Vector Notation: Vector $\mathbf{r}$, Unit vector $\hat{\mathbf{r}}$; Components r_i, $i = 1, 2, 3$.

Vector Calculus

For scalar field Φ or vector field $\mathbf{A}$:

$$\mathrm{Grad}\,\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r_i} \qquad \text{Gradient — normal to equipotential surfaces}$$

$$\mathrm{Div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_i}{\partial r_i} \qquad \text{Divergence — origin of flux lines}$$

$$\mathrm{Curl}\,\mathbf{A} = \nabla\wedge\mathbf{A} = \epsilon_{ijk}\frac{\partial A_k}{\partial r_j} \qquad \text{Rotation — flux lines form loops}$$

Integral theorems:

$$\int \nabla\cdot\mathbf{A}\, d^3\mathbf{r} = \int \mathbf{A}\cdot d\mathbf{S} \qquad \text{or more generally} \qquad \int \frac{\partial F}{\partial r_i}\, d^3\mathbf{r} = \int F\, dS_i$$

$$\int \nabla\wedge\mathbf{A}\cdot d\mathbf{S} = \int \mathbf{A}\cdot d\mathbf{r}$$

Flux of $\mathbf{A}$ across a surface $= \int \mathbf{A}\cdot d\mathbf{S}$.

Representation with field lines useful if $\nabla\cdot\mathbf{A} = 0$ in most places.

If $\nabla\wedge\mathbf{A} = 0$ then there exists Φ such that $\mathbf{A} = \nabla\,\Phi$. ($\Phi$ not unique: arbitrary constant.)

If $\nabla\cdot\mathbf{B} = 0$ then there exists $\mathbf{A}$ such that $\mathbf{B} = \nabla\wedge\mathbf{A}$. ($\mathbf{A}$ not unique: arbitrary gradient.)

ELECTROMAGNETISM Electrostatics 2

Coulomb's Law

Force on point charge q_1 due to q_2: $\mathbf{F}_{12} \;=\; \dfrac{\alpha}{4\pi}\dfrac{q_1 q_2 \hat{\mathbf{r}}_{12}}{r_{12}^2}$. Linearity. Divisibility.

Introduce electric field concept as intermediary between charges:

Charge $\rightarrow$ Field $\rightarrow$ Force on Charge.

Define *Electric Field* $\mathbf{E}$ by

$$\boxed{\mathbf{F} = q\mathbf{E}}$$

so that the electric field at $\mathbf{r}$ due to a point charge q at $\mathbf{r}_0$ can be written

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{\alpha}{4\pi}\frac{q(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}.$$

Easy to show that $\nabla\cdot\mathbf{E} = 0$ and $\nabla\wedge\mathbf{E} = 0$ (except possibly at $\mathbf{r}_0$). Hence can use field lines, and *Electrostatic Potential* Φ:

$$\boxed{\mathbf{E} = -\nabla\Phi} \qquad \boxed{\Phi = \frac{\alpha q}{4\pi|\mathbf{r}-\mathbf{r}_0|}}$$

Hence work done on this field to move a test charge from a to b

$$\int_a^b -\mathbf{F}\cdot d\mathbf{r} = \int_a^b q\nabla\Phi\cdot d\mathbf{r} = q\left(\Phi(b) - \Phi(a)\right).$$

Hence: $\boxed{\text{Electrostatic Energy } V = q\Phi}$

Gauss' Law

Consider the surface integral $Q = \int \mathbf{E} \cdot d\mathbf{S}$ due to a set of point charges q_i.

By linearity $Q = \sum_i Q_i$ where $Q_i = \int \mathbf{E}_i \cdot d\mathbf{S}$ where $\mathbf{E}_i$ is the field due to q_i.

Since $\nabla \cdot \mathbf{E}_i = 0$ except at the charge, $Q_i = 0$ if q_i is not inside S. If it is inside S, $Q_i = \alpha q_i$:

$$\boxed{\int \mathbf{E} \cdot d\mathbf{S} = \alpha \sum \text{Charge enclosed by } S}$$

Using the divergence theorem, and writing enclosed charge in terms of a charge density $\rho(\mathbf{r})$

$$\int \nabla \cdot \mathbf{E}\, d^3\mathbf{r} = \alpha \int \rho\, d^3\mathbf{r}$$

Units: these are fixed by the arbitrary constant α, for which there are at least three common choices:

System:	Gauss	Heaviside	S.I.
α:	4π	1	$1/\epsilon_0$

Hence:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \alpha\rho}$$

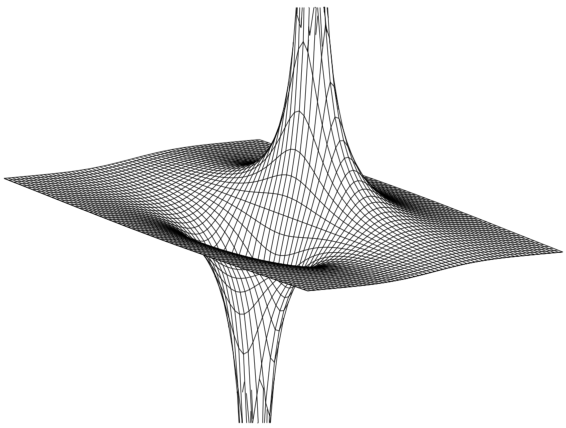
M.1

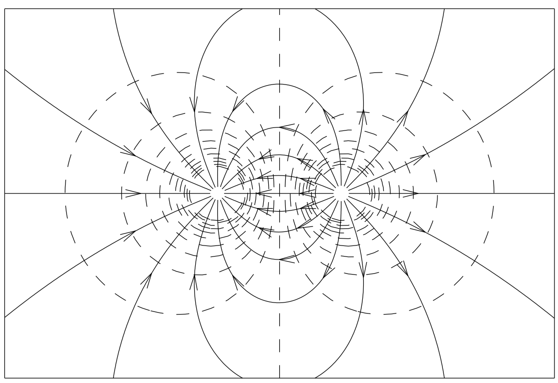
Example of Potential and Field: Two Point Charges

Consider a system of two charges q and $-q$.

$$\Phi = \Phi_1 + \Phi_2 = \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{-q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}$$

$$\mathbf{E} = -\nabla\Phi = -\left(\nabla\Phi_1 + \nabla\Phi_2\right)$$





The point dipole $\mathbf{p}$ is defined by the limit $\mathbf{r}_1 \to \mathbf{r}_2$ and $q(\mathbf{r}_1 - \mathbf{r}_2) \to \mathbf{p}$:

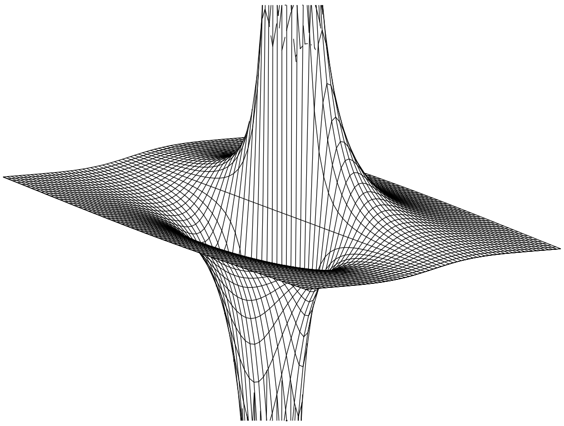
$$\Phi(\mathbf{r}) = \lim_{a\to 0, qa\to p} \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2 - a\mathbf{n}|} - \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|} = \mathbf{p} \cdot \nabla_2 \left(\frac{1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}\right)$$

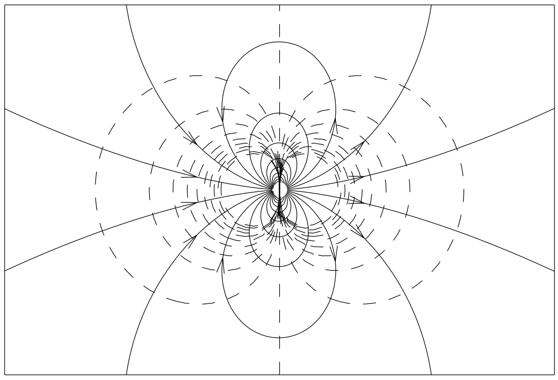
This is minus the gradient with respect to field position.

