FIRST YEAR MATHS FOR PHYSICS STUDENTS

NORMAL MODES AND WAVES

Hilary Term 2014

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Question Sheet 1: Normal Modes

[Questions marked with an asterisk (*) are optional]

1. Two identical pendula each of length l and with bobs of mass m are free to oscillate in the same plane. The bobs are joined by a massless spring with a small spring constant k, such that the tension in the spring is k times its extension.

(a) Show that the motion of the two bobs is governed by the equations

$$m\ddot{x} = -mgx/l + k(y-x)$$

and $$m\ddot{y} = -mgy/l - k(y-x)$$

(b) By looking for solutions where x and y vary harmonically at the same angular frequency ω, convert these differential equations into two ordinary simultaneous equations for the amplitudes of oscillation x_0 and y_0.

(c) Why do we not expect these equations to determine the absolute values of x_0 and y_0?

(d) Each of these new equations gives the ratio of x_0/y_0 in terms of ω. Find the values of ω that make these equations consistent.

(e) For each of these values of ω, find the ratio x_0/y_0. Describe the relative motions of the two pendula for each of these normal modes. Is one of the values of ω obvious?

(f) At $t = 0$, both pendula are at rest, with $x = A$ and $y = A$. Describe the subsequent motion of the two pendula.

2. Two coupled simple pendula are of equal length l, but their bobs have different masses m_1 and m_2. Their equations of motion are:

$$\ddot{x} = -\frac{g}{l}x - \frac{k}{m_1}(x-y)$$

and $$\ddot{y} = -\frac{g}{l}y + \frac{k}{m_2}(x-y)$$

(a) Use the standard method first to find the frequencies and the relative amplitudes of the bobs for the normal modes of the system.

(b) By taking suitable linear combinations of the two equations of motion, obtain two uncoupled differential equations for linear combinations of x and y. Hence again find the normal mode frequencies and the relative amplitudes. [Hint: One of these linear combinations is fairly obvious. For the other, it may be helpful to consider the centre of mass of the two bobs.]

3. Consider again the two identical pendula of question 1.

(a) At $t = 0$, both pendula are at rest, with $x = A$ and $y = 0$. They are then released. Describe the subsequent motion of the system. If $k/m = 0.105g/l$, show that

$$x = A\cos\Delta t\cos\bar{\omega}t$$

$$\text{and } y = A\sin\Delta t\sin\bar{\omega}t$$

where $\Delta = 0.05\sqrt{g/l}$ and $\bar{\omega} = 1.05\sqrt{g/l}$

Sketch x and y, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

(b) State a different set of initial conditions such that the subsequent motion of the pendula corresponds to that of a normal mode.

(c) The most general initial conditions are where each bob has a given initial displacement and a given initial velocity. Explain as fully as you can why the solution in this general case has the form:

$$x = \alpha_c\cos\omega_1 t + \alpha_s\sin\omega_1 t + \beta_c\cos\omega_2 t + \beta_s\sin\omega_2 t$$

$$\text{and} \quad y = \alpha_c\cos\omega_1 t + \alpha_s\sin\omega_1 t - \beta_c\cos\omega_2 t - \beta_s\sin\omega_2 t$$

where ω_1 and ω_2 are the normal mode angular frequencies and α_c, α_s , β_c and β_s are arbitrary constants. How are these arbitrary constants determined?

