Quantum Optics with Quantum Gases

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Abstract

Quantum optics with quantum gases represents a new field, where the quantum nature of both light and ultracold matter plays equally important role. Only very recently this ultimate quantum limit of light-matter interaction became feasible experimentally. In traditional quantum optics, the cold atoms are considered classically, whereas, in quantum atom optics, the light is used as an essentially classical axillary tool. On the one hand, the quantization of optical trapping potentials can significantly modify many-body dynamics of atoms, which is well-known only for classical potentials. On the other hand, atomic fluctuations can modify the properties of the scattered light.

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I. INTRODUCTION

The general goal of this direction of our research is to join two broad and intensively developing fields of modern physics: quantum optics and ultracold quantum gases. Such a unified approach will consider both light and matter at an ultimate quantum level, the level that only very recently became accessible experimentally [1].

Optics has become one of the most well-established and exciting disciplines in physics. Even classical optics, treating the light as classical electromagnetic waves, led to the important discoveries and technological breakthroughs. For example, up to now, the optical measurements provide us the highest level of precision. A new era in optics started in the 20th century with the creation of quantum theory and invention of a laser, when the concept of photons came into existence and became testable experimentally. Now, quantum optics, which studies nonclassical light (i.e., the light whose properties cannot be explained by classical optics, but well described using the quantum wave-particle dualism), also achieved a very high level of development.

In the last decades of the 20th century, the progress in laser cooling of atoms led to the foundation of a new field in atomic physics: atom optics. According to quantum mechanics, at low temperatures, massive particles behave similarly to the waves with the wavelengths larger for lower temperatures. Thus, the matter waves in atom optics behave analogously to the light waves in optics. The quantum properties of matter waves became well accessible, after the ultralow temperatures were reached and the first Bose-Einstein condensation (BEC) was achieved in 1995. An exiting demonstration of "quantum atom optics" was presented in 2002, when the phase transition between two different quantum states of matter waves (superfluid and Mott insulator) was achieved.

Thus, the role of light and matter in quantum atom optics and quantum optics is totally reversed. However at present, even in the most involved works, the role of light in quantum atom optics is essentially reduced to a classically auxiliary tool. One can create and manipulate intriguing atomic quantum states using the forces and potentials produced by classical light waves. For example, the light beams are used to form beam splitters, mirrors, and other "devices" known in optics, but now applied for matter waves. In this context, the periodic micropotentials (the famous optical lattices) are analogies of cavities in classical optics, as they enable storing and manipulating atomic states.
Quantum optics with quantum gases will close the gap between standard quantum optics and quantum atom optics. In contrast to the traditional works, it will address the phenomena, where the quantum nature of light and ultracold gases is equally important. So, the quantum optics with quantum gases can be considered as an ultimate quantum limit of light-matter interaction, which became experimentally feasible only recently, when a BEC was coupled to the light mode of a cavity [1].

As both light and atoms will be considered at the quantum level, quantum optics with quantum gases will enable the unprecedented control of light and matter. This will find applications in the following areas. Novel non-destructive detectors of atomic states using light scattering (currently unavailable). Quantum information processing: novel protocols will be developed using the multipartite entangled states naturally appearing at this level of interaction. Quantum interferometry and metrology: the multipartite entangled states of massive particles are considered as a resource to approach the ultimate Heisenberg limit, which can be used in the gravitational wave detectors and novel quantum nanolithography. Moreover, as the far off-resonant interactions can be considered, the general approaches can be applied to other fields: molecular physics (quantum molecular gases were recently obtained); semiconductor systems (BEC of exciton-polaritons); superconductor systems (circuit cavity quantum electrodynamics).

A natural way to couple the quantum light and quantum gas is to load ultracold atoms in a high-Q cavity. Theoretically, we contributed to this field [2, 3, 4, 5, 6, 7, 8], which has stimulated further theoretical research in other groups as well [9].

On the one hand, the properties of scattered light will be affected by the quantum characteristics of the atomic states, e.g., by the particular atom number fluctuations. Thus the light can serve as a non-destructive probe of the atomic state. The aim of this paper is to summarize our results on light scattering and outline some perspectives by presenting several particular cases and using simple physical models and interpretations.