Point Electric Dipole

For a point electric dipole at the origin:

$$\Phi = -\mathbf{p}\cdot\nabla\left(\frac{1}{4\pi\epsilon_0 r}\right) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} \qquad \mathbf{E} = -\nabla\Phi = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0 r^3}$$





The potential energy of the dipole in an external field is given by

$$V = \lim_{a\to 0} p\left(\frac{\Phi(\mathbf{r}+\mathbf{a}) - \Phi(\mathbf{r})}{a}\right) = \mathbf{p}\cdot\nabla\Phi = -\mathbf{p}\cdot\mathbf{E}.$$

In a uniform field there is no force, but a non-zero couple.

Poisson's Equation

Substitute $\mathbf{E} = -\nabla\Phi$ into the first Maxwell equation:

$$\boxed{\nabla^2\Phi = \frac{-\rho}{\epsilon_0} = -\alpha\rho}$$

For a given charge distribution ρ this determines Φ and hence $\mathbf{E}$. There is a formal solution using a Green function: consider the potential due to the charge in a small volume $d^3\mathbf{r}'$ at $\mathbf{r}'$:

$$d\Phi(\mathbf{r}) = \frac{\rho(\mathbf{r}')d^3\mathbf{r}'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

which we can integrate to give

$$\boxed{\Phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')d^3\mathbf{r}'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.}$$

The function $G(\mathbf{r},\mathbf{r}') = \dfrac{1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$ gives the potential at $\mathbf{r}$ due to a unit point charge at $\mathbf{r}'$, and satisfies

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')/\epsilon_0.$$

Laplace's equation

In a charge-free region the potential satisfies Laplace's equation: $\boxed{\nabla^2\Phi = 0}$

Solutions to this are sometimes called *harmonic functions.* Consider the solution in spherical co-ordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\Phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

If we separate variables $\Phi = R(r)C(\theta,\phi)$, the angular part will be *spherical harmonics.* Convenient to use a real set, and differently normalised to the Y_{lm}:

$$\int C_{l,m}C_{l',m'}\,d\Omega = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}.$$

These are much more memorable: $C_{0,0} = 1$, $C_{1,m} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\} = \hat{r}_m$

This leaves
$$\frac{d}{dr}r^2\frac{dR}{dr} = l(l+1)R.$$

This is *homogeneous* and can be solved by a power of r. Try $R = r^n$:

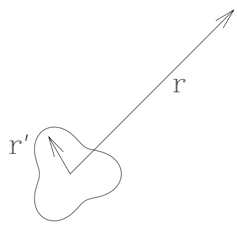
$$n(n+1) = l(l+1) \quad \text{so} \quad n = l \quad \text{or} \quad n = -l-1.$$

The potential due to any charge distribution, exterior to it, can thus be written

$$\Phi(r,\theta,\phi) = \sum_{l,m}\frac{Q_{l,m}}{4\pi\epsilon_0 r^{l+1}}C_{l,m}(\theta,\phi).$$

Multipole Expansion

Consider the potential Φ of a localised charge distribution ρ. We can use the Green function:



$$\Phi(\mathbf{r}) = \int\frac{\rho(\mathbf{r}')\,d^3\mathbf{r}'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

The generating function for Legendre polynomials is given by

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\Theta}} = \sum_l\frac{r_<^l}{r_>^{l+1}}P_l(\cos\Theta) = \sum_{l,m}\frac{r_<^l}{r_>^{l+1}}C_{l,m}(\theta,\phi)C_{l,m}(\theta',\phi')$$

so that in the exterior region ($r_> = r$) Φ becomes

$$\Phi(\mathbf{r}) = \sum_{l,m}\frac{Q_{l,m}}{4\pi\epsilon_0 r^{l+1}}C_{l,m}(\theta,\phi) \qquad \text{where} \qquad Q_{l,m} = \int(r')^l C_{l,m}(\theta',\phi')\rho(\mathbf{r}')\,d^3\mathbf{r}'$$

$$= \frac{q}{4\pi\epsilon_0 r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}\ldots \quad \text{where} \quad q = \int\rho\,d^3\mathbf{r}', \qquad \mathbf{p} = \int\mathbf{r}'\rho\,d^3\mathbf{r}'$$

$Q_{l,m}$ are the *multipole moments* of the charge distribution: monopole, dipole, quadrupole . .

Energy in the Electric field

For a pair of point charges the potential energy, defined as the work to bring them from ∞, is

$$V = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} = q_1\Phi(\mathbf{r}_1) = q_2\Phi(\mathbf{r}_2) = \frac{1}{2}\left(q_1\Phi(\mathbf{r}_1) + q_2\Phi(\mathbf{r}_2)\right).$$

Thus for a system of charges the energy is $V = \sum_{i>j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \sum_{i\neq j} \frac{1}{2}\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \sum_i \frac{1}{2} q_i \Phi(\mathbf{r}_i)$.

If we can subdivide the charges indefinitely, so that there are no point charges of finite magnitude, then we can re-write this as

$$V = \int \frac{1}{2}\rho(\mathbf{r})\Phi(\mathbf{r})\, d^3\mathbf{r} = \int \frac{1}{2}\epsilon_0 \Phi\nabla\cdot\mathbf{E}\, d^3\mathbf{r} = \int \frac{1}{2\alpha}\Phi\nabla\cdot\mathbf{E}\, d^3\mathbf{r}$$

(If there are finite point charges then this differs by the inclusion of the $i = j$ terms and diverges.) We can complete the differential under the integral (or *integrate by parts*):

$$\nabla\cdot(\Phi\mathbf{E}) = \Phi\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla\Phi = \Phi\nabla\cdot\mathbf{E} - E^2.$$

The integral extends over the charge distribution; if we extend it to all space the divergence integrates to a surface term which tends to zero ($\Phi \propto r^{-1}$, $\mathbf{E} \propto r^{-2}$) to leave

$$\boxed{V = \int \frac{1}{2}\epsilon_0 E^2(\mathbf{r})\, d^3\mathbf{r} = \int \frac{1}{2\alpha} E^2(\mathbf{r})\, d^3\mathbf{r}.}$$

This suggests that we can view the potential energy as residing in the field.

Biot-Savart Law

Ampère, Biot and Savart found a form analagous to Coulomb's law for the force between two current elements. We factor the expression into

Current $I_1(\mathbf{r}_1)$ $\rightarrow$ Field $\mathbf{B}(\mathbf{r}_2)$ $\rightarrow$ Force on Current $I_2(\mathbf{r}_2)$:

$$d\mathbf{B}(\mathbf{r}_2) = \frac{\gamma\alpha}{4\pi}\frac{I_1\, d\mathbf{r}_1 \wedge (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

$$d\mathbf{F}_2 = \beta\, I_2\, d\mathbf{r}_2 \wedge \mathbf{B}(\mathbf{r}_2)$$

Integration around the circuit gives the total $\mathbf{B}$.

(For historical reasons $\mathbf{B}$ is more correctly called the magnetic flux density or the magnetic induction; magnetic field is something else.)

Units

These are fixed by the constants β and γ, one of which is arbitrary but which have to satisfy $\beta\gamma c^2 = 1$:

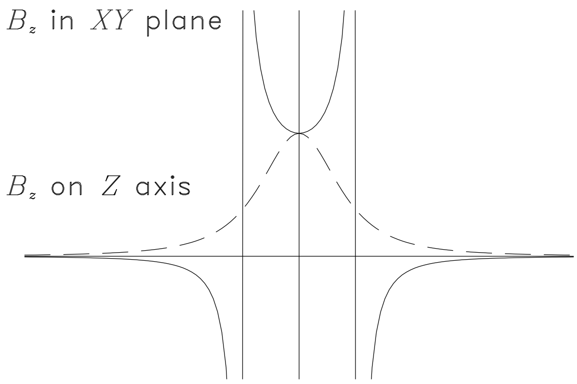
System:	Gauss	Heaviside	S.I.
β:	$1/c$	$1/c$	1
γ:	$1/c$	$1/c$	$\mu_0\epsilon_0$

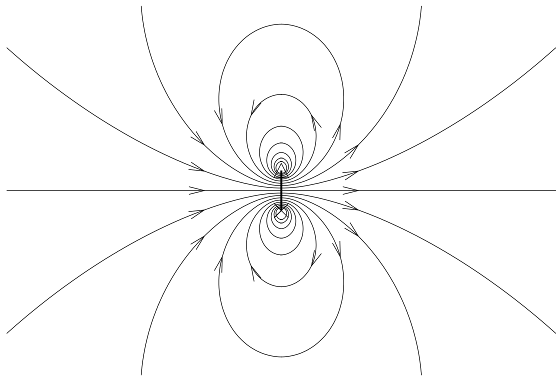
Example of Field: Circular Current Loop

It follows from Biot-Savart that $\nabla \cdot \mathbf{B} = 0$. $\nabla \wedge \mathbf{B}$ is also zero when integrated around the source circuit. (But both expressions diverge at points where there is a line current.) We can thus use a field-line representation. Carry out the integral for a circular current loop:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{r}_1 = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{pmatrix} \qquad d\mathbf{r}_1 = \begin{pmatrix} -a\sin\theta\, d\theta \\ a\cos\theta\, d\theta \\ 0 \end{pmatrix}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{r}_1 \wedge (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$