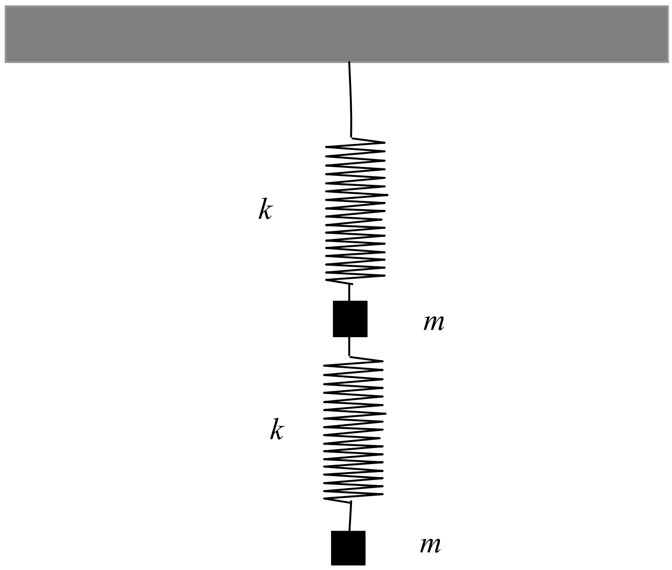
(d) At $t = 0$, both bobs are at their equilibrium positions: the first is stationary but the second is given an initial velocity v_0. Show that subsequently

$$x = \frac{v_0}{2}\left(\frac{1}{\omega_1}\sin\omega_1 t - \frac{1}{\omega_2}\sin\omega_2 t\right)$$

$$\text{and } y = \frac{v_0}{2}\left(\frac{1}{\omega_1}\sin\omega_1 t + \frac{1}{\omega_2}\sin\omega_2 t\right)$$

(e) For the initial conditions of part (d), and with $k/m = 0.105g/l$, describe as fully as possible the subsequent velocities of the two bobs.

4.

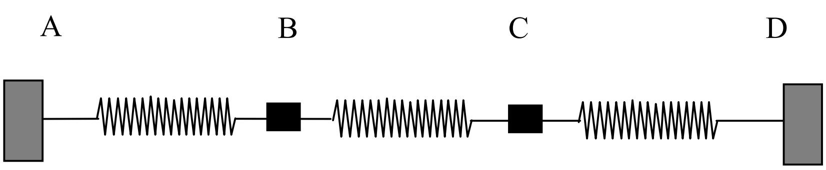


Two equal masses m are connected as shown with two identical massless springs, of spring constant k. Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by

$$\omega^2 = \left(3 \pm \sqrt{5}\right) k / 2m$$

Find the ratio of the amplitudes of the two masses in each separate mode.
Why does the acceleration due to gravity not appear in these answers?

5. AB, BC, and CD are identical springs with negligible mass, and stiffness constant k:



The masses m, fixed to the springs at B and C, are displaced by small distances x_1 and x_2 from their equilibrium positions along the line of the springs, and execute small oscillations. Show that the angular frequencies of the normal modes are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Sketch how the two masses move in each mode. Find x_1 and x_2 at times $t > 0$ if at $t = 0$ the system is at rest with $x_1 = a$, $x_2 = 0$.

6*. The setup is as for question 5, except that in this case the springs AB and CD have stiffness constant k_0, while BC has stiffness constant k_1. If C is clamped, B vibrates with frequency ν_0 = 1.81 Hz. The frequency of the lower frequency normal mode is ν_1 = 1.14 Hz. Calculate the frequency of the higher frequency normal mode, and the ratio k_1/k_0. (From French 5-7).

7*. Two particles, each of mass m, are connected by a light spring of stiffness k, and are free to slide along a smooth horizontal track. What are the normal frequencies of this system? Describe the motion in the mode of zero frequency. Why does a zero-frequency mode appear in this problem, but not in question 5, for example?

8*. A stretched massless string has its ends at $x = 0$ and $x = 3l$ fixed, and has equal masses attached at $x = l$ and $x = 2l$. Show that the equations of the transverse motion of the masses are approximately

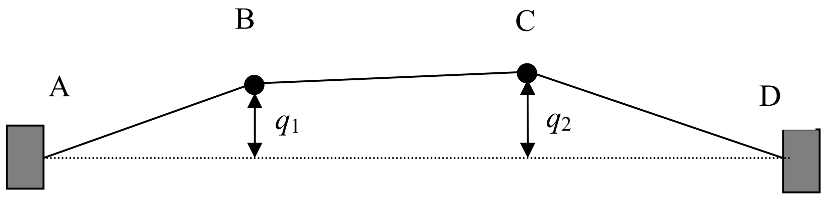
$$m\ddot{y}_1 = \frac{T}{l}(y_2 - 2y_1)$$

and $$m\ddot{y}_2 = \frac{T}{l}(y_1 - 2y_2)$$

where T is the tension in the string. (Gravity does not appear in this equation because (i) y_1 and y_2 refer to motion with respect to the equilibrium positions; or (ii) the motion takes place on a horizontal frictionless table.) Convince yourself that, for small oscillations, it is reasonable to neglect the changes in tension caused by the variation in length of the three sections of the string resulting from the transverse motion of the masses.

