Waves & Normal Modes

Matt Jarvis

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Contents

Chapter 1

Oscillations

Before we go into the main body of the course on waves and normal modes, it is useful to have a small recap on what we know about simple systems where we only have a single mass on a pendulum for example. This would all come under the remit of simple harmonic motion, which forms the basis of some of the problems that we will encounter in this course.

1.1 Simple Harmonic Motion - revision

First, consider Hooke's Law,

$$F = -kx, \tag{1.1}$$

where F is the force, x is the displacement with respect to the equilibrium position, and k is the constant of proportionality relating the two.

The usual aim is to solve for x as a function of time t in the oscillation of the spring or pendulum for example.

We know that $F = ma$, therefore

$$F = -kx = m\frac{d^2x}{dt^2}. \tag{1.2}$$

This equation tells us that we need to find a solution for which the second derivative is proportional to the negative of itself. We know that functions that obey this are the *sine*, *cosine* and exponentials. So we can try a fairly general solutions of the form,

$$x(t) = A\cos(\omega t + \phi) \qquad \text{or} \qquad x(t) = Ae^{i\omega t} \tag{1.3}$$

The phase ϕ just provides a linear shift on the time axis, the scale factor ω expands or contracts the curve on the time axis and the constant A gives the amplitude of the curve.

To check that this all works we can substitute Eq. 1.3 into Eq. 1.2, obtaining

$$-k\left[A\cos(\omega t+\phi)\right]=m\left[-\omega^2 A\cos(\omega t+\phi)\right] \tag{1.4}$$

$$\Longrightarrow (-k+m\omega^2)\left[A\cos(\omega t+\phi)\right]=0 \tag{1.5}$$

Since this equation must hold at all times, t, we must therefore have,

$$k-m\omega^2=0 \qquad \Longrightarrow \qquad \omega=\sqrt{\frac{k}{m}}. \tag{1.6}$$

You can also do this slightly more rigorously by writing the differential equation as

$$-kx=m.\frac{dv}{dt} \tag{1.7}$$

but this is awkward as it contains three variables, x, v and t. So you can't use the standard strategy of separation of variables on the two sides of the equation and then integrate. But we can write

$$F=ma=m.\frac{dv}{dt}=m.\frac{dx}{dt}.\frac{dv}{dx}=mv.\frac{dv}{dx}. \tag{1.8}$$

Which then leads to,

$$F=ma \qquad \Longrightarrow \qquad -kx=m\left(v\frac{dv}{dx}\right) \qquad \Longrightarrow \qquad -\int kx.dx=\int mv.dv \tag{1.9}$$

Integrating, we find

$$-\frac{1}{2}kx^2=\frac{1}{2}mv^2+E, \tag{1.10}$$

where the constant of integration, E happens to be the energy. It follows that

$$v=\pm\sqrt{\frac{2}{m}}\sqrt{E-\frac{1}{2}kx^2}, \tag{1.11}$$

which can be written as,

$$\frac{dx}{\sqrt{E}\sqrt{1-\frac{kx^2}{2E}}}=\pm\sqrt{\frac{2}{m}}\int dt. \tag{1.12}$$

A trig substitution turns the LHS into an arcsin or arccos function, and the result is

$$x(t) = A\cos(\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}} \tag{1.13}$$

which is the same result given in Eq. 1.3.

General solutions to Hooke's law can obviously also encompass combinations of trig functions and/or exponentials, for example,

$$x(t) = A\sin(\omega t + \phi) \tag{1.14}$$

$$x(t) = A\sin(\omega t + \phi) = A\cos\phi\sin\omega t + A\sin\phi\cos\omega t \tag{1.15}$$

therefore,

$$x(t) = A_1\sin\omega t + A_2\cos\omega t \tag{1.16}$$

is also a solution.

Finally, for the complex exponential solution

$$x(t) = Ce^{i\beta t} \tag{1.17}$$

$$F = -kx = Ce^{i\beta t} = m\frac{d^2x}{dt^2} = -m\beta^2 Ce^{i\beta t} \tag{1.18}$$

$$\Longrightarrow \beta^2 = \frac{k}{m} = \omega^2 \tag{1.19}$$

$$\Longrightarrow \quad x(t) = A'e^{i\omega t} + B'e^{-i\omega t} \tag{1.20}$$

Using Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.21}$$

$$x(t) = A'\cos\omega t + A'i\sin\omega t + B'\cos\omega t - B'i\sin\omega t \tag{1.22}$$

If $A = i(A' - B')$ and $B = (A' + B')$ then

$$x(t) = A\sin\omega t + B\cos\omega t \tag{1.23}$$

Chapter 2

Normal Modes

Many physical systems require more than one variable to quantify their configuration; for example a circuit may have two connected current loops, so one needs to know what current is flowing in each loop at each moment. Another example is a set of N coupled pendula each of which is a one-dimensional oscillator. A set of differential equations, one for each variable, will determine the dynamics of such a system. For a system of N coupled 1-D oscillators there exist N normal modes in which all oscillators move with the same frequency and thus have fixed amplitude ratios (if each oscillator is allowed to move in $x-$dimensions, then xN normal modes exist). The normal mode is for the whole system. Even though uncoupled angular frequencies of the oscillators are not the same, the effect of coupling is that all bodies can move with the same frequency. If the initial state of the system corresponds to motion in a normal mode then the oscillations continue in the normal mode. However, in general the motion is described by a linear combination of all the normal modes; since the differential equations are linear, such a linear combination is also a solution to the coupled linear equations.

A normal mode of an oscillating system is the motion in which all parts of the system move sinusoidally with the ***same frequency*** and with a ***fixed phase relation***.

The best way to illustrate the existence and nature of normal modes is to work through some examples, and to see what kind of motion is produced.