Lorentz Force

We have the same problems with line currents as point charges — infinite field strength and energy density — so we define a *current density* $\mathbf{J}$ as charge flow/unit time/unit area. In terms of $\mathbf{J}$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}_1) \wedge (\mathbf{r} - \mathbf{r}_1)\, d^3\mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}.$$

Instead of a line element of current we now have a volume element, which experiences a force

$$d\mathbf{F} = \mathbf{J} \wedge \mathbf{B}\, d^3\mathbf{r} = \beta \mathbf{J} \wedge \mathbf{B}\, d^3\mathbf{r}.$$

Putting this together with the electric force on a charge density ρ we obtain a force density $\mathbf{f}$:

$$\boxed{\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \wedge \mathbf{B} = \rho\mathbf{E} + \beta\mathbf{J} \wedge \mathbf{B}}$$

If the charge density consists of a single point charge q moving with velocity $\mathbf{v}$ then $\rho = q\,\delta(\mathbf{r})$ and $\mathbf{J} = q\mathbf{v}\,\delta(\mathbf{r})$ and the integrated force is

$$\boxed{\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) = q\,(\mathbf{E} + \beta\mathbf{v} \wedge \mathbf{B})}$$

Vector Potential

In the current loop example the field lines form loops around the current. This suggests that the curl equation may break down where there is a current, but that the div equation is really true:

$$\boxed{\nabla \cdot \mathbf{B} = 0} \qquad \boxed{\text{M.2}}$$

This implies that we can define a *Vector Potential* $\mathbf{A}$ such that:

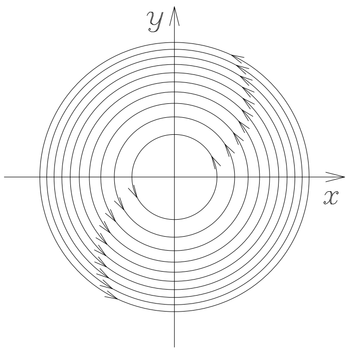
$$\boxed{\mathbf{B} = \nabla \wedge \mathbf{A}}$$

This does not define $\mathbf{A}$ uniquely, and we can add the supplementary condition $\nabla \cdot \mathbf{A} = 0$. We can get an expression for $\mathbf{A}$ from the integral for $\mathbf{B}$:

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \wedge (\mathbf{r} - \mathbf{r}')\, d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \int \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \wedge \mathbf{J}(\mathbf{r}')\, d^3\mathbf{r}' \\ &= \nabla \wedge \left[\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}' \right].\end{aligned}$$

This suggests we can identify the integral with $\mathbf{A}$. (It does satisfy $\nabla \cdot \mathbf{A} = 0$ when $\nabla \cdot \mathbf{J} = 0$.)

Example of Vector Potential: Uniform B Field



Consider a uniform $\mathbf{B}$ field in the z direction. The vector potential $\mathbf{A}$ is required to satisfy:

$$\nabla \wedge \mathbf{A} = B\hat{\mathbf{z}} \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0.$$

One possible solution is $\mathbf{A} = \frac{1}{2}\mathbf{B} \wedge \mathbf{r}$.

The origin appears to be a special place ($\mathbf{A}(0) = 0$), which in a uniform field is surprising. In fact it isn't: we can add obviously add to $\mathbf{A}$ any constant vector $\mathbf{A}_0$.
If we choose $\mathbf{A}_0 = -\frac{1}{2}\mathbf{B} \wedge \mathbf{r}_0$ then

$$\mathbf{A}' = \mathbf{A} + \mathbf{A}_0 = \frac{1}{2}\mathbf{B} \wedge (\mathbf{r} - \mathbf{r}_0).$$

This is an equally good vector potential, with the exactly the same form, but with $\mathbf{A} = 0$ shifted to the arbitrary point $\mathbf{r}_0$.

Ampère's Law

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' = \frac{\gamma\alpha}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}'}$$

The three components of the integral for $\mathbf{A}$ are analogous to the Green function solution for Φ, which suggests that

$$\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J} = -\gamma\alpha\mathbf{J}.}$$

Consider $\nabla\wedge\mathbf{B} = \nabla\wedge(\nabla\wedge\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$.
We know that $\nabla\cdot\mathbf{A} = 0$ if $\nabla\cdot\mathbf{J} = 0$, so

$$\boxed{\nabla\wedge\mathbf{B} = \mu_0\mathbf{J} = \gamma\alpha\mathbf{J}}$$

Integrating this result over a surface we obtain the integral form of this result which is known as Ampère's law:

$$\int \nabla\wedge\mathbf{B}\cdot d\mathbf{S} = \int \mathbf{B}\cdot d\mathbf{r} = \mu_0\int \mathbf{J}\cdot d\mathbf{S}$$

$$\Big(\text{Line integral of } \mathbf{B} \text{ around } S\Big) = \mu_0\Big(\text{Current through } S.\Big)$$

Summary of Electrostatics and Magnetostatics

The $\mathbf{E}$ and $\mathbf{B}$ fields are generated by charges and currents (with $\nabla\cdot\mathbf{J} = 0$):

$$\boxed{\nabla\cdot\mathbf{E} = \rho/\epsilon_0} \qquad \nabla\wedge\mathbf{B} = \mu_0\mathbf{J}$$

and also satisfy

$$\nabla\wedge\mathbf{E} = 0 \qquad \boxed{\nabla\cdot\mathbf{B} = 0.}$$

Hence we can define potentials by

$$\mathbf{E} = -\nabla\Phi \qquad \boxed{\mathbf{B} = \nabla\wedge\mathbf{A}}$$

with supplementary conditions

$$\Phi(\mathbf{r}) \to 0 \text{ as } \mathbf{r}\to\infty \qquad \nabla\cdot\mathbf{A} = 0 \text{ and } \mathbf{A}(\mathbf{r}) \to 0 \text{ as } \mathbf{r}\to\infty.$$

Hence the potentials satisfy

$$\nabla^2\Phi = -\rho/\epsilon_0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

which has the Green function solution

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.$$

Faraday's Law of Induction

Consider a wire loop moving in a $\mathbf{B}$ field. The charges in the wire are all subject to the force $\beta\mathbf{v}\wedge\mathbf{B}$ per unit charge, which creates a non-zero value of the integral around the wire — an e.m.f $\mathcal{E}$:

$$\mathcal{E} = \int \beta\mathbf{v}\wedge\mathbf{B}\cdot d\mathbf{r}.$$

Consider the motion over a time δt in which the wire element $d\mathbf{r}$ moves $\delta\mathbf{r} = \mathbf{v}\delta t$:

$$\mathcal{E} = -\frac{\beta}{\delta t}\int \mathbf{B}\cdot\delta\mathbf{r}\wedge d\mathbf{r}.$$

But $\delta\mathbf{r}\wedge d\mathbf{r}$ is the area swept out by the wire element in δt, so

$$\boxed{\mathcal{E} = -\beta\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = -\beta\frac{d\Psi}{dt}} \quad \text{where} \quad \boxed{\Psi = \int \mathbf{B}\cdot d\mathbf{S}}$$

Ψ is the flux linked to the circuit, and is proportional to the current in whatever circuit is generating the $\mathbf{B}$ field. The coefficient is called an inductance: either self-inductance L or mutual inductance M:

$$\beta\Psi_1 = LI_1 = \int \mathbf{B}_1(\mathbf{r}_1)\cdot d\mathbf{S}_1 = \int \mathbf{A}_1(\mathbf{r}_1)\cdot d\mathbf{r}_1$$

$$\beta\Psi_1 = MI_2 = \int \mathbf{B}_2(\mathbf{r}_1)\cdot d\mathbf{S}_1 = \int \mathbf{A}_2(\mathbf{r}_1)\cdot d\mathbf{r}_1$$

Another Maxwell Equation

But Faraday found that the same law applied whether the flux change was due to motion in a static $\mathbf{B}$ field, as above, or to a time-dependence in $\mathbf{B}$.