Find the frequencies and the ratio of amplitudes of the transverse oscillations for the normal modes of the two masses. Is the relative motion of the higher-frequency mode reasonable?

9. The figure shows two masses m at points B and C of a string fixed at A and D, executing small transverse oscillations. The tensions are assumed to be all equal, and in equilibrium AB = BC = CD = l.



If the (small) transverse displacements of the masses are denoted by q_1 and q_2, the equations of motion are

$$m\ddot{q}_1 = -k(2q_1 - q_2), \qquad m\ddot{q}_2 = -k(2q_2 - q_1) \qquad (1)$$

where $k = T / l$, and terms of order q_1^2, q_2^2 and higher have been neglected.

(a) Define the *normal coordinates* Q_1, Q_2 by

$$Q_1 = (q_1 + q_2)/\sqrt{2}, \qquad Q_2 = (q_1 - q_2)/\sqrt{2}$$

Show that $m\ddot{Q}_1 = -kQ_1$, $m\ddot{Q}_2 = -3kQ_2$, and hence that the general solution of (1) is

$$Q_1 = E\cos\omega_1 t + F\sin\omega_1 t, \qquad Q_2 = G\cos\omega_2 t + H\sin\omega_2 t \qquad (2)$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ are the *normal mode frequencies*. Hence find the general solution for q_1 and q_2.

(b) The forces on the RHS of (1) may be interpreted in terms of a potential energy function $V(q_1, q_2)$, as follows. We write the equations as

$$m\ddot{q}_1 = -\frac{\partial V}{\partial q_1}, \qquad m\ddot{q}_2 = -\frac{\partial V}{\partial q_2},$$

generalising "$m\ddot{x} = -\partial V/\partial x$". Show that V may be taken to be

$$V = k\left(q_1^2 + q_2^2 - q_1 q_2\right).$$

Derive the same result for V by considering the work done in giving each section of the string its deformation for equilibrium (e.g. for AB the work done is equal to $T\left(\sqrt{l^2 + q_1^2} - l\right)$, and expand in powers of q_1^2/l^2 *only*.

Show that, when written in terms of the variables Q_1 and Q_2, V becomes

$$V = \frac{1}{2}m\omega_1^2 Q_1^2 + \frac{1}{2}m\omega_2^2 Q_2^2$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ as before.

(c) Show that the *kinetic energy* of the masses is

$$K = \frac{1}{2}m\left(\dot{Q}_1^2 + \dot{Q}_1^2\right)$$

and hence that the total energy, in terms of Q_1 and Q_2, is

$$V + K = \left(\frac{1}{2}m\dot{Q}_1^2 + \frac{1}{2}m\omega_1^2 Q_1^2\right) + \left(\frac{1}{2}m\dot{Q}_2^2 + \frac{1}{2}m\omega_2^2 Q_2^2\right) = E_1 + E_2$$

where E_1 is the total energy of 'oscillator' Q_1 with frequency ω_1, and similarly for E_2.
What is the expression for the total energy when written in terms of $q_1, q_2, \dot{q}_1$, and $\dot{q}_2$? Discuss the similarities and differences.

(d) Find the equations of motion for Q_1 and Q_2 from Newton's law in the form

$$m\ddot{Q}_1 = -\frac{\partial V}{\partial Q_1}, \qquad m\ddot{Q}_2 = -\frac{\partial V}{\partial Q_2},$$

and hence re-derive solution (2).