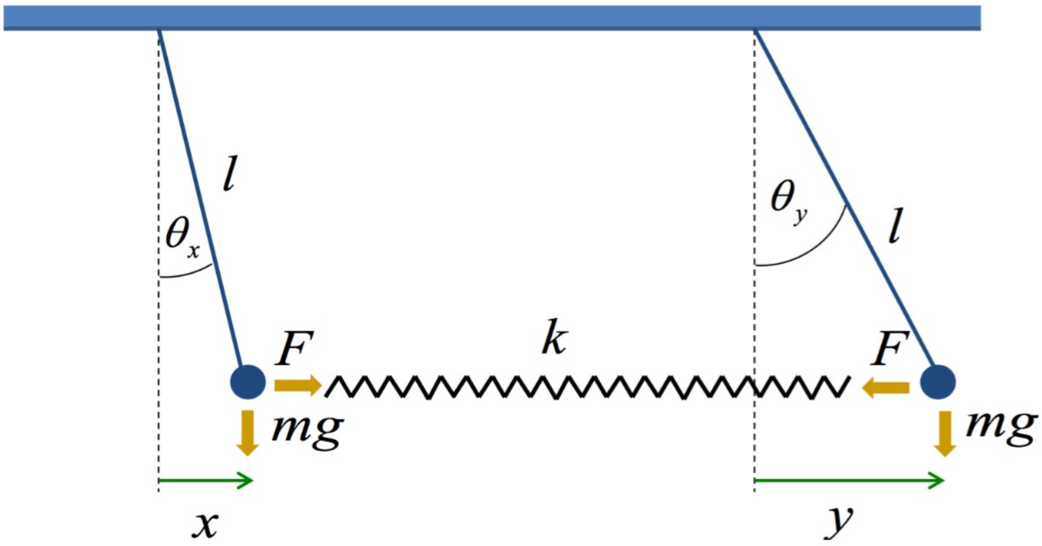

Figure 2.1: The Coupled Pendulum

2.1 The coupled pendulum

Rather than a single pendulum, now let us consider two pendula which are coupled together by a spring which is connected to the masses at the end of two thin strings. The spring has a spring constant of k and the length, l of each string is the same, as shown in Fig. 2.1

Unlike the simple pendulum with a single string and a single mass, we now have to define the equation of motion for the whole system together. However, we do this in exactly the same way as we would in any simple pendulum.

We first determine the forces acting on the first mass (left hand side). Like in the simple pendulum case, we assume that the displacements from the equilibrium positions are small enough that the restoring force due to gravity is given by $mg\sin\theta \approx mgx/l$ and acts tangentially to the arc of the pendulum. This force related to gravity produces the oscillatory motions if the pendulum is offset from the equilibrium position, i.e.

$$m\frac{d^2x}{dt^2} = m\ddot{x} = -mg\sin\theta_x \approx -mg\frac{x}{l}. \quad (2.1)$$

Likewise, for the second mass

$$m\frac{d^2y}{dt^2} = m\ddot{y} = -mg\sin\theta_y \approx -mg\frac{y}{l}. \quad (2.2)$$

However, in addition to this gravitational force we also have the force due to the spring that is connected to the masses. This spring introduces additional forces on the two masses, with the force acting in the opposite direction to the direction of the displacement, if we assume that the spring obeys Hooke's law, i.e.

$$F_{\text{spring}} = k(x - y), \tag{2.3}$$

then the equations of motion become modified:

$$\begin{aligned} m\ddot{x} = -mg\frac{x}{l} - F_{\text{spring}} &= -mg\frac{x}{l} - kx + ky \\ &= -mg\frac{x}{l} + k(y - x), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} m\ddot{y} = -mg\frac{y}{l} + F_{\text{spring}} &= -mg\frac{y}{l} - ky + kx \\ &= -mg\frac{y}{l} - k(y - x). \end{aligned} \tag{2.5}$$

We now have two equations and two unknowns. How can we solve these?

2.1.1 The Decoupling Method

The first method is quick and easy, but can only be used in relatively *symmetric* systems, e.g. where l and m are the same, as in this case. The underlying strategy of this method is to combine the equations of motion given in Eqs. 2.4 and 2.5 in ways so that x and y only appear in unique combinations.

In this problem we can simply try adding the equations of motion, which is one of the two useful combinations that we will come across.

$$\begin{aligned} m(\ddot{x} + \ddot{y}) = -mg\frac{x}{l} + k(y - x) - mg\frac{y}{l} - k(y - x) \\ = -\frac{mg}{l}(x + y) \quad . \end{aligned} \tag{2.6}$$

This type of equation should look familiar, where the left-hand side is a second derivative of the displacement with respect to time, and the right-hand side is a constant multiplied by a displacement in x and y. It becomes more clear if we define

$$q_1 = x + y \quad \text{and} \quad \ddot{q}_1 = \ddot{x} + \ddot{y}. \tag{2.7}$$

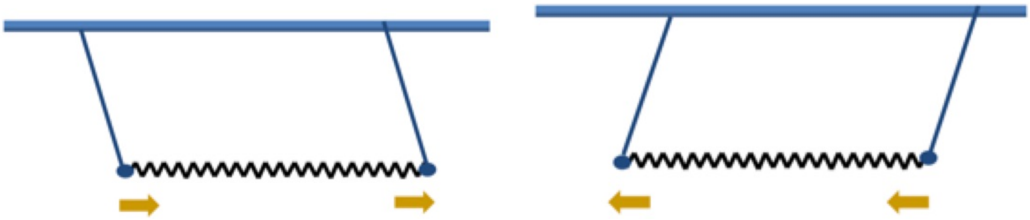

Figure 2.2: The centre of mass motion of the coupled pendulum as described by $q_1 = x + y$.

Eq. 2.6 then becomes,

$$m\ddot{q}_1 = -\frac{mg}{l}q_1 \quad \Longrightarrow \quad \ddot{q}_1 = -\frac{g}{l}q_1 \tag{2.8}$$

This is now easily recognisable as the the usual SHM equation, with $\ddot{q}_1 = -\omega^2 q_1$, where

$$\boxed{\omega_1 = \sqrt{\frac{g}{l}}.} \tag{2.9}$$

We can then immediately write a solution for the system as

$$q_1 = A_1 \cos(\omega_1 t + \phi_1), \tag{2.10}$$

where A_1 and ϕ_1 are arbitrary constants set by the initial or boundary conditions.

The motion described by $q_1 = x + y$ tells us about the coupled motion of the two pendula in terms of how they oscillate together around a centre of mass. There is no dependence on the spring whatsoever, i.e, it does not contract or expand, as ω_1 does not have a term which involves k. This motion can be visualised as shown in Fig. 2.2.

Given what we have learnt, the obvious other combination of x and y is to subtract Eq. 2.5 from Eq. 2.4:

$$\begin{aligned} m(\ddot{x} - \ddot{y}) &= -mg\frac{x}{l} + k(y - x) + mg\frac{y}{l} + k(y - x) \\ &= -m\left(\frac{g}{l} + \frac{2k}{m}\right)(x - y) \end{aligned} \tag{2.11}$$

As before, let us define a second coordinate,

$$q_2 = x - y \quad \text{and} \quad \ddot{q}_2 = \ddot{x} - \ddot{y}. \tag{2.12}$$

This gives us the following,

$$\ddot{q}_2 = -\left(\frac{g}{l} + \frac{2k}{m}\right)q_2. \tag{2.13}$$

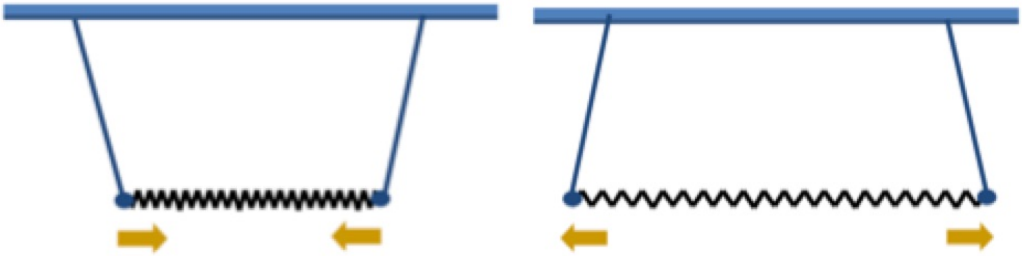

Figure 2.3: The relative motion of the coupled pendulum as described by $q_2 = x - y$.