In the latter case the e.m.f. cannot be due to the $\mathbf{v}\wedge\mathbf{B}$ force, and has to be interpreted as an electric field caused by the time variation of $\mathbf{B}$.

The integral of this electric field around the circuit has to give the e.m.f:

$$\mathcal{E} = \int \mathbf{E}\cdot d\mathbf{r} = -\beta\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S}$$

where now the time-dependence is due to $\mathbf{B}$ not the motion of the path of integration:

$$\int \nabla\wedge\mathbf{E}\cdot d\mathbf{S} = -\beta\int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

This has to be true for an arbitrary circuit, so we have the third Maxwell equation

$$\boxed{\nabla\wedge\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} = -\beta\frac{\partial\mathbf{B}}{\partial t}} \qquad \boxed{\text{M.3}}$$

Energy in the magnetic field

An alternative derivation of the energy density in the electric field considers a parallel-plate capacitor, modelled as two sheets of charge q and $-q$, area A separation d.
The field, by Gauss's law, is $E = q/(\epsilon_0 A)$, so that the potential difference is $\Delta\Phi = qd/\epsilon_0 A$.
Thus the capacitance is $C = \epsilon_0 A/d$. The stored energy is then

$$V = \frac{1}{2}C\Delta\Phi^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)\left(E^2 d^2\right) = \left(\frac{1}{2}\epsilon_0 E^2\right)(Ad)\,.$$

We can apply a similar technique to energy stored in the $\mathbf{B}$ field by considering the energy stored in an inductance $\frac{1}{2}LI^2 = \frac{1}{2}I\beta\Psi$.

$$V = \frac{\beta}{2}I\int \mathbf{A}\cdot d\mathbf{r} = \frac{\beta}{2}\int \mathbf{A}\cdot\mathbf{J}\,d^3\mathbf{r}$$

But we can substitute $\nabla\wedge\mathbf{B}/\mu_0$ ($\nabla\wedge\mathbf{B}/\gamma\alpha$) for $\mathbf{J}$, and integrate by parts:

$$\nabla\cdot(\mathbf{A}\wedge\mathbf{B}) = (\nabla\wedge\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\wedge\mathbf{B}).$$

The divergence can be turned into a surface integral, which $\rightarrow$ zero: ($\mathbf{A}\propto r^{-2}$, $\mathbf{B}\propto r^{-3}$)

$$V = \int \frac{1}{2\mu_0}B^2 d^3\mathbf{r} = \int \frac{\beta}{2\gamma\alpha}B^2 d^3\mathbf{r}.$$

This suggests that we can view the energy as residing in the $\mathbf{B}$ field.

Conservation of Charge

Earlier we considered only currents where charge did not build up in any place ($\nabla\cdot\mathbf{J} = 0$). In general this condition does not hold, and the current flow leads to changes in ρ.

Consider an arbitrary volume $\mathcal{V}$ in a current flow. The total charge inside $\mathcal{V}$ at time t is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r},t)\,d^3\mathbf{r}$$

The total rate at which charge leaves $\mathcal{V}$ is found by integrating the current density over the bounding surface S:

$$-\frac{dQ}{dt} = \int_{\mathcal{V}} -\frac{\partial\rho}{\partial t}\,d^3\mathbf{r} = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_{\mathcal{V}} \nabla\cdot\mathbf{J}\,d^3\mathbf{r}.$$

This must hold for an arbitrary volume, so

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0}$$

This ensures the local conservation of charge, and is the prototype for other local conservation laws, which have a density term, a flux term, and in general a loss term as well if the quantity is not absolutely conserved but can transform into something else.

Maxwell's Displacement Current

The only field equation we have so far involving $\mathbf{J}$ is $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} = \gamma\alpha\mathbf{J}$.
Taking the divergence of this gives $\nabla \cdot \mathbf{J} = 0$, which was the constraint under which we derived it. This is not true in general. Maxwell's solution to this problem involved the introduction of another term to link it to M.1 where ρ appears as a source:

$$\nabla \wedge \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \qquad \nabla \wedge \mathbf{B} - \gamma \frac{\partial \mathbf{E}}{\partial t} = \gamma\alpha\mathbf{J}.$$

This completes the set of four Maxwell equations:

$$\boxed{\begin{array}{lll}
\nabla \cdot \mathbf{E} = \rho/\epsilon_0 & \nabla \cdot \mathbf{E} = \alpha\rho & \text{M.1} \\
\nabla \cdot \mathbf{B} = 0 & \nabla \cdot \mathbf{B} = 0 & \text{M.2} \\
\nabla \wedge \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 & \nabla \wedge \mathbf{E} + \beta \dfrac{\partial \mathbf{B}}{\partial t} = 0 & \text{M.3} \\
\nabla \wedge \mathbf{B} - \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} & \nabla \wedge \mathbf{B} - \gamma \dfrac{\partial \mathbf{E}}{\partial t} = \gamma\alpha\mathbf{J} & \text{M.4}
\end{array}}$$

Vector and Scalar Potentials

Since $\mathbf{E}$ no longer has vanishing curl in general, we cannot represent it as $-\nabla\Phi$;
however the representation $\mathbf{B} = \nabla \wedge \mathbf{A}$ is still valid, from $\nabla \cdot \mathbf{B} = 0$.

$$\nabla \wedge \mathbf{E} = -\beta \frac{\partial}{\partial t} (\nabla \wedge \mathbf{A}) = -\beta \nabla \wedge \left(\frac{\partial \mathbf{A}}{\partial t}\right) \quad \rightarrow \quad \nabla \wedge \left(\mathbf{E} + \beta \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

This implies that we can always find $\mathbf{A}$ and Φ such that:

$$\boxed{\begin{array}{l} \mathbf{B} = \nabla \wedge \mathbf{A} \\ \mathbf{E} = -\nabla\Phi - \dfrac{\partial \mathbf{A}}{\partial t} \end{array}} \qquad \boxed{\begin{array}{l} \mathbf{B} = \nabla \wedge \mathbf{A} \\ \mathbf{E} = -\nabla\Phi - \beta \dfrac{\partial \mathbf{A}}{\partial t} \end{array}}$$

Writing the $\mathbf{E}$ and $\mathbf{B}$ fields in this form ensures that the homogeneous Maxwell equations (the ones with no source terms: M.2 and M.3 above) are identically satisfied.

Gauge Transformations

A and Φ are not uniquely defined by the requirement that the above differentials give the **E** and **B** fields. Consider the following transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \qquad \Phi' = \Phi - \beta\,\frac{\partial\chi}{\partial t}.$$

The curl of $\nabla\chi$ vanishes, so the **B** field is identical, and the two two extra terms in **E** cancel, so the **E** field is also identical.
This is an example of a *gauge transformation,* and indicates that the potentials as defined have an unphysical degree of freedom.
(Another example of a useful function with an unphysical degree of freedom is the Q.M. wave function, for which the overall phase is arbitrary.)

We are free to specify an extra relation between **A** and Φ; this is called choosing a gauge. One useful choice (of several) is the Lorentz gauge:

$$\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = 0 \qquad \nabla\cdot\mathbf{A} + \gamma\,\frac{\partial\Phi}{\partial t} = 0.$$

We can always choose χ to bring **A** and Φ to this form, and even this χ is not unique, so that there are still *restricted gauge transformations* within the Lorentz gauge.

Poynting's Theorem

We can find a local conservation law for energy in the electromagnetic field.
Take $\mathbf{B}\cdot$ (M3) and subtract $\mathbf{E}\cdot$ (M4):

$$\Big(\mathbf{B}\cdot(\nabla\wedge\mathbf{E}) - \mathbf{E}\cdot(\nabla\wedge\mathbf{B})\Big) + \left(\beta\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} + \gamma\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\right) + \gamma\alpha\mathbf{E}\cdot\mathbf{J} = 0$$

We can combine the terms in the first bracket as $\nabla\cdot(\mathbf{E}\wedge\mathbf{B})$.
We can rewrite the partial derivatives using $\mathbf{E}\cdot\dfrac{\partial\mathbf{E}}{\partial t} = \dfrac{1}{2}\dfrac{\partial E^2}{\partial t}$ and similarly for the **B** terms.
Thus we have a conservation law with a loss term:

$$\nabla\cdot\mathbf{S} + \frac{\partial u}{\partial t} + \mathbf{E}\cdot\mathbf{J} = 0$$

where:

$\mathbf{S} = \dfrac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B} = \dfrac{1}{\gamma\alpha}\mathbf{E}\wedge\mathbf{B}$ is Poynting's vector and represents the energy flux in the field;
$u = \dfrac{\epsilon_0}{2}E^2 + \dfrac{1}{2\mu_0}B^2 = \dfrac{1}{2\alpha}E^2 + \dfrac{\beta}{2\gamma\alpha}B^2$ is the energy density in the field, and is the sum of the magnetic and electric field energies already found;
$\mathbf{E}\cdot\mathbf{J}$ represents the rate per unit volume at which the EM field does work on the current — a loss term from the EM field energy.

