2EM1

Mostly Dipoles

1) By taking the appropriate limit show that the torque on a point dipole in a uniform electric field $\mathbf{E}$ is $\Gamma = \mathbf{p} \wedge \mathbf{E}$. (The torque of a force $\mathbf{F}$ applied at $\mathbf{r}$ is $\mathbf{r} \wedge \mathbf{F}$.) By considering the work done $\delta U = -\Gamma \cdot \delta\boldsymbol{\phi}$ in rotating the dipole against this torque through an infinitesimal angle $\delta\boldsymbol{\phi}$, show that this can be derived from the potential energy $U = -\mathbf{p} \cdot \mathbf{E}$. (Note that the change in $\mathbf{p}$ when it is rotated is $\delta\mathbf{p} = \delta\boldsymbol{\phi} \wedge \mathbf{p}$.)

2) (Mods Physics 3 1994 question 2) Write down an expression for the electrostatic potential at a distance r from a point charge q.

An electrostatic quadrupole consists of a charge $+2q$ at the origin and charges $-q$ on the z axis at the two points $z = \pm a$. Using polar coordinates, r and θ, where θ defines the angle between r and the z axis, expand the potential of these charges in a power series in a/r. Show that in the limit when $r \gg a$, the potential may be expressed as

$$\Phi = \frac{A}{r^3}\left(3\cos^2\theta - 1\right),$$

and find A.

Sketch the θ dependence of the radial component of the electric field E_r, and identify the surfaces on which E_r is zero.

3) (Mods Mathematical Physics 1993 question 6) Find the general solution of Laplace's equation in spherical polar co-ordinates in the absence of ϕ dependence, expressing the result in terms of the Legendre polynomials $y = P_n(\mu)$ ($\mu = \cos\theta$) that satisfy the equation

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + n(n+1)y = 0.$$

Solutions finite at $\mu = \pm 1$ exist for n equal to a positive integer or zero, and $P_n(1) = 1$.

Find the electrostatic potential of a thin uniformly charged ring of radius a and total charge Q along a line perpendicular to the plane of the ring, and passing through its centre.

Hence obtain the solution for all space.

4) Find the electrostatic potential Φ and the electric field E_r due to a charge Q uniformly distributed through a sphere of radius a. Show that the electrostatic energy of the charge distribution is

$$\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 a}.$$

(There are several ways of doing this. You could solve Poisson's equation directly assuming spherical symmetry, with boundary conditions $\Phi(0)$ finite, Φ and E_r continuous at $r = a$, $\Phi \to 0$ as $r \to \infty$. You could use Gauss's law for E_r directly, and find Φ by integration. You could find the potential of a spherical shell and integrate over shells (this is essentially a Green function approach). Similarly the last part can be found by integrating either $(1/2)\rho\Phi$ or $(1/2)\epsilon_0 E^2$.)

5) Write down integral expressions for $\mathbf{A}$ and $\mathbf{B}$ due to a circular current loop radius a in the XY-plane. Carry out the $\mathbf{B}$ integration on the z-axis to find

$$B_z = \frac{\mu_0}{2\pi}\frac{I\pi a^2}{(z^2+a^2)^{3/2}}.$$

In the limit $r \gg a$ show that the $\mathbf{A}$ integral becomes (to order $(a/r)^2$)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r} \int \left(1 + \frac{\mathbf{r} \cdot \mathbf{s}}{r^2}\right) \, d\mathbf{s}.$$

Show that the first term imtegrates to zero, and that the first non-zero term is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \wedge \mathbf{r}}{r^3}$$

where the magnetic dipole of the loop is given by

$$\mathbf{m} = I \int \frac{1}{2} \mathbf{s} \wedge d\mathbf{s}.$$

You might find it helpful to know that

$$I \int s_i \, ds_j = \epsilon_{ijk} m_k.$$