We have SHM again, but this time with the q_2 coordinate, i.e. $\ddot{q_2} = -\omega^2 q_2$, where in this case

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}. \tag{2.14}$$

Again, we can write a solution immediately as

$$q_2 = A_2 \cos(\omega_2 t + \phi_2), \tag{2.15}$$

with A_2 and ϕ_2 arbitrary constants defined by the initial and boundary conditions.

In this case the q_2 represents the relative motion of the coupled pendulum. As should be clear from the dependence of ω_2 on the spring constant k, this motion must describe how the motion of the system depends on the compression and expansion of the spring.

The variables q_1 and q_2 are the ***modes*** or ***normal coordinates*** of the system. In any normal mode, only one of these coordinates is active at any one time. However, in a coupled system both of these normal modes can be *excited.*

It is actually more common to define the normal coordinates with a normalising factor of $1/\sqrt{2}$, such that

$$q_1 = \frac{1}{\sqrt{2}}(x + y) \quad \text{and} \quad q_2 = \frac{1}{\sqrt{2}}(x - y). \tag{2.16}$$

The factor of $1/\sqrt{2}$ is chosen to give the standard form for the kinetic energy in terms of the normal modes. We will come to this later.

We now have the two normal modes that describe the system, and the general solution of the coupled pendulum is just the sum of these two normal modes.

$$q_1 + q_2 = x + y + x - y = 2x$$

which, using Eqs. 2.10 and 2.15, and incorporating the factor of 2 into A_1 and A_2, leads to,

$$x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \tag{2.17}$$

and

$$q_1 - q_2 = x + y - x + y = 2y$$

which leads to,

$$y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \tag{2.18}$$

The constants are then just set by the initial conditions.

2.1.2 The Matrix Method

This method is a bit more involved but in principle it can be used to solve any set up. But first of all we will use the example of the coupled pendulum shown in Fig. 2.1. One strategy that we can use is to look for simple kinds of motion where the masses all move with the same frequency, building up to a general solution by combining these simple kinds of motion. So in the matrix method we start by having a guess at the solutions to the system.

By rearranging Eqs. 2.4 and 2.5, we find

$$\begin{aligned} m\ddot{x} + \frac{mg}{l}x + kx - ky = 0 \\ = \ddot{x} + x\left(\frac{g}{l} + \frac{k}{m}\right) - \frac{k}{m}y = 0 \end{aligned} \tag{2.19}$$

and similarly,

$$\ddot{y} + y\left(\frac{g}{l} + \frac{k}{m}\right) - \frac{k}{m}x = 0. \tag{2.20}$$

We can write this as a matrix in the following way,

$$\begin{bmatrix} \frac{d^2}{dt^2} + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \left(\frac{g}{l} + \frac{k}{m}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.21}$$

Given this equation, we could multiply both sides of the equation by the inverse of the matrix, which would lead to $(x, y) = (0, 0)$. This is obviously a solution and would mean that the pendula just hang there and do not move. However, we want to find a general solution which would involve describing how the masses move, not just when they are stationary.

We are expecting oscillatory solutions, so let's just try one, such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t}, \tag{2.22}$$

where X and Y are complex constants (although we are just interested in the real ($\Re$) solutions). Substituting this trial solution into Eq. 2.21 we find,

$$\begin{bmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.23}$$

This is the **Eigenvector Equation** with $-\omega^2$ being the **eigenvalues**.

Therefore we need to find the non-trivial solutions, and the only way to escape the trivial solution is to ensure that both X and Y must be zero when the inverse of the matrix does not exist.

To find the inverse of a matrix involves finding cofactors and determinants, which can be a bit messy. However, the key thing to remember is that the inverse of a matrix $\mathbf{A}$ is given by,

$$\mathbf{A^{-1}} = \frac{1}{|\mathbf{A}|} \times \mathbf{C^T}, \tag{2.24}$$

where $|\mathbf{A}|$ is the determinant of the matrix $\mathbf{A}$ and $\mathbf{C^T}$ is the transposed matrix of the cofactors.

So determining the inverse of a matrix always involves dividing by the determinant of that matrix. Therefore, if the determinant is zero then the inverse does not exist. This is what we want in order to solve Eqn. 2.23. So setting the determinant to zero,

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0. \tag{2.25}$$

we find the solution to be

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \tag{2.26}$$

therefore,

$$\begin{aligned} \left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \\ \implies -\omega^2 + \frac{g}{l} + \frac{k}{m} = \pm\frac{k}{m} \end{aligned} \tag{2.27}$$

So we get two solutions for ω,

$$\boxed{\omega_1^2 = \frac{g}{l} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + \frac{2k}{m}} \tag{2.28}$$

The $\pm$ ambiguity in ω is ignored as you get sinusoidal solutions.

To complete the solution we substitute the values for ω back into the eigenvector equation.

For $\omega_1^2 = g/l$,

$$\begin{bmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.29}$$

which leads to $X_1 = Y_1$. So (X, Y) is proportional to the vector $(1, 1)$.

For $\omega_2^2 = g/l + 2k/m$,

$$\begin{bmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.30}$$

which leads to $X_2 = -Y_2$. So (X, Y) is proportional to the vector $(1, -1)$.

We can now write the general solution as the sum of the two solutions that we have found,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = X_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_1 t} + X_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_2 t} \tag{2.31}$$

X_1 and X_2 are complex constants, lets define them as $A_1 e^{i\phi_1}$ when $X = Y$, and $A_2 e^{i\phi_2}$ when $X = -Y$. Substituting back into Eq. 2.31, we find

$$x = A_1 e^{i\phi_1} e^{i\omega_1 t} + A_2 e^{i\phi_2} e^{i\omega_2 t} \tag{2.32}$$

$$y = A_1 e^{i\phi_1} e^{i\omega_1 t} - A_2 e^{i\phi_2} e^{i\omega_2 t} \tag{2.33}$$

Using Euler's formula (Eq. 1.21) and just considering the real components and removing the imaginary parts,

$$x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \tag{2.34}$$

$$y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \tag{2.35}$$

which are the same normal modes and frequencies that we found from the decoupling method. These are the general solutions of a coupled pendulum.