Wave Equations

In the Maxwell equations we can substitute the potentials that satisfy the homogeneous pair:

$$-\nabla^2\Phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho/\epsilon_0 \qquad -\nabla^2\Phi - \beta\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \alpha\rho$$

$$\nabla\wedge(\nabla\wedge\mathbf{A}) + \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} + \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\Phi = \mu_0\mathbf{J} \qquad \nabla\wedge(\nabla\wedge\mathbf{A}) + \beta\gamma\frac{\partial^2\mathbf{A}}{\partial t^2} + \gamma\frac{\partial}{\partial t}\nabla\Phi = \gamma\alpha\mathbf{J}.$$

Using the Lorentz gauge these uncouple to give

$$\mu_0\epsilon_0\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \rho/\epsilon_0 \qquad \beta\gamma\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \alpha\rho$$

$$\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \qquad \beta\gamma\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \gamma\alpha\mathbf{J}.$$

This is the wave equation, with wave speed $1/\sqrt{\mu_0\epsilon_0} = 1/\sqrt{\gamma\beta}$, which we identify as c. Similarly we can obtain wave equations for $\mathbf{E}$ and $\mathbf{B}$. We take the curl of one of M.3 and M.4, and combine with the t-derivative of the other. In a source-free region:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = 0$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = 0.$$

Duality

Maxwell's Equations have a fairly obvious $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$ symmetry. This is more explicit if we postulate the existence of magnetic charge density ρ_m and current density $\mathbf{J}_m$:

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0 \qquad \frac{\partial\mathbf{E}}{\partial t} - c^2\nabla\wedge\mathbf{B} = -\mathbf{J}/\epsilon_0$$

$$\nabla\cdot\mathbf{B} = \mu_0\rho_m \qquad \frac{\partial\mathbf{B}}{\partial t} + \nabla\wedge\mathbf{E} = -\mu_0\mathbf{J}_m$$

We define the duality transformation $\mathcal{D}$ by

$$\mathcal{D}\mathbf{E} = c\mathbf{B} \qquad \mathcal{D}\rho = \rho_m/c \qquad \mathcal{D}\mathbf{J} = \mathbf{J}_m/c$$

$$\mathcal{D}\mathbf{B} = -\mathbf{E}/c \qquad \mathcal{D}\rho_m = -c\rho \qquad \mathcal{D}\mathbf{J}_m = -c\mathbf{J}$$

The equations are invariant under this transformation. But in practice ρ_m and $\mathbf{J}_m$ are zero. So dual solutions exist provided we can find sources for them. For example

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}(3\mathbf{p}\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}} - \mathbf{p}) \quad\longrightarrow\quad \mathbf{B} = \frac{\mu_0}{4\pi r^3}(3\mathbf{m}\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}} - \mathbf{m}).$$

Note that $\mathcal{D}^2 = -1$; corresponding to reversing the definition of positive/negative charge.

Plane Waves I

We shall look for solutions to Maxwell's equations in free space in the form of plane waves:

each component of $\mathbf{E}, \mathbf{B} \propto \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi) \propto \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi) + \exp -i(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$

Complex notation: we shall use just the negative frequency component of the wave (sometimes called the *analytic signal)*, writing this as $\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$; the physical field is found by *adding the complex conjugate* rather than taking the real part (a factor of 2 difference).

$$\mathbf{E} = \boldsymbol{\mathcal{E}} \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t) + \text{ c.c.} \qquad \mathbf{B} = \boldsymbol{\mathcal{B}} \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t) + \text{ c.c.}$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are complex vectors, incorporating the phase ϕ.

When we substitute these into Maxwell's equations $\partial/\partial t \to -i\omega$ and $\nabla \to i\mathbf{k}$:

$$\begin{array}{ll} \mathbf{k}\cdot\boldsymbol{\mathcal{E}} = 0 & \mathbf{k}\cdot\boldsymbol{\mathcal{B}} = 0 \\ \mathbf{k}\wedge\boldsymbol{\mathcal{E}} = \omega\boldsymbol{\mathcal{B}} & \mathbf{k}\wedge\boldsymbol{\mathcal{B}} = -\dfrac{\omega}{c^2}\boldsymbol{\mathcal{E}} \end{array}$$

The scalar product equations follow from the vector product equations, and tell us that the waves are *transverse*: $\mathbf{E}$ and $\mathbf{B}$ lie in the plane perpendicular to $\mathbf{k}$.

Eliminating either $\boldsymbol{\mathcal{E}}$ or $\boldsymbol{\mathcal{B}}$ from the cross product equations gives:

$$k^2 = \omega^2/c^2$$

so the waves have phase velocity c.

Plane Waves II

We then write $\mathbf{k} = \hat{\mathbf{k}}(\omega/c)$:

$$\hat{\mathbf{k}}\wedge\boldsymbol{\mathcal{E}} = c\boldsymbol{\mathcal{B}} \qquad c\hat{\mathbf{k}}\wedge\boldsymbol{\mathcal{B}} = -\boldsymbol{\mathcal{E}}$$

These determine the relative magnitude of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ and their relative orientations and phase. We separate these factors by defining the magnitudes $\mathcal{E}$ and $\mathcal{B}$ ($\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^\star = \mathcal{E}^2$) and unit complex polarization vectors $\mathbf{e}$ and $\mathbf{b}$ ($\boldsymbol{\mathcal{E}} = \mathcal{E}\mathbf{e}$, hence $\mathbf{e}\cdot\mathbf{e}^* = 1$).

Dot either equation with its complex conjugate; *e.g.*

$$\mathcal{E}^2\left(\hat{\mathbf{k}}\wedge\mathbf{e}\right)\cdot\left(\hat{\mathbf{k}}\wedge\mathbf{e}^\star\right) = \mathcal{E}^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}\,\mathbf{e}\cdot\mathbf{e}^\star - |\hat{\mathbf{k}}\cdot\mathbf{e}|^2\right) = \mathcal{E}^2 = c^2\mathcal{B}^2$$

Thus $\mathcal{E} = c\mathcal{B}$. Dividing this factor out we get $\hat{\mathbf{k}}\wedge\mathbf{e} = \mathbf{b}$ and $\hat{\mathbf{k}}\wedge\mathbf{b} = -\mathbf{e}$.

Summary

$$\mathbf{e}\cdot\mathbf{e}^\star = \mathbf{b}\cdot\mathbf{b}^\star = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
$$\mathbf{e}\cdot\hat{\mathbf{k}} = \mathbf{b}\cdot\hat{\mathbf{k}} = \mathbf{e}\cdot\mathbf{b} = 0$$
$$\hat{\mathbf{k}}\wedge\mathbf{e} = \mathbf{b} \qquad \hat{\mathbf{k}}\wedge\mathbf{b} = -\mathbf{e}$$
$$\mathbf{e}\wedge\mathbf{b}^\star = \mathbf{e}^\star\wedge\mathbf{b} = \hat{\mathbf{k}}$$

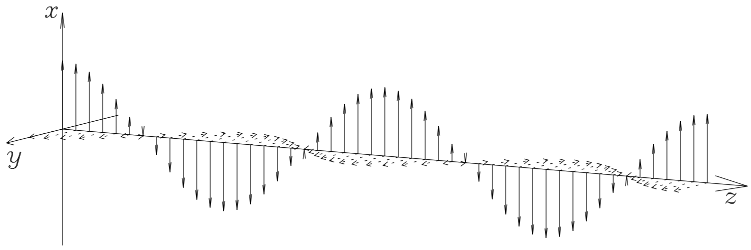
Linear Polarization

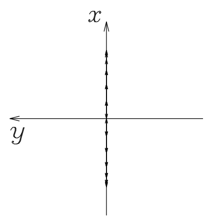
The simplest solution to these equations is $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ and either $\mathbf{e} = \hat{\mathbf{x}}$, $\mathbf{b} = \hat{\mathbf{y}}$ or $\mathbf{e} = \hat{\mathbf{y}}$, $\mathbf{b} = -\hat{\mathbf{x}}$. These are *linearly polarized.* The first of these cases is

$\mathbf{E} = \hat{\mathbf{x}}\,\mathcal{E}\,\exp(kz - \omega t) +$ c.c. Explicitly

$$\mathbf{E} = \hat{\mathbf{x}}\,2\mathcal{E}\,\cos(kz - \omega t) \qquad \mathbf{B} = \hat{\mathbf{y}}\,\frac{2\mathcal{E}}{c}\,\cos(kz - \omega t)$$

The second case is dual to the first.