On the other hand, the quantum nature of trapping potentials provided by the cavity field can significantly modify the many-body dynamics and quantum phase transitions, well known only for classical potentials. Moreover, the interaction between particles via a cavity field provides a new type of the long range correlations that has not been studied in the standard condensed matter problems, and can also lead to novel strongly correlated systems. This part of the problem is outside of the scope of the present paper (for the generalized
Bose-Hubbard model taking into account the quantized potentials see [4, 5]).

II. THEORETICAL MODEL

We consider $N$ two-level atoms trapped in a deep optical lattice with $M$ sites formed by strong laser beams. A region of $K \leq M$ sites is illuminated by two additional light modes (Fig. 1). We will be interested in a situation, where one mode plays a role of the probe, while another one represents the scattered light, which is collected by a cavity and measured.

Interestingly, the many-body quantum problem can be significantly simplified, if one assumes the atomic tunneling much slower than fast light dynamics, which is a reasonable approximation. Then, the full problem to describe light scattering reduces to simple equations, which have a very transparent physical interpretation. The interpretation has a direct classical analogy, keeping however essentially quantum features.

The Heisenberg equation for the annihilation operator of the cavity light mode $a_1$, where $a_0$ is the classical probe amplitude, with the frequencies $\omega_{1,0}$ and mode functions $u_{1,0}(r)$ is

$$\dot{a}_1 = -i\left(\omega_1 + \frac{g^2 \hat{D}_{11}}{\Delta_{1a}}\right)a_1 - i\frac{g^2 \hat{D}_{10}}{\Delta_{1a}}a_0 - \kappa a_1 + \eta(t),$$

(1)

with $\hat{D}_{lm} \equiv \sum_{i=1}^{K} u_l^*(r_i) u_m(r_i) \hat{n}_i$,

where $l, m = 0, 1$, $g$ is the atom-light coupling constant, $\Delta_{1a} = \omega_l - \omega_a$ are the large cavity-atom detunings, $\kappa$ is the cavity relaxation rate, $\eta(t) = \eta e^{-i\omega_p t}$ gives the external probe and $\hat{n}_i$ are the atom number operators at a site with coordinate $r_i$. We also introduce the operator of the atom number at illuminated sites $\hat{N}_K = \sum_{i=1}^{K} \hat{n}_i$.

In a classical limit, Eq. (1) corresponds to the Maxwell’s equation with the dispersion frequency-shifts of cavity mode $g^2 \hat{D}_{11}/\Delta_{1a}$ and the coupling coefficient between the light modes $g^2 \hat{D}_{10}/\Delta_{1a}$. For a quantum gas those quantities are operators, which will lead to striking results: atom number fluctuations will be directly reflected in such measurable frequency- and angle-dependent observables. Thus, the cavity transmission-spectra and angular distributions of scattered light will reflect atomic statistics.
III. FREQUENCY DEPENDENCES

As a first example, we consider the simplest case: the atoms are coupled to the single cavity mode (Fig. 2). Thus, we are interested in the transmission spectra of a cavity around ultracold atoms. The general Eq. (1) is reduced to

\[
\dot{a}_1 = -i \left( \omega_0 + \frac{g^2 \hat{D}_{11}}{\Delta_{1a}} \right) a_0 - \kappa a_0 + \eta(t),
\]

which in the steady state has the solution for the cavity photon number as

\[
a_1^\dagger a_1 = \frac{|\eta|^2}{(\Delta_p - g^2 \hat{D}_{11}/\Delta_{1a})^2 + \kappa^2},
\]

where \(\Delta_p = \omega_p - \omega_1\) is the probe-cavity detuning. Theoretically, the simplest situation is realized for the traveling-wave cavity (Fig. 2). In this case, \(\hat{D}_{11} = \hat{N}_K\). Thus the operator-valued frequency dispersion shift, independently of the cavity-lattice angle, is simply given by the atom number at \(K\) illuminated sites, which is a fluctuating quantity.

The transmission spectra are shown in Fig. 2. In the Mott insulator state (MI), where all atom number fluctuations are totally suppressed, one sees the classical transmission contour given by a Lorentzian. In contrast, in the superfluid state (SF), where the atom number fluctuations are strong, and \(N_K\) can take any value, one sees any possible Lorentzians with any possible dispersion frequency shifts. The frequency distance between different Lorentzians is given by a fluctuation produced by a single atom and is equal to \(g^2/\Delta_{1a}\).

