2EM0

Vectors and Vector Calculus

1. Vectors

There are two types of binary product, the scalar and vector products.

a) Scalar Product $\mathbf{a}\cdot\mathbf{b}$. This is defined by $\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z$.
Show that $\mathbf{a}\cdot\mathbf{b}$ is invariant under a rotation of co-ordinates $\mathbf{a} \to \mathbf{a}' = \mathbf{R}\mathbf{a}$ where $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

b) Vector Product $\mathbf{a}\wedge\mathbf{b}$. This is defined by $(\mathbf{a}\wedge\mathbf{b})_i = \epsilon_{ijk} a_j b_k$.
Show that under a rotation $\mathbf{a}'\wedge\mathbf{b}' = \mathbf{R}(\mathbf{a}\wedge\mathbf{b})$ if $\det\mathbf{R} = 1$. (Note that $\epsilon_{ijk}\mathbf{R}_{ii'}\mathbf{R}_{jj'}\mathbf{R}_{kk'} = \epsilon_{i'j'k'}\,\mathrm{det}\mathbf{R}$.)
Show that $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$ provided $\mathbf{a}$ and $\mathbf{b}$ commute.

There are two types of triple product.

c) Scalar Triple Product $\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}$.
Show that $\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} = \mathbf{a}\wedge\mathbf{b}\cdot\mathbf{c}$, and that $\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} = \mathbf{b}\cdot\mathbf{c}\wedge\mathbf{a}$ if all three vectors commute.

d) Vector Triple Product $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, provided all vectors commute.
Show that $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) \neq (\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$.

There are quite a lot of quadruple products, which can be handled by treating them as a product of two binary products or a vector and a triple product. In e) – g) we assume all vectors commute.

e) Show that $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) = \mathbf{a}\cdot\mathbf{c}\;\mathbf{b}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d}\;\mathbf{b}\cdot\mathbf{c}$.

f) By considering two ways of splitting up $(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d})$ prove $\mathbf{c}(\mathbf{a}\wedge\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\cdot\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}\wedge\mathbf{d}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}\wedge\mathbf{d})$.

g) By considering two ways of splitting up $\mathbf{a}\wedge(\mathbf{b}\wedge(\mathbf{c}\wedge\mathbf{d}))$ prove $\mathbf{a}\wedge\mathbf{c}\;\mathbf{b}\cdot\mathbf{d} - \mathbf{a}\wedge\mathbf{d}\;\mathbf{b}\cdot\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}\wedge\mathbf{d}) - \mathbf{c}\wedge\mathbf{d}\;\mathbf{a}\cdot\mathbf{b}$.

2. Gradients

In the following, $\phi(\mathbf{r})$ is an arbitrary scalar function of $\mathbf{r}$, and $f(r)$ is an arbitrary function of r, while $\mathbf{p}$ is a constant vector. Prove the following.

a) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$.

b) $\nabla(\mathbf{p}\cdot\mathbf{r}) = \mathbf{p}$.

c) $\nabla f(r) = (\mathbf{r}/r)\frac{df}{dr} = \hat{\mathbf{r}}\frac{df}{dr}$.

d) Combine a) – c):

$$\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = \frac{\mathbf{p}}{r^3} - \frac{3\mathbf{p}\cdot\mathbf{r}\,\mathbf{r}}{r^5}.$$

3. Divergences and Curls

In the following $\mathbf{A}(\mathbf{r})$ is an arbitrary vector field and $\mathbf{p}$ is a constant vector. Prove the following.

a) $\nabla\cdot(\mathbf{A}\phi) = \mathbf{A}\cdot(\nabla\phi) + \phi\,\nabla\cdot\mathbf{A}$.

b) $\nabla\cdot\mathbf{r} = 3$

c) $\nabla\wedge(\mathbf{p}\wedge\mathbf{r}) = 2\mathbf{p}$.

d) $\nabla\wedge(\mathbf{A}\phi) = \phi\nabla\wedge\mathbf{A} - \mathbf{A}\wedge\nabla\phi$.

e) Combine a) – d):

$$\nabla\wedge\left(\frac{\mathbf{p}\wedge\mathbf{r}}{r^3}\right) = -\frac{\mathbf{p}}{r^3} + \frac{3\mathbf{p}\cdot\mathbf{r}\,\mathbf{r}}{r^5}.$$