The advantage of using the complex exponential is only evident if there is a mixture of single and double derivatives as in the case of a damped pendulum discussed later. In the undamped case just discussed it would be equally simple to start with a normal mode trial solution proportional to $\cos(\omega t + \phi)$.

2.1.3 Initial conditions and examples

In this section we will go through an example using initial conditions, and the motions that these produce in the coupled pendulum. Further examples will be given in the lectures.

Let's consider the case where, $x(t = 0) = a$, $y(t = 0) = 0$ and the initial velocity of the masses $\dot{x} = \dot{y} = 0$. Before we go into the maths, what does this look like? We know from SHM of a single pendulum that the velocity of the pendulum is zero at its maximum displacement from equilibrium. But in this case the second pendulum that is displaced in the $y-$direction starts at it's equilibrium position. This must mean that the spring connecting the two is either stretched or compressed, depending on the definition of the direction of x and y. It is useful to sketch these initial conditions so that you know roughly what to expect in terms of the excitation of the normal modes.

Using our solutions to the coupled pendulum from Eqs. 2.17 and 2.18, for $t = 0$ we get

$$\begin{aligned} x(0) &= A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = a \\ y(0) &= A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = 0 \end{aligned} \tag{2.36}$$

Adding the Eqs. 2.36 together we find $2A_1 \cos\phi_1 = a$, subtracting we find $2A_2 \cos\phi_2 = a$. So A_1 and A_2 must have non-zero solutions.

We also have information about the initial velocity of the two masses. This therefore leads us to looking at the first differential of Eqs. 2.17 and 2.18 with respect to time. At time $t = 0$,

$$\begin{aligned} \dot{x}(0) &= -A_1\omega_1 \sin(\phi_1) - A_2\omega_2 \sin(\phi_2) = 0 \\ \dot{y}(0) &= -A_1\omega_1 \sin(\phi_1) + A_2\omega_2 \sin(\phi_2) = 0 \end{aligned} \tag{2.37}$$

Again adding and subtracting Eqs. 2.37, we find $A_1 \sin\phi_1 = 0$ and $A_2 \sin\phi_2 = 0$. As both A_1 and A_2 have non-zero values, then $\phi_1 = \phi_2 = 0$. Therefore, $2A_1 \cos\phi_1 = a = 2A_1$ and $2A_2 \cos\phi_2 = a = 2A_2$, thus $A_1 = A_2 = a/2$.

Substituting these back into Eqs. 2.17 and 2.18, we find

$$x = \frac{a}{2}(\cos\omega_1 t + \cos\omega_2 t) \tag{2.38}$$

$$y = \frac{a}{2}(\cos\omega_1 t - \cos\omega_2 t) \tag{2.39}$$

In this case both normal modes are excited as $q_1 = (x + y) \neq 0$ and $q_2 = (x - y) \neq 0$. As you will see in lectures, we also have cases where only one normal mode is active.

But let us continue with this system and try and see what it is actually doing. We can rewrite Eqs. 2.38 and 2.39 using some trig identities as,

$$x = a \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \tag{2.40}$$

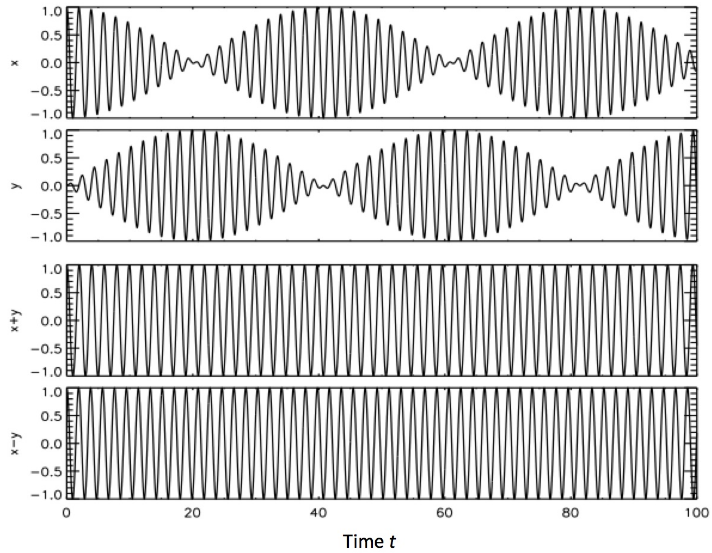

Figure 2.4: Oscillatory pattern for a system where both normal modes are excited.

$$y = -a \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right) \tag{2.41}$$

This shows that we have two distinct frequencies that the system is oscillating in. A high-frequency mode where the frequency is $(\omega_1 + \omega_2)/2$ and a low-frequency mode with $(\omega_1 - \omega_2)/2$. This means that if we were to visualise the oscillations in x and y as a function of time, then we would see a broad envelope defined by the low-frequency term, but within this envelope we see much higher frequency oscillations (Fig. 2.4).

We can also calculate the period of the oscillations in the usual way. For example the period of the envelope is

$$T_{\mathrm{env}} = \frac{2\pi}{\omega} = \frac{4\pi}{\omega_1 - \omega_2}. \tag{2.42}$$

Here we have a 'beat' frequency, where one complete period of the envelope is equal to 2 beats.

'Beats' is when energy is transferred between pendula.

2.1.4 Energy of a coupled pendulum

So we have now found the solutions to the coupled pendulum in terms of how the two pendula oscillate with time. We can also determine how the energy in the system varies between different types of energy, the two main forms of energy being kinetic and potential energy.

We know that for a perfect system with no friction, then the total energy of the system is given by the sum of the potential and kinetic energies, $U = KE + PE = T + V$.

The total kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \tag{2.43}$$

and the potential energy comes from two sources in the coupled pendulum, namely the spring and gravity.

$$V_{\text{spring}} = \frac{1}{2}k(y - x)^2 \tag{2.44}$$

$$V_{\text{gravity}} = mgh = mg(l - l\cos\theta) = lmg(1 - \cos\theta) \tag{2.45}$$

Using $2\sin^2\theta = 1 - \cos 2\theta$ and $\sin\theta = x/l$ we find for the x-pendulum

$$mgh = \frac{mgl}{2l^2}x^2 \tag{2.46}$$

and for the y-pendulum

$$mgh = \frac{mgl}{2l^2}y^2 \tag{2.47}$$

so that

$$V_{\text{gravity}} = \frac{mg}{2l}(x^2 + y^2). \tag{2.48}$$

The total potential energy is just the sum of the gravitational and spring components,

$$V_{\text{total}} = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy. \tag{2.49}$$

This is not the only way to find the total energy of the system. We can also use our knowledge of how the energy is related to the force, and therefore the equations of motion.