One can show, that the transmission spectrum represents exactly the atom number distribution function. Thus, the quantum phase transition from the SF to MI is displayed in the transmission spectrum as a shrinking to a single Lorentzian. Note, that for a standing-wave cavity similar conditions can be easily analyzed.

Moreover, taking into account the quantum and dynamical nature of the probe field, other atom number related quantities and their distribution functions, e.g. the atom number difference between odd and even sites, can be accessed by the transmission spectra. Interestingly, using that atom number difference one can distinguish between the SF state (where the total atom number is fixed) and the coherent-state approximation to the SF (where the total atom number is infinite and thus unfixed). In SF, the frequency distance
between the Lorentzians will be twice as that in the coherent state, since for the fixed atom
number, the atom number difference between odd and even sites changes with the step 2,
while for the coherent state it changes with the step 1. Thus for SF, the splitting is \(2g^2/\Delta_{1a}\),
while for the coherent state it is \(g^2/\Delta_{1a}\).

IV. ANGULAR DISTRIBUTIONS

Let us now consider the configuration with the probe wave \(a_0\). However, in this section,
we neglect the dispersion frequency shift in Eq. (1), assuming it is smaller than the cavity
relaxation rate or probe-cavity detuning \(\Delta_{01}\) (the consideration of the external probe \(\eta\) is
also not necessary here). In this case, the stationary solution is even simpler:

\[
a_1 = C \hat{D}_{10},
\]

with the constant \(C \equiv ig_0^2a_0/|\Delta_{1a}(i\Delta_{01} - \kappa)|\). Thus, the light amplitude is proportional
to the coupling coefficient between two modes, which depends on the atom numbers and,
therefore, is a fluctuating quantity.

Equation (1) shows that the expectation value of the field amplitude is simply proportional
to \(\langle \hat{D}_{10} \rangle\), which depends on the mean atom numbers \(\langle \hat{n}_i \rangle\). Thus, if in different atomic
states the mean atom number is the same (as it is in MI and SF), the amplitude of the
scattered light will be also identical. However, the mean photon number \(\langle a_1^\dagger a_1 \rangle\) is proportional
to the second moment \(\langle \hat{D}_{10}^\dagger \hat{D}_{10} \rangle\), which depends on the density-density correlations
\(\langle \hat{n}_i \hat{n}_j \rangle\). Those correlations are different for various atomic states, even if the mean atom
numbers are the same. Thus, in contrast to the light amplitude, the photon number carries
information about number statistics in the atomic state.

We will demonstrate the difference in the light scattering from different atomic states by
considering the angle-dependent quantity \(R(\theta_0, \theta_1)\) (where \(\theta_0\) and \(\theta_1\) are the angles between
the lattice and light beams, cf. Fig. 1), which is proportional to the difference ("noise")
between the photon number and classical intensity (the latter is just the amplitude squared):
$$R(\theta_0, \theta_1) \equiv \langle \hat{D}^* \hat{D} \rangle - |\langle \hat{D} \rangle|^2 =$$

$$= \langle \delta \hat{n}_a \delta \hat{n}_b \rangle \sum_{i=1}^{K} u_i^*(r_i) u_0(r_i) \rangle^2 + \langle \langle \delta \hat{n}^2 \rangle - \langle \delta \hat{n}_a \delta \hat{n}_b \rangle \rangle \sum_{i=1}^{K} |u_i^*(r_i) u_0(r_i)|^2.$$

(5)

Here we assumed that all pair correlations and on-site fluctuations are the same for all lattice sites. For a 1D lattice of period $d$ and atoms trapped at $x_m = md$ ($m = 1, 2, ..., M$) the mode functions are $u_{0,1}(r_m) = \exp(i mk_{0,1}x d)$ for traveling and $u_{0,1}(r_m) = \cos(mk_{0,1}x d)$ for standing waves with $k_{0,1} = |k_{0,1}| \sin \theta_{0,1}$ (cf. Fig. 1).

