Normal Modes and Waves

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2019 Problem Set

1 Normal Modes

(1.1) Standard Solution Method. Two identical pendula each of length l and with bobs of mass m are free to oscillate in the same plane. The bobs are joined by a massless spring with a small spring constant k, such that the tension in the spring is k times its extension.

(a) Show that the motion of the two bobs is governed by the equations

$$m\frac{\mathrm{d}^2\mathbf{x}_1}{\mathrm{d}t^2} = -\frac{mg}{l}x_1 + k(x_2 - x_1) \tag{1}$$

and

$$m\frac{\mathrm{d}^2\mathbf{x}_2}{\mathrm{d}t^2} = -\frac{mg}{l}x_2 + k(x_1 - x_2) \tag{2}$$

(b) Write these equations in the form

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{x}$$

and write down the $\mathbf{K}$ matrix.

(c) Substitute a normal mode solution $\mathbf{x} = \mathbf{a}\,f(t)$ and show that this satisfies the equation of motion provided $\mathbf{a}$ is an eigenvector of $\mathbf{K}$. Find and solve the corresponding equation for $f(t)$.

(d) How many eigenvectors does $\mathbf{K}$ have? Find them and write down a general solution for the problem.

(e) At $t = 0$, both pendula are at rest, with $x_1 = A$ and $x_2 = A$. Describe the subsequent motion of the two pendula.

(1.2) Fitting Initial Conditions. Consider the two coupled pendula of question 1.1.

(a) At $t = 0$, both pendula are at rest, with $x_1 = A$ and $x_2 = 0$. They are then released. Describe the subsequent motion of the system. If $k/m = 0.105g/l$, show that

$$x_1 = A\cos\bar{\omega}t\cos\Delta t$$

and

$$x_2 = A\sin\bar{\omega}t\sin\Delta t$$

where $\bar{\omega} = 1.05\sqrt{g/l}$ and $\Delta = 0.05\sqrt{g/l}$.

Sketch x_1 and x_2, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

(b) State a different set of initial conditions such that the subsequent motion of the pendula corresponds to that of a normal mode.

(c) At $t = 0$, both bobs are at their equilibrium positions: the first is stationary but the second is given an initial velocity v_0. Show that subsequently

$$\mathbf{x} = \frac{v_0}{2\omega_1}\begin{pmatrix}1\\1\end{pmatrix}\sin\omega_1 t + \frac{v_0}{2\omega_2}\begin{pmatrix}-1\\1\end{pmatrix}\sin\omega_2 t$$

(d) For the initial conditions of part (c), and with $k/m = 0.105g/l$, sketch the subsequent positions and velocities of the two bobs.

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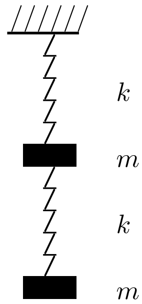

Figure 1: Example for Question 1.3

(1.3) Non-trivial Eigenvectors. Two equal masses m are connected as shown in Figure 1 with two identical massless springs, of spring constant k.

(a) Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by $\omega_i^2 = q_i(k/m)$ where $q_1 = [3 - \sqrt{5}]/2$ and $q_2 = [3 + \sqrt{5}]/2$

(b) Find the corresponding eigenvectors.

(c) Why does the acceleration due to gravity not appear in these answers?

(1.4) Zero Eigenvalue. Two particles, each of mass m, are connected by a light spring of stiffness k, and are free to slide along a frictionless horizontal track.

(a) Find the normal frequencies and eigenvectors of this system.

(b) Why does a zero-frequency mode appear in this problem? Write down the general solution.

(1.5) Energy in the equal mass case. The system of question 1.3 with equal springs ($k_1 = k_2 = k$) (shown in Figure 1) is excited by pulling the lower mass down a distance $2a$ and releasing from rest.

(a) Show that the initial condition is $\mathbf{x}(0) = \begin{pmatrix} a \\ 2a \end{pmatrix}$.

(b) Show that the system energy is ka^2.

(c) Find the energies in each of the normal modes. You can use the eigenvalues

$$\kappa_1 = 2k[1 - \cos(\pi/5)] \quad \text{and} \quad \kappa_2 = 2k[1 - \cos(3\pi/5)]$$

and normalised eigenvectors

$$\mathbf{c}_1 = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\pi/5) \\ \sin(2\pi/5) \end{pmatrix} \quad \text{and} \quad \mathbf{c}_2 = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(3\pi/5) \\ \sin(6\pi/5) \end{pmatrix}.$$

However you should give the result numerically; otherwise it's a complex and unilluminating string of trig functions!