We know,

$$F_x = -\frac{\partial V_x}{\partial x} \quad \text{and} \quad F_y = -\frac{\partial V_y}{\partial y}, \tag{2.50}$$

therefore

$$\begin{aligned} F_x &= -\frac{\partial V_x}{\partial x} = m\ddot{x} = -mg\frac{x}{l} + k(y-x) \\ F_y &= -\frac{\partial V_y}{\partial y} = m\ddot{y} = -mg\frac{y}{l} - k(y-x). \end{aligned} \tag{2.51}$$

Integrating these equations we find,

$$\begin{aligned} V(x,y) &= mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y) + C \\ V(x,y) &= mg\frac{y^2}{2l} + \frac{1}{2}ky^2 - kxy + f(x) + C \end{aligned} \tag{2.52}$$

Neglecting the arbitrary constant C, as this just relates to how we define the zero potential, then we find that the total potential energy is

$$V_{\text{total}} = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy. \tag{2.53}$$

as before.

So we have expressions for the kinetic energy and the potential energy, so we can now write the total energy of the system as

$$U = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy \tag{2.54}$$

but this doesn't really convey much about what is going on in terms of the energy associated with each normal mode. So why don't we go back and rewrite this in terms of the normal coordinates q_1 and q_2, where we defined

$$q_1 = \frac{1}{\sqrt{2}}(x+y) \quad \text{and} \quad q_2 = \frac{1}{\sqrt{2}}(x-y). \tag{2.55}$$

and found that

$$\omega_1 = \sqrt{\frac{g}{l}}. \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}. \tag{2.56}$$

Substituting these into Eq. 2.54, and with a bit of rearranging, we obtain

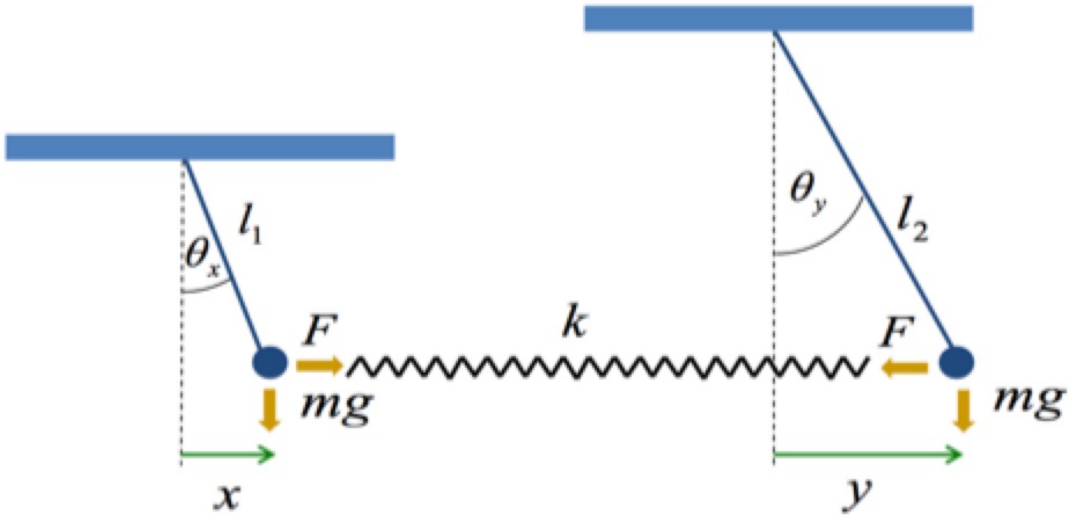

Figure 2.5: The Unequal Coupled Pendulum

$$U = \left(\frac{1}{2}m\dot{q_1}^2 + \frac{1}{2}m\omega_1^2 q_1^2\right) + \left(\frac{1}{2}m\dot{q_2}^2 + \frac{1}{2}m\omega_2^2 q_2^2\right). \tag{2.57}$$

The first term here is then the energy associated with the first normal mode and the second term is the energy associated with the second normal mode. Therefore the total energy of the system is simply the sum of the energies in each mode.

We have gone through most of the mathematics that we will use when looking at coupled systems, although we have started with the simplest example. Let us now have a look at a slightly more complicated system, where the two pendula that are coupled by a spring are no longer the same.

2.2 Unequal Coupled Pendula

In Fig. 2.5 we show a system where there are still two pendula coupled by a spring, with spring constant k, but this time the length of the pendula are different, with the left-hand pendulum having length l_1 and the right-hand pendulum having a length of l_2.

This alters the equations of motion and therefore also the solutions to the system. So the equations of motions of this system are similar to Eqs. 2.4 and 2.5, and are:

$$m\ddot{x} = -mg\frac{x}{l_1} + k(y-x), \tag{2.58}$$

and

$$m\ddot{y} = -mg\frac{y}{l_2} - k(y-x). \tag{2.59}$$

Unlike in the previous coupled pendulum, this system is not symmetric. Therefore let us go straight to the matrix method in order to solve for x and y.

As before, we equate Eqs. 2.58 and 2.59 to zero and write them in a matrix format.

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l_2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.60}$$

As before, we define x and y in terms of complex constants and assume an oscillatory solution of the form $e^{i\omega t}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t}, \tag{2.61}$$

and substituting to form the Eigenfunction equation,

$$\begin{bmatrix} -\omega^2 + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l_2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.62}$$

Again we have the trivial solution for $x = y = 0$, but for the non-trivial solutions we adopt the same process and set the determinant of the matrix to zero, i.e.

$$\begin{vmatrix} -\omega^2 + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l_2} + \frac{k}{m} \end{vmatrix} = 0. \tag{2.63}$$

Calculating the determinant, we find

$$\left(-\omega^2 + \frac{g}{l_1} + \frac{k}{m}\right)\left(-\omega^2 + \frac{g}{l_2} + \frac{k}{m}\right) - \left(\frac{k}{m}\right)^2 = 0 \tag{2.64}$$

Expanding this and solving for ω^2 gives the following,

$$\omega_{1,2}^2 = \frac{1}{2}\left[(\beta_1^2+\beta_2^2) + \frac{2k}{m} \pm \sqrt{(\beta_1^2-\beta_2^2)^2 + \left(\frac{2k}{m}\right)^2}\right] \tag{2.65}$$

where $\beta_{1,2}^2 = \frac{g}{l_{1,2}}$.

We can check whether this agrees with the previous solution for the equal coupled pendulum from Section 2.1, by setting $l_1 = l_2 = l$ and checking that we get the same values for ω_1 and ω_2 that are given in Eq. 2.28.