Thus, Eq. (5) shows that the difference between quantum and classical light scattering $R(\theta_0, \theta_1)$ depends on the on-site $\langle \delta \hat{n}^2 \rangle$ and pair fluctuations $\langle \delta \hat{n}_a \delta \hat{n}_b \rangle$. In MI, both type of fluctuations are zero, so MI shows precisely classical light scattering. In the SF state, $\langle \delta \hat{n}^2 \rangle = n(1 - 1/M)$ and $\langle \delta \hat{n}_a \delta \hat{n}_b \rangle = -N/M^2$. Thus both angle-dependent terms contribute to the difference. Note, that if SF state is approximated by a coherent state, where the pair fluctuations are neglected and $\langle \delta \hat{n}^2 \rangle = n$, only the second term contributes to the difference.

Figure 3 shows angular distributions for the intensity of classical scattering (the lattice period is $d = \lambda_{0,1}/2$) and the classical-quantum difference $R$ for the simplest configuration, where both modes are traveling waves. In SF, except for the isotropic background in $R$, the noise is suppressed at the directions of the classical diffraction maxima. Such suppression corresponds to the suppressed total atom number fluctuations in SF state, in contrast to the coherent state (a totally isotropic background is observed for the coherent state approximation).

Figure 4 shows similar situation, but the probe is the traveling wave, whereas the cavity mode is the standing wave. Figure 5 shows the situation, where both the probe and cavity are the standing waves. One sees, that the noise displays the angular distribution much richer than the classical diffraction. This is a consequence of the second-order interference (the interference of intensities similar to the Hanbury Brown and Twiss effect). The new features can appear at the angles of the first order diffraction maxima, although classically, only the zero order diffraction is possible.

Thus, the measurement of angular distribution of scattered light can distinguish between different atomic quantum states (MI, SF, and coherent in the above examples).
V. CONCLUSIONS

We have shown that various atomic quantum states can be distinguished by analyzing the frequency and angular distributions of the scattered light, even if the mean atomic density is identical for the atomic states. We have demonstrated the simplified model. This model could be improved in the following directions. Here, the trapping potentials are assumed to be very deep, so the atoms are narrowly localized at each lattice site. However, one can take into account the finite width of the atomic wave function (e.g., the one given by the Wannier functions), which will lead to the modification of both classical and quantum scattering. Moreover, we assumed the uniform dependence of all fluctuations in space. Various spatial dependences of the correlations as \( \langle \hat{n}_i \hat{n}_j \rangle (r_i - r_j) \) will manifest themselves in the angular distribution of the scattered light. Here, we presented the calculations of some expectation values, which assumes the repeated measurement. To fully characterize the quantum measurement process, taking into account the measurement back-action should be done. A single-run measurement of scattered photons will lead to the reduction of the atomic state as well, due to the entanglement between light and matter \[6\].

Fig. 1. General setup. Atoms in a deep lattice (the trapping beams are not shown) are illuminated by a probe wave $a_0$ at angle $\theta_0$. Additional probing through a mirror $\eta$ is possible. The scattered light $a_1$ is collected by a cavity at angle $\theta_1$ and measured by a detector.

Fig. 2. Transmission spectra of a traveling wave cavity. (a) Setup. (b) Cavity photon numbers at different probe-cavity detunings. Single Lorentzian for the Mott insulator state and many Lorentzians for the superfluid state, $\kappa = 0.1g^2/\Delta_0$, $N = M = 30$, $K = 15$.

Fig. 3. Intensity angular distributions for two traveling waves, the probe is transverse to the lattice ($\theta_0 = 0$), the lattice period is $d = \lambda/2$. (a) Setup. (b) Intensity distribution of the classical diffraction. (c) Noise quantity $R(\theta_1)$ for the superfluid state. $N = M = K = 30$.

Fig. 4. Intensity angular distributions for scattering of a traveling probe into a standing-wave cavity. The probe is at $\theta_0 = 0.1\pi$. (a) Setup. (b) Intensity distribution of the classical diffraction. (c) Noise quantity $R(\theta_1)$ for the superfluid state. $N = M = K = 30$, $d = \lambda/2$.

Fig. 5. Intensity angular distributions for scattering of a standing-wave probe into a standing-wave cavity. The probe is at $\theta_0 = 0.1\pi$. (a) Setup. (b) Intensity distribution of the classical diffraction. (c) Noise quantity $R(\theta_1)$ for the superfluid state. $N = M = K = 30$, $d = \lambda/2$. 
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Fig. 5