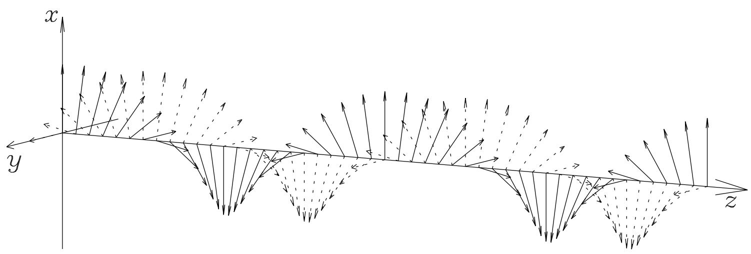


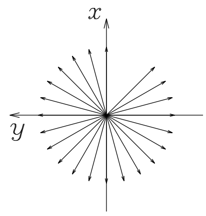
Left Circular Polarization

Superposing two equal orthogonal linearly polarized solutions with a $\pi/2$ phase difference gives $\mathbf{e} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$, $\mathbf{b} = (\hat{\mathbf{y}} - i\hat{\mathbf{x}})/\sqrt{2} = -i\mathbf{e}$. The fields are given by

$$\mathbf{E} = \sqrt{2}\mathcal{E}\,\left(\hat{\mathbf{x}}\cos(kz - \omega t) - \hat{\mathbf{y}}\sin(kz - \omega t)\right)$$
$$\mathbf{B} = \frac{\sqrt{2}\mathcal{E}}{c}\left(\hat{\mathbf{y}}\cos(kz - \omega t) + \hat{\mathbf{x}}\sin(kz - \omega t)\right)$$

This describes a *circularly polarized* wave. In each plane $\mathbf{E}$ and $\mathbf{B}$ rotate anti-clockwise looking at the source: left circularly polarized. Note that the dual solution differs only in phase.





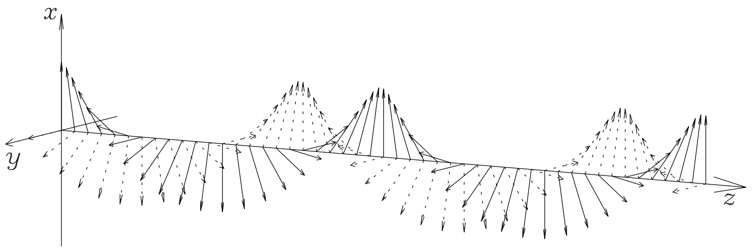
Right Circular Polarization

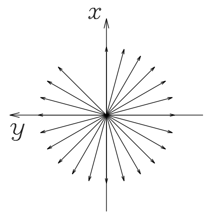
Alternatively we can superpose with the opposite phase difference:
$\mathbf{e} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})/\sqrt{2}$, $\mathbf{b} = (\hat{\mathbf{y}} + i\hat{\mathbf{x}})/\sqrt{2} = i\mathbf{e}$. The fields are given by

$$\mathbf{E} = \sqrt{2}\mathcal{E}\ (\hat{\mathbf{x}}\cos(kz - \omega t) + \hat{\mathbf{y}}\sin(kz - \omega t))$$

$$\mathbf{B} = \frac{\sqrt{2}\mathcal{E}}{c}\,(\hat{\mathbf{y}}\cos(kz - \omega t) - \hat{\mathbf{x}}\sin(kz - \omega t))$$

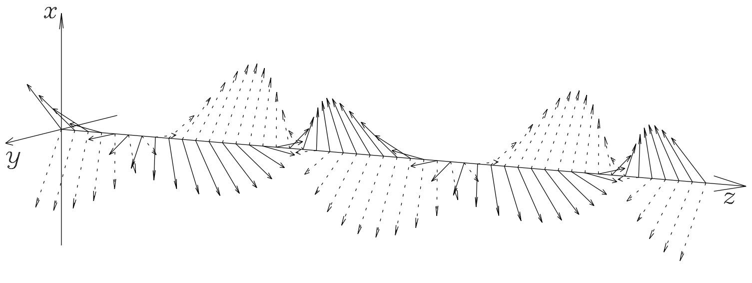
In each plane the vectors rotate clockwise looking at the source: right circularly polarized.

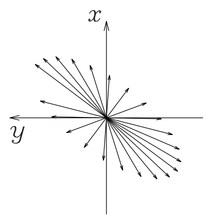




Elliptical Polarization

A general superposition of linearly polarized states with arbitrary amplitudes and phases results in elliptical polarization. This plot was generated with $\mathbf{e} = [(36 + i20)\hat{\mathbf{x}} + (48 - i15)\hat{\mathbf{y}}]/65$
The $\mathbf{E}$ vector rotates clockwise, but traces out an ellipse.





Transport Properties

We shall calculate the energy density and energy flux in the plane waves above. These are position- and time-dependent, but we shall consider the time-average, which is also independent of position. This is because the waves are plane — time averaging is equivalent to phase averaging, and the only $\mathbf{r}$-dependence is in the phase.

Consider $\mathbf{E}\cdot\mathbf{E} = (\boldsymbol{\mathcal{E}}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t) + \text{ c.c.})^2 = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^\star + \boldsymbol{\mathcal{E}}^\star\cdot\boldsymbol{\mathcal{E}} +$ time-dependent terms.

Clearly in this notation any quadratic product involving fields $\mathbf{F}$ and $\mathbf{G}$ can be written as the sum of the products of $\boldsymbol{\mathcal{F}}$ with $\boldsymbol{\mathcal{G}}^\star$ and $\boldsymbol{\mathcal{F}}^\star$ with $\boldsymbol{\mathcal{G}}$.

Energy Density: $\bar{u} = \frac{1}{2}\epsilon_0(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^\star + \boldsymbol{\mathcal{E}}^\star\cdot\boldsymbol{\mathcal{E}}) + \frac{1}{2}(\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{B}}^\star + \boldsymbol{\mathcal{B}}^\star\cdot\boldsymbol{\mathcal{B}})/\mu_0 = \epsilon_0\mathcal{E}^2 + \mathcal{B}^2/\mu_0 = 2\epsilon_0\mathcal{E}^2$.
The energy density has equal contributions from the $\mathbf{E}$ and $\mathbf{B}$ fields.

Poynting's Vector: $\bar{\mathbf{S}} = (\boldsymbol{\mathcal{E}}\wedge\boldsymbol{\mathcal{B}}^\star + \boldsymbol{\mathcal{E}}^\star\wedge\boldsymbol{\mathcal{B}})/\mu_0 = (\mathcal{E}\mathcal{B}/\mu_0)(\mathbf{e}\wedge\mathbf{b}^* + \mathbf{e}^*\wedge\mathbf{b}) = \bar{u}c\,\hat{\mathbf{k}}$.
The energy propagates with velocity c in the wave direction.

These properties are independent of polarization — but angular momentum transport depends on the degree of circular polarization . . .

Degree of Circular Polarization

There are some products of $\mathbf{e}$, $\mathbf{b}$ and $\hat{\mathbf{k}}$ not defined above, including $\mathbf{e}\cdot\mathbf{e}$, $\mathbf{e}\cdot\mathbf{b}^\star$, $\mathbf{e}\wedge\mathbf{e}^\star$.
These are all related, and allow us to define a degree of circular polarization.
Consider $\mathbf{e}\wedge\mathbf{e}^\star$: the vector is pure imaginary, since it is equal to minus its complex conjugate.
Both vectors lie in the plane perpendicular to $\hat{\mathbf{k}}$, so the vector product lies along $\hat{\mathbf{k}}$.
The magnitude is less than unity:

$$(\mathbf{e}\wedge\mathbf{e}^\star)\cdot(\mathbf{e}^\star\wedge\mathbf{e}) = 1 - \mathbf{e}\cdot\mathbf{e}\,\mathbf{e}^\star\cdot\mathbf{e}^\star = 1 - |\mathbf{e}\cdot\mathbf{e}|^2.$$