So now that we have the equations for ω_1 and ω_2, we can find $x(t)$ and $y(t)$ in the usual way, i.e. by substituting Eq. 2.65 into the matrix given in 2.62. Doing this, we find

$$\left(\frac{Y_1}{X_1}\right)_{\pm} = \frac{-\omega_{\pm}^2 + \beta_1^2 + (k/m)}{k/m} \tag{2.66}$$

$$\left(\frac{X_2}{Y_2}\right)_{\pm} = \frac{-\omega_{\pm}^2 + \beta_2^2 + (k/m)}{k/m} \tag{2.67}$$

Substituting in for $\omega_{\pm}$ we find,

$$\left(\frac{Y_1}{X_1}\right)_{\pm} = \frac{m}{2k}\left[(\beta_1^2-\beta_2^2) \pm \sqrt{(\beta_1^2-\beta_2^2)^2 + (2k/m)^2}\right] \tag{2.68}$$

$$\left(\frac{X_2}{Y_2}\right)_{\pm} = -\frac{m}{2k}\left[(\beta_1^2-\beta_2^2) \pm \sqrt{(\beta_1^2-\beta_2^2)^2 + (2k/m)^2}\right] \tag{2.69}$$

and therefore

$$\left(\frac{Y_1}{X_1}\right)_{+} = -1\left(\frac{X_2}{Y_2}\right)_{-} \tag{2.70}$$

similarly,

$$\left(\frac{Y_1}{X_1}\right)_{-} = -1\left(\frac{X_2}{Y_2}\right)_{+} \tag{2.71}$$

and we can define this ratio of the amplitudes,

$$r = \left(\frac{Y_1}{X_1}\right) = \frac{m}{2k}\left[(\beta_1^2-\beta_2^2) + \sqrt{(\beta_1^2-\beta_2^2)^2 + (2k/m)^2}\right] \tag{2.72}$$

therefore $rX_1 = Y_1$ and $X_2 = -rY_2$.

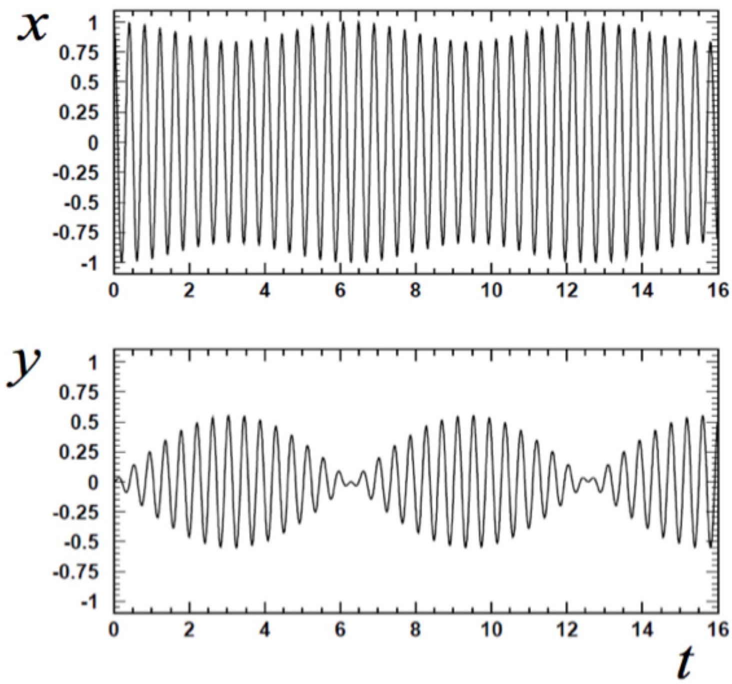

Figure 2.6: Oscillatory pattern for the unequal coupled pendulum. Both normal modes are excited again and a 'Beats' solution is apparent but in this case with $r < 1$ there is an incomplete transfer of energy between the pendula.

We can therefore write the general solution as a function of this amplitude ratio,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ r \end{bmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{bmatrix} -r \\ 1 \end{bmatrix} A_2 \cos(\omega_2 t + \phi_2) \tag{2.73}$$

So again we can set up some initial conditions and solve for these. As we did in Sec. 2.1 let us consider $x(t=0) = a$; $y(t=0) = 0$; $\dot{x}(t=0) = \dot{y}(t=0) = 0$, to see what difference the unequal pendulum length makes.

Without going through all of the maths (which you should try), we find that $A_1 = a/(1+r^2)$; $A_2 = -ra/(1+r^2)$; $\phi_1 = \phi_2 = 0$.

Hence, substituting these in to the general solution we find

$$x(t) = a(\cos\omega_1 t + r^2 \cos\omega_2 t)/(1+r^2) \tag{2.74}$$

$$y(t) = ar(\cos\omega_1 t - \cos\omega_2 t)/(1+r^2) \tag{2.75}$$

which can be written in terms of the two active frequencies $(\omega_1 - \omega_2)/2$ and $(\omega_1 + \omega_2)/2$ as before. Doing this we get

$$x(t) = a\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) - a\left(\frac{1-r^2}{1+r^2}\right)\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right) \tag{2.76}$$

$$y(t) = -2a\left(\frac{r}{1+r^2}\right)\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right) \tag{2.77}$$

So by having unequal pendula we introduce additional terms into the amplitude of the oscillations of each normal mode. If you think about the coupled pendulum and how it would oscillate this is unsurprising.

As a check that these solutions reduce to the solution given in Sec. 2.1.3 for when the pendula have equal lengths, we can set $\beta_1 = \beta_2$, which results in $r = 1$. Substituting this into Eqs. 2.76 and 2.77, we find that they reduce to the solutions given in Eqs. 2.40 and 2.41.

2.3 The Horizontal Spring-Mass system

We now move away from pendula and consider other systems which produce normal modes (although we will return to the pendula in Sec. 2.7). Here we look at the horizontal spring-mass system, where three springs with spring constant αk, k, and αk are connecting two masses of mass m to two fixed points at either end, as shown in Fig. 2.7.

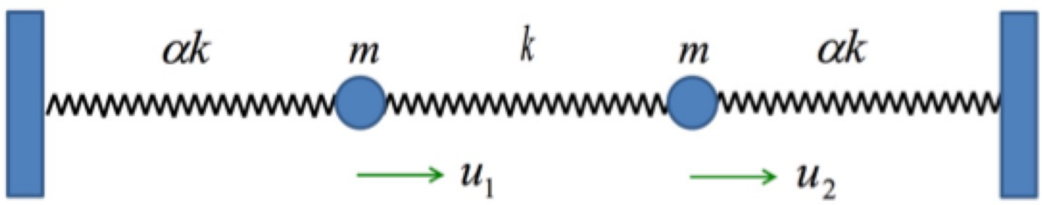

Figure 2.7: The horizontal spring-mass system.

As usual we first set up the equations of motion:

$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2) \tag{2.78}$$

$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2) \tag{2.79}$$

In this case the coordinates of the two masses are denoted by u_1 and u_2. We have what looks like a symmetrical system, so we can probably use the decoupling method. So let's use this first and then check that it all looks fine with the matrix method.

