CP4: MULTIPLE INTEGRALS AND VECTOR CALCULUS (OXFORD PHYSICS FIRST YEAR)

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These problem sets are a small evolution of those that have been used for the previous few years. This reflects the inclusion of introductory material on continuous statistical distributions to the course, as well as some small rearrangements to better line up the questions with the order material is currently taught.

The course syllabus is as follows:

Double integrals and their evaluation by repeated integration in Cartesian, plane polar and other specified coordinate systems. Jacobians. Probability theory and general probability distributions. Line, surface and volume integrals, evaluation by change of variables (Cartesian, plane polar, spherical polar coordinates and cylindrical coordinates only unless the transformation to be used is specified). Integrals around closed curves and exact differentials. Scalar and vector fields. The operations of grad, div and curl and understanding and use of identities involving these. The statements of the theorems of Gauss and Stokes with simple applications. Conservative fields.

Recommended Reading Material:

The absolute minimum:

- *Mathematical Methods for Physics and Engineering,* K F Riley, M P Hobson and S J Bence. This is the main maths textbook for the first year and contains all the necessary material, although it can be quite brief at times.

You may also consider:

- *Introduction to Electrodynamics,* D J Griffiths. This books has a very nice introductory chapter on Div, Grad, Curl and the basics of vector calculus. You will need it for your 2nd year anyway, so why not get it now?

Additional information:

Course notes are available for download here (and elsewhere):

https://www2.physics.ox.ac.uk/contacts/people/fender

and contain all of the course material plus plenty of worked examples. The problem sheets below will be set and marked by your college tutors.

PROBLEM SET 1

1. Which of the following can be described by vectors: (a) temperature; (b) magnetic field; (c) acceleration; (d) force; (e) molecular weight; (f) area; (g) angle of polarization.

2. (a) For the folliwing integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y \, \mathrm{d}x\mathrm{d}y, \qquad \int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, \mathrm{d}y\mathrm{d}x, \qquad \int_0^1 \int_{-y}^{y^2} x \, \mathrm{d}x\mathrm{d}y. \tag{1}$$

   (b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x \, \mathrm{d}x\mathrm{d}y. \tag{2}$$

3. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at (1, -1, 2).

4. (a) A mass distribution in the positive x region of the xy-plane and in the shape of a semicircle of radius a, centred on the origin, has mass per unit area k. Find, using polar coordinates,
   (i) its mass M, (ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass, (iii) its moments of inertia about the x and y axes.
   (b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp[-(x^2 + y^2)/a^2] \text{ for x} \geq 0, \; \sigma = 0 \text{ for x} > 0.$$

   where a is a constant. Comment on the comparisons between the two sets of answers. Note that

$$\int_0^{\infty} \exp(-\lambda u^2) \, \mathrm{d}u = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$$

   (c) Evaluate the following integral:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \arctan(y/x) \, \mathrm{d}x\mathrm{d}y.$$

5. If X is a continuous random variable with probability density function (PDF) $f(x) = ce^{-x}$ for $x \geq 0$ and zero otherwise, (a) Find c (b) Find the cumulative distribution function $F(x)$ (c) Find $P(1 < X < 3)$

6. Let X and Y be two jointly continuous random variables with a joint PDF $f(x, y) = cx^2y$ for $0 \leq y \leq x \leq 1$ and zero otherwise. (a) Sketch the region in the (x, y) plane for which the PDF is non-zero (b) Find c (c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$ (d) Find $P(Y \leq X/2)$

PROBLEM SET 2

1. Find $\vec{\nabla}\,\phi$ in the cases: (a) $\phi = ln \mid \vec{r}\mid$; (b) $\phi = r^{-1}$, where $r = \mid \vec{r}\mid$.