We can therefore write $\mathbf{e}\wedge\mathbf{e}^\star = -i\eta\,\hat{\mathbf{k}}$ where η is real and $-1\le\eta\le 1$.
The other products are then related:

$$\mathbf{e}\cdot\mathbf{b}^\star = \mathbf{e}\cdot(\hat{\mathbf{k}}\wedge\mathbf{e}^\star) = \hat{\mathbf{k}}\cdot(\mathbf{e}^\star\wedge\mathbf{e}) = i\eta$$

$$\mathbf{b}\wedge\mathbf{b}^\star = (\hat{\mathbf{k}}\wedge\mathbf{e})\wedge\mathbf{b}^\star = \mathbf{e}(\hat{\mathbf{k}}\cdot\mathbf{b}^\star) - \hat{\mathbf{k}}(\mathbf{e}\cdot\mathbf{b}^\star) = -i\eta\,\hat{\mathbf{k}}$$

For the cases considered above:

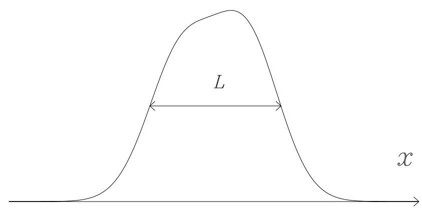
Linearly Polarized: $\eta = 0$;
LH Circularly Polarized: $\eta = i\mathbf{e}^\star\cdot\mathbf{b} = \frac{i}{2}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})\cdot(-i\hat{\mathbf{x}} + \hat{\mathbf{y}}) = 1$;
RH Circularly Polarized: $\eta = -1$;
Arbitrary Elliptical Polarization above: $\eta = -120/169$.

We define η to be the degree of circular polarization.

Finite Plane Wave

Consider a coherent beam of radiation in the z direction

$$\mathbf{E} = \boldsymbol{\mathcal{E}}(\mathbf{r})\, e^{i(kz-\omega t)} \;+\; \text{c.c.} \qquad \mathbf{B} = \boldsymbol{\mathcal{B}}(\mathbf{r})\, e^{i(kz-\omega t)} \;+\; \text{c.c.}$$



where the complex vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ vary in the XY plane with length scale $L \gg \lambda$. The wave equation shows that the variation in the z direction is even slower:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\boldsymbol{\mathcal{E}} + \left(2ik\frac{\partial \boldsymbol{\mathcal{E}}}{\partial z} + \nabla^2 \boldsymbol{\mathcal{E}}\right) = 0.$$

The second term is $\sim (1/kL)^2$ w.r.t the first. So $\boldsymbol{\mathcal{E}} \approx \boldsymbol{\mathcal{E}}^{(0)}(x, y)$ satisfies the wave eq'n to 1st order in a region $\sim L$ in all directions. However, the error in M1 and M2 is 1st order:

$$\nabla \cdot \mathbf{E} = \left(i\mathbf{k} \cdot \boldsymbol{\mathcal{E}}^{(0)} + \nabla \cdot \boldsymbol{\mathcal{E}}^{(0)}\right) \exp i(kz - \omega t) \;+\; \text{c.c.}$$

To correct it we add small longitudinal terms: $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(1)}$, $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}^{(0)} + \boldsymbol{\mathcal{B}}^{(1)}$ where

$$\boldsymbol{\mathcal{E}}^{(1)} = \frac{i}{k}\hat{\mathbf{z}}\, \nabla \cdot \boldsymbol{\mathcal{E}}^{(0)} \qquad \text{and similarly} \qquad \boldsymbol{\mathcal{B}}^{(1)} = \frac{i}{k}\hat{\mathbf{z}}\, \nabla \cdot \boldsymbol{\mathcal{B}}^{(0)}.$$

This also corrects the first-order error in M3 & M4 if $\hat{\mathbf{k}} \wedge \boldsymbol{\mathcal{E}} = c\boldsymbol{\mathcal{B}}$ and $c\hat{\mathbf{k}} \wedge \boldsymbol{\mathcal{B}} = -\boldsymbol{\mathcal{E}}$. These $\sim 1/kL$ terms in $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ lead to additional terms in the transport properties.

Solutions to the Wave Equations

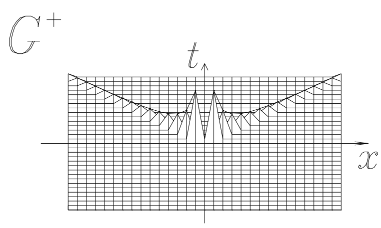
The inhomogeneous wave equation (or wave equation with sources) has a Green function solution similar to that for Poisson's equation.

Consider the wave equation for Φ:

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{\rho(\mathbf{r}, t)}{\epsilon_0}$$

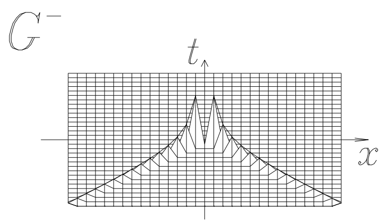
The Green function will give the solution for a point charge at $\mathbf{r}'$ which only exists for an instant at t'; the full solution is then obtained by integrating over $\mathbf{r}'$ and t'.

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \frac{\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')}{\epsilon_0}$$



This has *two* solutions which tend to zero as $\mathbf{r} \to \infty$:

$$G^+ = \frac{\delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|} \qquad G^- = \frac{\delta\left(t' - t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$$



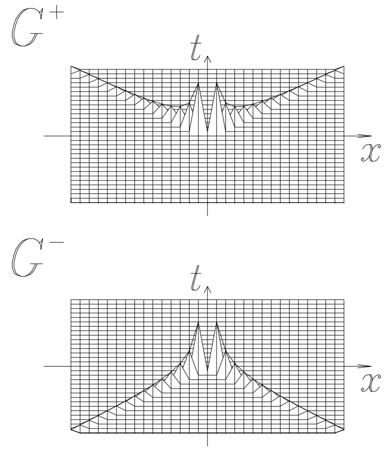
These represent outgoing (G^+) and incoming (G^-) spherical pulses diverging from or converging to $\mathbf{r}'$, t'.

Advanced and Retarded Potentials

These solutions represent different physical situations:

G^+ represents a source distribution generating Φ and radiating it to future times;

G^- represents an incoming Φ precisely chosen to be absorbed by the 'source' distribution.



Choosing the *retarded potential* G^+ and carrying out the t' integral we find:

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\mathbf{r}', t')]_{t'=t-|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}(\mathbf{r}', t')]_{t'=t-|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}'.$$

t' is known as the retarded time.

These reduce to the earlier results when ρ or $\mathbf{J}$ are time-independent.

Radiation by sources

We shall consider a localised distribution of charge density ρ and current density $\mathbf{J}$ which oscillates with angular frequency ω:

$$\mathbf{J}(\mathbf{r}, t) = \boldsymbol{\mathcal{J}}(\mathbf{r}) e^{-i\omega t} + \text{c.c.} \qquad \rho(\mathbf{r}, t) = \tilde{\rho}(\mathbf{r}) e^{-i\omega t} + \text{c.c.}$$

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J} \quad \text{hence} \quad i\omega\tilde{\rho}(\mathbf{r}) = \nabla \cdot \boldsymbol{\mathcal{J}}.$$

The potentials $\boldsymbol{\mathcal{A}} e^{-i\omega t}$ and $\varphi e^{-i\omega t}$ are then given by the retarded Green function:

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \int \frac{\boldsymbol{\mathcal{J}}(\mathbf{r}') e^{ikR}}{4\pi\epsilon_0 c^2 R} \, d^3\mathbf{r}' \qquad \text{where} \quad R = |\mathbf{r}-\mathbf{r}'|, \quad k = \frac{\omega}{c}$$

The scalar potential is related to $\boldsymbol{\mathcal{A}}$ by the Lorentz gauge condition $\dfrac{i\omega}{c^2}\varphi = \nabla \cdot \boldsymbol{\mathcal{A}}$.