2.3.1 Decoupling method

In the decoupling method we define new coordinates which describe the coupled motion of the two masses, with the usual coordinates being $u_1 + u_2$ and $u_1 - u_2$, along with a normalising factor of $1/\sqrt{2}$ that ensures that the energies all work out as expected. So as before, we have

$$q_1 = \frac{1}{\sqrt{2}}(u_1 + u_2) \quad \text{and} \quad q_2 = \frac{1}{\sqrt{2}}(u_1 - u_2) \tag{2.80}$$

and the acceleration for these new coordinates defined as,

$$\ddot{q}_1 = \frac{1}{\sqrt{2}}(\ddot{u}_1 + \ddot{u}_2) \quad \text{and} \quad \ddot{q}_2 = \frac{1}{\sqrt{2}}(\ddot{u}_1 - \ddot{u}_2). \tag{2.81}$$

As before, we add Eqs. 2.78 and 2.79,

$$\begin{aligned} m(\ddot{u}_1 + \ddot{u}_2) = m\ddot{q}_1 = -\alpha k(u_1 + u_2) - ku_1 + ku_2 + ku_1 - ku_2 \\ = -\alpha k(u_1 + u_2) \end{aligned} \tag{2.82}$$

$$\Longrightarrow \ddot{q}_1 = \frac{-\alpha k}{m} q_1 \tag{2.83}$$

Then subtract Eq. 2.79 from 2.78

$$\begin{aligned} m(\ddot{u}_1 - \ddot{u}_2) = m\ddot{q}_2 = -\alpha k(u_1 - u_2) - ku_1 + ku_2 - ku_1 + ku_2 \\ = -(\alpha + 2)k(u_1 - u_2) \end{aligned} \tag{2.84}$$

$$\Longrightarrow \ddot{q}_2 = -\frac{(\alpha + 2)k}{m} q_2 \tag{2.85}$$

We can see that by using the decoupling method we have immediately found two equations that have the usual SHM structure, as such we can easily obtain the values for ω:

$$\omega_1^2 = \frac{\alpha k}{m} \quad \text{and} \quad \omega_2^2 = \frac{\alpha + 2}{m} k. \tag{2.86}$$

2.3.2 The Matrix Method

As stated earlier, the matrix method can be used for all systems, not only symmetric systems, so we can check that what we obtain using the decoupling method is in agreement with an independent method (this is basically just practice in using the Matrix Method).

So we follow exactly the same process as we did for the coupled pendula, and

1) write out the equations of motion with a homogeneous matrix equation:

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{\alpha k}{m} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{\alpha k}{m} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.87}$$

2) Substitute in the trial solution:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t}. \tag{2.88}$$

$$\begin{bmatrix} -\omega^2 + \frac{\alpha k}{m} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{\alpha k}{m} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.89}$$

3) Demand that the resulting operator matrix is singular, i.e. $|\mathbf{A}| = 0$:

$$\begin{vmatrix} -\omega^2 + \frac{\alpha k}{m} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{\alpha k}{m} + \frac{k}{m} \end{vmatrix} = 0 \tag{2.90}$$

4) So calculate the determinant:

$$\left(-\omega^2 + \frac{\alpha k}{m} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \tag{2.91}$$

5) Equate to find ω. In this case we can factorise,

$$\left(\omega^2 - \frac{\alpha k}{m}\right)\left(\omega^2 - \frac{(\alpha + 2)k}{m}\right) = 0 \tag{2.92}$$

So we have the same solutions as we found using the decoupling method.

$$\omega_1^2 = \frac{\alpha k}{m} \quad \text{and} \quad \omega_2^2 = \frac{\alpha + 2}{m} k \tag{2.93}$$

6) To complete the solution we substitute the values for ω back into the eigenvector equation (Eq. 2.89).

For $\omega_1^2 = \alpha k/m$:

$$\begin{bmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.94}$$

For $\omega_2^2 = (\alpha + 2)k/m$:

$$\begin{bmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.95}$$

These are exactly the same as we found for the coupled pendulum in Sec. 2.1, with $X = Y$ and $X = -Y$, we therefore have the same form for the general solution:

$$x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \tag{2.96}$$

$$y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \tag{2.97}$$

2.3.3 Energy of the horizontal spring-mass system

Unlike in the case of the coupled pendulum, we do not have any gravitational component to the potential energy, and all energy in this system is contained within the springs. So let us now just follow through the same method as we used for the coupled pendulum and determine the total energy of the system.

Again using

$$q_1 = \frac{1}{\sqrt{2}}(u_1 + u_2) \quad \text{and} \quad q_2 = \frac{1}{\sqrt{2}}(u_1 - u_2).$$

and

$$\dot{q}_1 = \frac{1}{\sqrt{2}}(\dot{u}_1 + \dot{u}_2) \quad \text{and} \quad \dot{q}_2 = \frac{1}{\sqrt{2}}(\dot{u}_1 - \dot{u}_2).$$

The total kinetic energy of the system is simply:

$$K = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2) = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) \tag{2.98}$$

For the potential energy, we have the component for the two springs which are fixed to the wall, and a relative displacement term for the middle spring.

$$V = \frac{1}{2}\alpha k u_1^2 + \frac{1}{2}k(u_2 - u_1)^2 + \frac{1}{2}\alpha k u_2^2 \tag{2.99}$$

Substituting in q_1 and q_2, we find

$$V = \frac{1}{2}\alpha k q_1^2 + \frac{1}{2}k q_2^2(\alpha + 2) \tag{2.100}$$

Using the values for $\omega_{1,2}^2$ that we found in the last section $\omega_1^2 = \alpha k/m$ and $\omega_2^2 = (\alpha + 2)k/m$:

$$V = \frac{1}{2}m\omega_1^2 q_1^2 + \frac{1}{2}m\omega_2^2 q_2^2 \tag{2.101}$$

Finally, combining the Kinetic Energy and the Potential Energy, we find the total energy

$$U = \left(\frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\omega_1^2 q_1^2\right) + \left(\frac{1}{2}m\dot{q}_2^2 + \frac{1}{2}m\omega_2^2 q_2^2\right) \tag{2.102}$$

Again, the sum of the energies in each normal mode.

2.3.4 Initial Conditions

As with the coupled pendulum it is possible to use the general solution to find the motion of the system given a set of initial conditions. If one were to use the same set of initial conditions as stated in Sec. 2.1.3 then we also find a similar solution here, i.e.

$$u_1 = a\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right), \tag{2.103}$$

$$u_2 = a \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right), \tag{2.104}$$

where both normal modes are excited and we have a 'Beats' solution, defined by the low-frequency envelope $(\omega_1 - \omega_2)/2$.

