COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

PROBLEM SET 0: Complex numbers

Problems and solutions courtesy Julia Yeomans via Michael Barnes
Comments and corrections to John Magorrian

1. Change to polar form

(i) $-i$, (ii) $\frac{1}{2}-\frac{\sqrt{3}i}{2}$, (iii) $-3-4i$, (iv) $1+i$, (v) $1-i$, (vi) $(1+i)/(1-i)$.

2. For (a) $z_1 = 1+i$, $z_2 = -3+2i$ and (b) $z_1 = 2e^{\frac{i\pi}{4}}$, $z_2 = e^{\frac{-3i\pi}{4}}$ find

(i) z_1+z_2, (ii) z_1-z_2, (iii) z_1z_2, (iv) z_1/z_2, (v) $|z_1|$, (vi) z_1^*.

3. For $z = x+iy$ find the real and imaginary parts of

(i) z^2, (ii) $1/z$, (iii) i^{-5}, (iv) $(2+3i)/(1+6i)$, (v) $e^{\frac{i\pi}{6}} - e^{-\frac{i\pi}{6}}$.

4. Draw in the complex plane.

(i) $3-2i$,
(ii) $4e^{-\frac{i\pi}{6}}$,
(iii) $|z-1| = 1$,
(iv) $\text{Re}\,(z^2) = 4$,
(v) $\arg\,(z+3i) = \pi/4$,
(vi) $|z+1|+|z-1| = 8$,
(vii) $z = te^{it}$ for real values of the parameter t,
(viii) $\arg\left(\frac{z-4}{z-1}\right) = \frac{3\pi}{2}$.

5. Use de Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Deduce that

$$\cos(\pi/8) = \left(\frac{2+\sqrt{2}}{4}\right)^{1/2}$$

and write down an expression for $\cos(3\pi/8)$.

6. Express $\sin^6\theta$ as a sum of terms in $\cos n\theta$ for integer n.

7. Find (i) $(1+i)^9$, (ii) $(1-i)^9/(1+i)^9$.

8. Find all the values of the following roots

(i) $\sqrt[4]{\frac{-1-\sqrt{3}i}{2}}$,

(ii) $(-8i)^{\frac{2}{3}}$,

(iii) $\sqrt[8]{16}$.

9. (i) Show that the sum of the n n^{th} roots of any complex number is zero.

(ii) By considering the roots of $z^{2n+1}+1=0$, with n a positive integer, show that

$$\sum_{k=-n}^{n} \cos\left(\frac{2k+1}{2n+1}\right)\pi = 0.$$

10. Find the roots of the equation $(z-1)^n+(z+1)^n=0$. Hence solve the equation $x^3+15x^2+15x+1=0$.

11. Show that

$$\sum_{n=0}^{\infty} 2^{-n}\cos n\theta = \frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}.$$

12. Prove that

$$\sum_{r=1}^{n} {}^{n}C_r \sin 2r\theta = 2^n \sin n\theta \cos^n\theta.$$

Hint: express the left-hand side as $\text{Im}\Big\{e^{in\theta}\sum_{r=1}^{n} {}^{n}C_r e^{i(2r-n)\theta}\Big\}$.

13. Find the real and imaginary parts of:

(i) $e^{3ln2-i\pi}$, (ii) $\ln i$, (iii) $\ln(-e)$, (iv) $(1+i)^{iy}$, (v) $\sin(i)$,
(vi) $\cos(\pi-2i\ln 3)$, (vii) $\tanh(x+iy)$, (viii) $\tan^{-1}(\sqrt{3}i)$, (ix) $\sinh^{-1}(-1)$.