2. If $F = x^2 z + e^{y/x}$ and $G = 2z^2 y - xy^2$, find $\vec{\nabla}(F+G)$ and $\vec{\nabla}(FG)$ at (1, 0, -2).

3. The pair of variables $(x,\, y)$ are each functions of the pair of variables $(u,\, v)$ and *vice versa*. Consider the matrices:
$$A = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$
(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.
(b) Verify this property explicitly for the case in which $(x,\, y)$ are Cartesian coordinates and u and v are the polar coordinates $(r,\, \theta)$.
(c) Assuming the result that the determinant of a matrix and the determine of its inverse are reciprocals, deduce the relation between the Jacobians
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

4. (a) Using the change of variable $x+y=u, x-y=v$ evaluate the double integral $\int_R (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$ where R is the region bounded by the straight lines $y=x$, $y=x+2$, $y=-x$ and $y=-x+2$.
(b) Given that $u = xy$ and $v = y/x$, show that $\partial(u,v)/\partial(x,y) = 2y/x$. Hence evaluate the integral
$$\int \exp(-xy)\,\mathrm{d}x\mathrm{d}y$$
over the region $x>0$, $y>0$, $xy<1$, $1/2 < y/x < 2$.

5. A vector field $\vec{A}(\vec{r})$ is defined by its components
$$(4x - y^4, -4xy^3 - 3y^2, 4).$$
Evaluate the line integral $\int \vec{A}\cdot d\vec{l}$ between the points with position vectors (0, 0, 0) and (1, 2, 0) along the following paths:
(a) the straight line from (0,0,0) to (1,2,0);
(b) on the path of straight lines joining (0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 2, 1) and (1, 2, 0) in turn.
Show that $\vec{A}$ is conservative and find a scalar function $V(\vec{r})$ such that $\vec{A} = \vec{\nabla}V$.

6. A vector field $\vec{A}(\vec{r})$ is defined by its components
$$(3x^2 + 6y, -14yz, 20xz^2).$$
Evaluate the line integral $\int \vec{A}\cdot d\vec{l}$ from (0, 0, 0) to (1, 1, 1) along the following paths
(a) $x=t$, $y=t^2$, $z=t^3$;
(b) on the path of straight lines joining (0, 0, 0), (1, 0, 0), (1, 1, 0) and (1, 1, 1) in turn;
(c) the straight line joining the two points.
Is $\vec{A}$ convservative?

7. Show that the surface area of the surved portion of a hemisphere of radius a is $2\pi a^2$ by
(a) directly integrating the element of area $a^2 \sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$ over the surface of the hemisphere.
(b) projecting onto an integral taken over the $x-y$ plane.

8. (a) Find the area of the plane $x - 2y + 5z = 13$ cut out by a cylinder $x^2 + y^2 = 9$.
(b) A uniform lamina is made of that part of the plane $x+y+z=1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for this area.

PROBLEM SET 3

1. Spherical polar coodinates are defined in the usual way. Show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta.$$

2. A solid hemisphere of uniform density k occupies the volume of $x^2 + y^2 + z^2 \leq a^2, z \geq 0$. Using symmetry arguments wherever possible, find
(a) its total mass M, (b) the position $(\overline{x}, \overline{y}, \overline{z})$ of its centre-of-mass, and (c) its moments and products of inertia, $I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}$ where

$$I_{zz} = \int k(x^2 + y^2)\,\mathrm{d}V \quad I_{xy} = -\int k\,xy\,\mathrm{d}V, \quad \text{etc}$$

3. Air is flowing with a speed 0.4 m s^{-1} in the direction of the vector (-1, -1, 1). Calculate the volume of air flowing per second through the loop which consists of straight lines joining, in turn, the following (1, 1, 0), (1, 0, 0), (0, 0, 0), (0, 1, 1). (1, 1, 1) and (1, 1, 0).

4. If $\vec{n}$ is the unit normal to the surface S, evaluate $\int \vec{r} \cdot \vec{n}\,\mathrm{d}S$ over (a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$; (b) the surface of a sphere of radius a centred at the origin.

5. Evaluate $\int \vec{A} \cdot \vec{n}\mathrm{d}S$ for the following cases:

   (a) $\vec{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

   (b) $\vec{A} = (x^2 + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

   (c) $\vec{A} = (6z, 2x + y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9,\ x = 0,\ y = 0,\ z = 0$ and $y = 8$.

