2EM3

Mostly Radiation

1) Show that in a plane wave $\mathbf{E}\cdot\mathbf{B}$ is zero at all times, independent of polarization.

2) Using the electric dipole $\mathbf{E}$ and $\mathbf{B}$ fields given in the lecture, find the time-averaged Poynting's vector in the far zone.

$$\left[\bar{\mathbf{S}} = \frac{2k^4 c\hat{\mathbf{r}}(|\boldsymbol{\mathcal{P}}|^2 - |\boldsymbol{\mathcal{P}}\cdot\hat{\mathbf{r}}|^2)}{(4\pi\epsilon_0)(4\pi r^2)}\right]$$

[Can you show that the same result is obtained *without* making the far-zone approximation? Note that the contribution to $\bar{\mathbf{S}}$ of fields in quadrature is zero.]
Integrate over a sphere to find the total rate of radiation of energy.

$$\left[W = \frac{4}{3}\frac{ck^4|\boldsymbol{\mathcal{P}}|^2}{4\pi\epsilon_0}\right]$$

You may find it helpful to use the result for integrating products of components of the unit vector $\hat{\mathbf{r}}$ over a sphere:

$$\int \hat{r}_i\hat{r}_j\,d\Omega = \frac{4\pi}{3}\delta_{ij}.$$

Knowing W, we can write Poynting's vector as $\mathbf{S} = W\,g(\theta,\phi)\hat{\mathbf{r}}/r^2$ where $g(\theta,\phi)$ is a normalised angular distribution, $\int g\,d\Omega = 1$. For the case of a linear dipole oscillating along the z axis, $\boldsymbol{\mathcal{P}} = \mathcal{P}\hat{\mathbf{z}}$, show that

$$g(\theta,\phi) = \frac{3}{8\pi}\sin^2\theta.$$

For the case of a dipole rotating in the xy plane $\boldsymbol{\mathcal{P}} = \mathcal{P}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$ show that

$$g(\theta,\phi) = \frac{3}{16\pi}(1+\cos^2\theta).$$

Sketch these angular distributions, and find the polarization of the radiation for these two cases in various directions, in particular in the xy plane and along the z axis.

3) Show that the first-order corrected $\mathbf{E}$ and $\mathbf{B}$, i.e. $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(1)}$ and $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}^{(0)} + \boldsymbol{\mathcal{B}}^{(1)}$, satisfy the Maxwell Equations M3 and M4 to first order.

Calculate the transport properties to first order. Show that the first order terms make no correction to the energy density to first order:

$$\bar{u} = \epsilon_0(\boldsymbol{\mathcal{E}}^{(0)}\cdot\boldsymbol{\mathcal{E}}^{(0)\star}) + (\boldsymbol{\mathcal{B}}^{(0)}\cdot\boldsymbol{\mathcal{B}}^{(0)\star})/\mu_0 = 2\epsilon_0(\mathcal{E}^{(0)})^2.$$

Show that Poynting's vector acquires four extra terms at this order:

$$\bar{\mathbf{S}} = \bar{u}c\hat{\mathbf{k}} + \frac{i\epsilon_0 c}{k}\left(\boldsymbol{\mathcal{E}}^{(0)}\nabla\cdot\boldsymbol{\mathcal{E}}^{(0)\star} + c^2\boldsymbol{\mathcal{B}}^{(0)}\nabla\cdot\boldsymbol{\mathcal{B}}^{(0)\star} - \boldsymbol{\mathcal{E}}^{(0)\star}\nabla\cdot\boldsymbol{\mathcal{E}}^{(0)} - c^2\boldsymbol{\mathcal{B}}^{(0)\star}\nabla\cdot\boldsymbol{\mathcal{B}}^{(0)}\right)$$

4) Derive the magnetic dipole radiation fields from the vector potential given in the lecture, and show that they are dual to the electric dipole fields.