle p_1^2 + p_2^2 \rangle = 2Z^2\hbar^2/(a_0^2)$ and $\langle 1/r_{12} \rangle = 5Z/(8a_0)$, show that the mean energy as a function of an effective nuclear charge Z is

$$E(Z) = -2E_R (4Z - Z^2 - 5Z/8).$$

Complete the variational procedure, and hence obtain an upper limit for the ground state energy, and a lower limit for the ionization energy of helium.

- (5) **Spectroscopic notation and Helium Wavefunctions.** Two of the energy levels in Helium have the standard notation $1s2s\ ^1S_0$ and $1s2s\ ^3S_1$. Explain every part of this notation. The difference in the notation for the two levels stresses a difference in the spin part of the wavefunction, but the spatial part is also different: what is the important distinguishing feature between the two spatial wavefunctions?
- (6) **Variational method.** Apply the variational method to find an upper limit on the ground state of a particle in the potential

$$V = \begin{cases} mgx & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases}$$

(This is the bouncing ball potential: the particle falls under gravity to $x = 0$ and then bounces back again.)

- (a) What are the boundary conditions for the wavefunction at $x = 0$ and $x \rightarrow \infty$? Show that a suitable wavefunction form is

$$\psi(x) \propto \begin{cases} x \exp(-ax) & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Find the expectation value of the Hamiltonian. (Don't forget to normalize the wavefunction!).

- (b) Complete the variational calculation to find an upper bound for the ground state energy. You should find

$$E_0 \leq 3 \left(\frac{9\hbar^2 g^2 m}{32} \right)^{1/3}.$$