6. The vector $\vec{A}$ is a function of position $\vec{r} = (x, y, z)$and has components(xy^2, x^2, yz). Calculate the surface integral $\int \vec{A} \cdot \mathrm{d}\vec{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = 1$. Show that the integral $\int \vec{A} \cdot \mathrm{d}\vec{S}$ taken outwards over the whole surface is not zero. Show that it equals $\int \vec{\nabla} \cdot \vec{A}\,\mathrm{d}V$ calculated over the volume of the prism. Why?

7. If $\vec{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find (a) $\vec{\nabla} \cdot \vec{A}$; (b) $\vec{A} \cdot \vec{\nabla}\phi$; (c) $\vec{\nabla} \cdot (\phi\vec{A})$; (d) $\vec{\nabla} \cdot (\vec{\nabla}\phi)$.

8. The magnetic field $\vec{B}$ at a distance r form a straight wire carrying a current I has a magnitude $\mu_0 I/(2\pi r)$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\vec{\nabla} \cdot \vec{B} = 0$.

PROBLEM SET 4

1. O is the origin and A, B, C are points with position vectors $\vec{a} = (1, 0, 0), \vec{b} = (1, 1, 1)$ and $\vec{c} = (0, 2, 0)$ respectively. Find the vector area $\vec{S}$ of the loop OABCO (a) by drawing the loop in projection onto the yz, zx and xy planes and calculating the components of $\vec{S}$, and (b) by filling the loop with (e.g. 2 or 3) plane polygons, ascribing a vector area to each and taking the resultant. Calculate the projected area of the loop (a) when seen from the direction which makes it appear as large as possible, and (b) when seen from the direction of the vector (0, -1, 1)? What are the corresponding answers for the loop OACBO?

2. Calculate the solid angle of a cone of half-angle α.

3. Sketch the vector fields $\vec{A} = (x, y, 0)$ and $\vec{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

4. The vector $\vec{A}(\vec{r}) = (y, -x, z)$. Verify Stokes' Theorem for the hemispherical surface $|\,\vec{r}\,| = 1, z \geq 0$.

5. $\vec{A} = (y, -x, 0)$. Find $\int \vec{A} \cdot d\vec{l}$ for a closed loop on the surface of the cylinder $(x-3)^2 + y^2 = 2$.

6. A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\vec{v}(\vec{r})$ as if it were a rigid body. Calculate $\vec{\nabla} \cdot \vec{v}$ and interpret the result. Calculate $\vec{\nabla} \times \vec{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\vec{v} = \vec{\nabla}\phi$? If so, what is ϕ?

7. Without using index notation (i.e. by writing out in components), prove: (a) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$; and (b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

8. If $\phi = 2xyz^2$, $\vec{F} = (xy, -z, x^2)$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals: (a) $\int_C \phi \, \mathrm{d}\vec{r}$; (b) $\int_C \vec{F} \times \mathrm{d}\vec{r}$.

9. To what scalar of vector quantities do the following sexpressions in index notation correspond (expand and sum where possible): $a_i b_j c_i$; $a_i b_j c_j d_i$; $\delta_{ij} a_i a_j$; $\delta_{ij} \delta_{ij}$; $\epsilon_{ijk} a_i b_k$; and $\epsilon_{ijk} \delta_{ij}$.

10. Use index notation to find the grad of the following scalar functions of the position $\vec{r} = (x, y, z)$: (a) $|\,\vec{r}\,|^n$, (b) $\vec{a} \cdot \vec{r}$.

11. Use index notation to find the div and curl of the following vector functions of position $\vec{r} = (x, y, z)$: (a) $\vec{r}$; (b) $|\,\vec{r}\,|^n\, \vec{r}$; (c) $(\vec{a} \cdot \vec{r})\vec{b}$; and (d) $\vec{a} \times \vec{r}$. Here, $\vec{a}$ and $\vec{b}$ are fixed vectors.

12. Use index notation to prove: (a) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$; and (b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ (i.e. repeat question 8 above by hopefully by a faster method).