There are two length scales in this problem: the source size a and the wavelength $2\pi/k$. We shall consider sources much smaller than a wavelength, $ka \ll 1$, and find the fields at large distances, $a \ll r$. (But note: can still have $kr \ll 1$ or $kr \gg 1$.) Thus $R \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$. We can then expand in a power series in r', retaining only the first order corrections:

$$\frac{e^{ikR}}{R} = \frac{e^{ikr}}{r} \frac{r e^{ik(R-r)}}{R} \approx \frac{e^{ikr}}{r} \left(1 + \frac{\mathbf{r} \cdot \mathbf{r}'(1-ikr)}{r^2} \right)$$

Electric Dipole Radiation I

Retaining just the lowest term gives the *electric dipole approximation:*

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \frac{e^{ikr}}{4\pi\epsilon_0 c^2 r} \int \boldsymbol{\mathcal{J}}(\mathbf{r}')\, d^3\mathbf{r}'.$$

The volume-integrated current represents nett charge flow in the source, and is in fact the rate of change of the electric dipole moment — hence the name.
Consider the tensor $\mathcal{J}_i r'_j$ and take the divergence with the i index:

$$\frac{\partial \mathcal{J}_i r'_j}{\partial r'_i} = r'_j \nabla \cdot \boldsymbol{\mathcal{J}} + \mathcal{J}_i \delta_{ij} = \mathbf{r}'\, \nabla \cdot \boldsymbol{\mathcal{J}} + \boldsymbol{\mathcal{J}}.$$

The integrand in $\boldsymbol{\mathcal{A}}$ is thus part of a divergence, which integrates to a surface integral over a surface outside the source. Thus the divergence integrates to zero, leaving

$$\int \boldsymbol{\mathcal{J}}(\mathbf{r}')\, d^3\mathbf{r}' = -\int \mathbf{r}' \nabla \cdot \boldsymbol{\mathcal{J}}(\mathbf{r}')\, d^3\mathbf{r}' = -i\omega \int \mathbf{r}'\,\tilde{\rho}\, d^3\mathbf{r}' = -i\omega \boldsymbol{\mathcal{P}}.$$

where the electric dipole of the source is given by $\mathbf{p} = \boldsymbol{\mathcal{P}} e^{-i\omega t}$+c.c.

Electric Dipole Radiation II

Thus the electric dipole potentials are given by

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \frac{-i\omega \boldsymbol{\mathcal{P}}\, e^{ikr}}{4\pi\epsilon_0 c^2 r} \qquad \varphi = \frac{-ic^2}{\omega} \nabla \cdot \boldsymbol{\mathcal{A}} = \frac{\boldsymbol{\mathcal{P}} \cdot \mathbf{r}\, e^{ikr}}{4\pi\epsilon_0 r^3} \Big(1 - ikr\Big)$$

The fields are quite complicated.

In the near zone $kr \ll 1$ the dominant term in the potential is $\varphi \approx \boldsymbol{\mathcal{P}} \cdot \mathbf{r}/4\pi\epsilon_0 r^3$, which is a dipole potential, and gives a typical dipole $\mathbf{E}$ field.

In the far zone $kr \gg 1$ the fields fall off as $1/r$, and are orthogonal to each other and to the radius vector.

$$\boldsymbol{\mathcal{E}} = \frac{k^2(\boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{r}}\, \hat{\mathbf{r}})\, e^{ikr}}{4\pi\epsilon_0 r} + \frac{3\boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{r}}\, \hat{\mathbf{r}} - \boldsymbol{\mathcal{P}}}{4\pi\epsilon_0 r^3} (1 - ikr)\, e^{ikr}$$

$$\boldsymbol{\mathcal{B}} = \frac{k^2(\hat{\mathbf{r}} \wedge \boldsymbol{\mathcal{P}})\, e^{ikr}}{4\pi\epsilon_0 c r} \left(1 - \frac{1}{ikr}\right).$$

Magnetic Dipole Radiation I

Some sources have a symmetry such that the electric dipole is always zero.
We then need the first-order term neglected above:

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \frac{\mu_0(1-ikr)e^{ikr}}{4\pi r^3}\int \mathbf{r}\cdot\mathbf{r}'\boldsymbol{\mathcal{J}}(\mathbf{r}')\,d^3\mathbf{r}'.$$

The integrand then involves the tensor $\mathcal{J}_i r'_j$. The symmetric and antisymmetric parts of this are best treated separately:

$$\mathcal{J}_i r'_j = \frac{1}{2}\left(J_i r'_j + J_j r'_i\right) + \frac{1}{2}\left(J_i r'_j - J_j r'_i\right)$$

The symmetric term gives the *electric quadrupole radiation field*, but we shall look only at the antisymmetric *magnetic dipole* term.

The magnetic dipole is defined by $\mathbf{m} = \boldsymbol{\mathcal{M}}e^{-i\omega t}+$ c.c., where $\boldsymbol{\mathcal{M}} = \int \frac{1}{2}\mathbf{r}'\wedge\boldsymbol{\mathcal{J}}(\mathbf{r}')\,d^3\mathbf{r}'$.

The integrand in $\boldsymbol{\mathcal{A}}$ involves

$$\frac{1}{2}\Big(\mathbf{r}\cdot\mathbf{r}'\,\boldsymbol{\mathcal{J}}(\mathbf{r}') - \mathbf{r}\cdot\boldsymbol{\mathcal{J}}(\mathbf{r}')\,\mathbf{r}'\Big) = \left(\frac{1}{2}\mathbf{r}'\wedge\boldsymbol{\mathcal{J}}(\mathbf{r}')\right)\wedge\mathbf{r}$$

which integrates to $\boldsymbol{\mathcal{M}}\wedge\mathbf{r}$.

Magnetic Dipole Radiation II

Thus the magnetic dipole potentials are given by

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{\mathcal{M}}\wedge\mathbf{r}\,e^{ikr}}{r^3}(1-ikr) \qquad \varphi(\mathbf{r}) = 0$$

The $\boldsymbol{\mathcal{M}}\wedge\mathbf{r}/r^3$ factor is a typical magnetic dipole vector potential, so that the near-zone fields are those of a magnetic dipole.

In the far zone we have

$$\boldsymbol{\mathcal{B}} = \frac{k^2\mu_0\,\hat{\mathbf{r}}\wedge(\boldsymbol{\mathcal{M}}\wedge\hat{\mathbf{r}})\,e^{ikr}}{4\pi r} \qquad \boldsymbol{\mathcal{E}} = \frac{k^2\mu_0 c\,(\boldsymbol{\mathcal{M}}\wedge\hat{\mathbf{r}})\,e^{ikr}}{4\pi r}$$

This is the dual of the electric dipole solution, with $\mathbf{E}\to c\mathbf{B}$, $\mathbf{B}\to -\mathbf{E}/c$ and $\boldsymbol{\mathcal{P}}/\epsilon_0 \to \mu_0 c\boldsymbol{\mathcal{M}}$.

Liénard-Wiechert Potentials

For a point charge q in arbitrary motion we can obtain an explicit form for the potentials. The result is not quite what you might expect . . .

At time t the charge is at $\mathbf{r}_q(t)$; thus we have

$$\rho(\mathbf{r}', t') = q\delta(\mathbf{r}' - \mathbf{r}_q(t')) \qquad \text{and} \qquad \mathbf{J}(\mathbf{r}', t') = q\mathbf{v}\delta(\mathbf{r}' - \mathbf{r}_q(t')).$$

Putting ρ into the integral with G^+ we obtain

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{r}' - \mathbf{r}_q(t'))\, \delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \, dt'$$

We do the integrals in the reverse order as compared with the derivation of the retarded potentials — we do the spatial integrals first. This just has the effect of replacing $\mathbf{r}'$ with $\mathbf{r}_q(t')$:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c}\right)}{|\mathbf{r} - \mathbf{r}_q(t')|} \, dt'$$

The t' integral is of the form

$$\int h(t')\delta(f(t'))\, dt' = \left. \frac{h(t')}{|df/dt'|} \right|_{f(t')=0}$$

The derivative of the δ-function argument is

$$\frac{d}{dt'}\left(t' - t + \frac{\sqrt{(\mathbf{r} - \mathbf{r}_q(t')) \cdot (\mathbf{r} - \mathbf{r}_q(t'))}}{c}\right) = 1 - \frac{(\mathbf{r} - \mathbf{r}_q(t')) \cdot \mathbf{v}(t')}{c|\mathbf{r} - \mathbf{r}_q(t')|}.$$

and we must evaluate this at the retarded time $t' = t - \frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c}$.

Introducing the unit vector from the retarded position of the charge towards $\mathbf{r}$ $\mathbf{n}(t') = (\mathbf{r} - \mathbf{r}_q(t'))/|\mathbf{r} - \mathbf{r}_q(t')|$ the potentials can be written

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_q(t')|(1 - \mathbf{v}(t') \cdot \mathbf{n}/c)}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v}(t')}{|\mathbf{r} - \mathbf{r}_q(t')|(1 - \mathbf{v}(t') \cdot \mathbf{n}/c)}$$

This differs from our expectation by the $(1-\mathbf{v}\cdot\mathbf{n}/c)$ factor, which we can understand in terms of the time taken for the retarded Green function to sweep over the charge.

Feynman gives a very elegant form for the derived fields (*Lectures* I 28-2 and II 21-1 – 21-11).