(1.6) Unequal mass case. In the coupled pendulum example of question 1 the two masses are unequal, m_1 and m_2.

(a) Modify the equations of motion and write in matrix form using the same $\mathbf{K}$ matrix as before and a mass matrix $\mathbf{M}$ which you should define.

(b) Solve the equations by finding the eigenvalues and eigenvectors of $\mathbf{K}$ with respect to the metric $\mathbf{M}$.

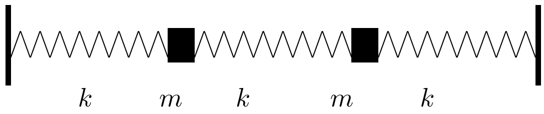

Figure 2: Example for Question 1.7

(1.7) Transverse oscillations; transition to Waves. Consider transverse oscillations of the system shown in Figure 2. The fixed side walls are separated by a distance $3l$, and the two masses divide the distance into three equal spaces of length l. The three springs are all identical with natural length $l_0 \ll l$, so that there is a equilibrium tension T_0 in all three springs.

(a) Show that small transverse displacements y_1, y_2 lead to extensions of the springs which are quadratic expressions of the displacements. Hence argue that for suficiently small displacements the dominant restoring force is the transverse component of the tension T_0 .

(b) Hence show that the linear approximation to the equations of motion is

$$m\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}t^2} = -\mathbf{K}\mathbf{y} \quad \text{where} \quad \mathbf{K} = \begin{pmatrix} 2T_0/l & -T_0/l \\ -T_0/l & 2T_0/l \end{pmatrix}$$

(c) Now consider a much larger system with total length $(N+1)l$ divided into equal sections by N equal masses, with tension T in all springs. Deduce the form of the $\mathbf{K}$ matrix for this case.

(d) Hence show that the general row ($i \neq 1$ or N) of the eigenvector equation $\mathbf{Kc} = \kappa\mathbf{c}$ is $c_{i+1} + c_{i-1} = (2 - \kappa l/T)c_i$, where c_i denotes the i'th element of the vector $\mathbf{c}$. Show that $c_i = \sin(i\phi)$ satisfies this equation and show that κ must then be given by $\kappa = (2T/l)(1 - \cos\phi)$. Show that this form also satisfies the $i = 1$ row of the equation, and that to satisfy the $i = N$ row of the equation we need $\phi = n\pi/(N+1)$ for integer n.

(e) Now consider very large N, and $n \ll N$. Define $L = (N+1)l$ as the total length of the string, and $k_n = n\pi/L$ as the rate of change of phase with distance along the string. Show that $\kappa \approx Tk_n^2 l/m$. Hence deduce that for these low-n modes ω_n and k_n are related by $\omega_n^2/k_n^2 = Tl/m$.

2 Waves I

(2.1) Standing and travelling waves.

(a) Outline the differences between a travelling wave and a standing wave.

(b) Convince yourself that $y_1 = A\sin(kx - \omega t)$ corresponds to a travelling wave. Which way does it move and with what velocity? What are the amplitude a, wavelength λ, wavenumber $\bar{\nu}$, wavevector, period T, frequency ν, and angular frequency of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y_1}{\partial t^2}$$

provided that ω and k are suitably related.

(d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t).$$

Convince yourself that this superposition of two travelling waves is a standing wave.

(e) Show that the standing wave $y = A\sin(kx)\sin(\omega t)$ can be written as a superposition of two travelling waves.

(2.2) Derivation of The Wave Equation for a string.

(a) A string of uniform linear density ρ is stretched to a tension T. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c^2 = T/\rho$.

(b) Show that the equation is linear and homogeneous, of the form $\mathcal{L}y = 0$ where $\mathcal{L}$ is a linear differential operator.

(c) What does this imply for solutions of the equation?

(2.3) Solution by separation of variables. Transverse waves are excited on a string stretched between two fixed points at $x = 0$ and $x = L$.

(a) Outline the solution of the wave equation using the method of separation of variables. Explain carefully how the boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$ determine the sign of the separation constant.

(b) Show that there exist two classes of separated solutions which satisfy the wave equation and the boundary conditions: $y = \sin(n\pi x/L)\sin(n\pi ct/L)$ and $y = \sin(n\pi x/L)\cos(n\pi ct/L)$ for positive integer n. Hence write down a general solution for y.

(c) Show that this general solution is periodic in time and find the period.

(d) Suppose the string is plucked at its midpoint and released from rest at $t = 0$:

$$y(x,0) = \begin{cases} 2ax/L & \text{for } 0 \le x < L/2, \\ a & \text{for } x = L/2 \\ 2a(L-x)/L & \text{for } L/2 < x \le L. \end{cases}$$

Show that the solution must now be of the form

$$y(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

where the coefficients A_n satisfy

$$y(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right).$$

The A_n can be found by the method of Fourier Series, which is not on the first year maths course (see optional question at end).

(e) Suppose instead that the initial conditions are $y(x,0) = 0$ and $y_t(x,0) = V(x)$, where y_t denotes $\partial y/\partial t$. Write down the form of the solution in this case, and the equation from which the coefficients can be determined.

(2.4) Fitting Initial Conditions. Suppose instead that the initial conditions for question 2.3 (c) are $y(x,0) = \sin(\pi x/L) + 2\sin(2\pi x/L)$ and $y_t(x,0) = 0$.

(a) Find an explicit expression for $y(x,t)$.

(b) Make rough sketches of $y(x,t)$ at the following times: $t = 0$, $t = L/4c$, $t = L/2c$, $t = 3L/4c$, $t = L/c$.

Note that this solution is neither a standing wave (no fixed nodes) nor a travelling wave (no net progression).

(2.5) Dispersion

(a) What is meant by a dispersive medium and what is the dispersion relation? Define the phase velocity $v_p = \omega/k$ and the group velocity $v_g = \partial\omega/\partial k$. Explain carefully what travels at each velocity.

(b) Show that an alternative expression for v_g is

$$v_g = v_p + k\frac{\partial v_p}{\partial k}.$$

(c) Evaluate v_p and v_g as a functions of k for the following cases:

i. Long wavelength surface waves on water $\omega = \sqrt{gk}$ (where g is the acceleration due to gravity).

ii. Short wavelength ripples on water $\omega = \sqrt{\sigma k^3/\rho}$ (where σ is the surface tension and ρ the density).

iii. In the crossover region where both effects are important $\omega^2 = gk + \sigma k^3/\rho$.

iv. Guided electromagnetic waves in a waveguide (with a non-zero longitudinal component of either E or B) $\omega^2 = \omega_0^2 + c^2k^2$ (where c is the speed of light).

(d) In the first two cases but not the other two you should have found $v_g = \alpha v_p$, where the constant α is different in the two cases. What type of dispersion relation leads to this result?

(e) In the fourth case you should have found $v_p v_g = c^2$, so that either v_p or v_g is greater than c. Which is it, and why does this *not* allow signalling faster than the speed of light?

(2.6) Stationary Phase or Group Velocity. In the long wavelength limit of question 2.5(c)i., v_p and v_g are decreasing functions of k, while in the short-wavelength limit of 2.5(c)ii. they increase with k. Thus in the cross-over region of question 2.5(c)iii. both pass through minima.

(a) At the minimum of v_p we have, using the result of question 2.5(b), $v_p = v_g$. Show this occurs at $k^2 = g\rho/\sigma$. Verify this using the dispersive wavepacket plotter on the course web page. (For the values used in the DWP this occurs at 364 m^{-1}), and describe the propagation of a wavepacket centred around this frequency (for example, $k_{\min} = 340\ \mathrm{m}^{-1}$, $k_{\max} = 390\ \mathrm{m}^{-1}$. (Don't forget that the length unit used in this section of the DWP is 10 cm.)

(b) The minimum of v_g is more difficult to calculate algebraically: in fact it occurs at

$$k^2 = \frac{2\sqrt{3}-3}{3}\frac{g\rho}{\sigma}$$

Verify this using the DWP. (Note that the displayed value of v_g is evaluated for the centre frequency of the wavepacket, so that it can be read out at steps of 5 m^{-1} in the cross-over region by using appropriate combinations of k_{min} and k_{max}.) Around this frequency v_g is essentially constant, making the envelope approximation particularly accurate.

(2.7) Energy Density.

(a) Show that the kinetic energy density u_K and the potential energy density u_P for a transverse wave on a string of linear density ρ and at tension T are given by

$$u_K = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

and

$$u_P = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

(b) Evaluate these for the wave $y = A\sin(kx - \omega t)$ where k and ω are such that y satisfies the wave equation.

(c) Show that u_k=u_P.