2.4 Vertical spring-mass system

The final ideal oscillating system that we are going to look at is the vertical spring-mass system (Fig. 2.8).

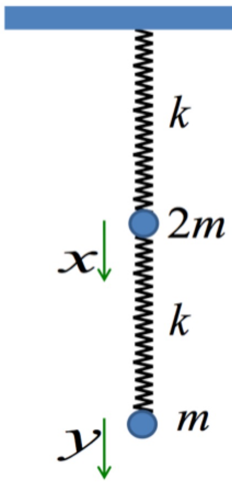

Figure 2.8: The vertical spring-mass system

As usual the first thing that we need to do is to write down the equations of motion.

$$2m\ddot{x} = -kx - k(x - y) = k(y - 2x) \tag{2.105}$$

$$m\ddot{y} = -ky + kx = k(x - y) \tag{2.106}$$

One thing to notice here is that we have unequal masses, this probably means that the decoupling method is not going to work (try it for yourself). So let us skip straight to using the matrix method.

2.4.1 The matrix method

Write the equations of motion as a homogeneous matrix equation:

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.107}$$

Following the previous examples, substitute in the trial solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t}. \tag{2.108}$$

$$\begin{bmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & -\omega^2 + \frac{k}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.109}$$

Demand that the resulting operator matrix is singular, i.e. $|\mathbf{A}| = 0$ and get the eigenvalue equation:

$$\left(-\omega^2 + \frac{k}{m}\right)^2 - \frac{1}{2}\left(\frac{k}{m}\right)^2 = 0 \tag{2.110}$$

From this we can easily show, using the equation to solve a quadratic, that the normal frequencies ω_1 and ω_2 are

$$\omega_{1,2}^2 = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right) \tag{2.111}$$

We can now obtain the normal modes of the vertical spring-mass system by substituting the values for $\omega_{1,2}$ in to the eigenvector equation.

$$\left(-\omega_{1,2}^2 + \frac{k}{m}\right) X - \left(\frac{k}{2m}\right) Y = 0, \tag{2.112}$$

For normal mode 1, (ω_1) yields, $X/Y = -1/\sqrt{2}$, and for normal mode 2 (ω_2) yields $X/Y = 1/\sqrt{2}$.

We can easily visualise the motion that these two normal modes represent, as they are again analogous to the coupled pendulum, with one representing the centre of mass motion, where both X and Y are moving in the same direction, or relative motion where they are moving in opposite directions (Fig. 2.9).

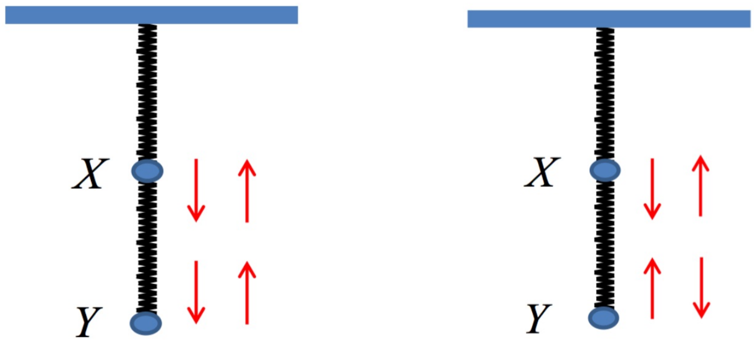

Figure 2.9: The vertical spring-mass system normal modes. On the left-hand side is the centre of mass motion described by $X/Y = 1/\sqrt{2}$ and on the right-hand side is the relative motion described by $X/Y = -1/\sqrt{2}$.

2.5 Interlude: Solving inhomogeneous differential equations

In the next section we will tackle a problem which involves dealing with an inhomogeneous 2nd order differential equation, where the simple solution, obtained by setting the determinant of the matrix is zero, only makes up a part of the general solution. As a precursor to this, in this section we will go through solving a simple pair of differential equations that have derivatives in both the x and y coordinates.

Let us consider the problem:

$$\frac{dx}{dt} + \frac{dy}{dt} + y = t \tag{2.113}$$

$$-\frac{dy}{dt} + 3x + 7y = e^{2t} - 1 \tag{2.114}$$

We first want to find the complementary function (CF). This is analogous to what we have done previously, and provides us with one set of solutions to the above equations, namely when the RHS is equal to zero.

To get the CF we write these equations in matrix format and set the RHS to zero:

$$\begin{bmatrix} \frac{d}{dt} & \frac{d}{dt} + 1 \\ 3 & 7 - \frac{d}{dt} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.115}$$

We then try a solution to this, as usual let's try $x = Xe^{\omega t}$ and $y = Ye^{\omega t}$ (note that I've omitted the complex form for this, as we are just looking at the first order differential equation). We substitute these into the matrix and find the determinant and set it equal to zero.

$$\begin{vmatrix} \omega & \omega+1 \\ 3 & 7-\omega \end{vmatrix} = 0 \tag{2.116}$$

From this we find the eigenvalues $\omega = 1$ and $\omega = 3$

In the usual way, we then substitute these values of ω back into the eigenvector equation to find the relation between X and Y.

After a bit of simple maths we find that for $\omega = 1$, $X = -2Y$ and for $\omega = 3$, $X = -4Y/3$.

Hence the CF is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = X_a \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} e^t, \quad \text{for} \ \ \omega = 1 \tag{2.117}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = X_b \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} e^{3t}, \quad \text{for} \ \ \omega = 3 \tag{2.118}$$

But the RHS is not actually equal to zero, and the CF only makes up a part of the general solution. In this case we need to find the particular integral (PI) as well.

So from the differential equations Eqns. 2.113 and 2.114, we know that the general solution must have a linear solution as Eq. 2.113 is related to t, and an exponential solution from Eq. 2.114. Let's deal with the linear part first and try,

$$\begin{bmatrix} \frac{d}{dt} & \frac{d}{dt}+1 \\ 3 & 7-\frac{d}{dt} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -1 \end{bmatrix} \tag{2.119}$$

and try the following as solutions,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{bmatrix}. \tag{2.120}$$

With this trial solution $\frac{dx}{dt} = X_1$ and $\frac{dy}{dt} = Y_1$, therefore substituting these into Eq. 2.119, we obtain,

$$X_1 + Y_1 + Y_0 + Y_1 t = t$$

Equating the terms with and without t, it follows that

$$Y_1 = 1 \quad \text{and} \quad X_1 + Y_1 + Y_0 = 0 \implies X_1 + Y_0 = -1 \tag{2.121}$$

From the second differential equation, we also have

$$3(X_0 + X_1 t) + 7(Y_0 + Y_1 t) - Y_1 = -1 \tag{2.122}$$

it therefore follows that

$$3X_0 + 7Y_0 = 0 \quad \text{and} \quad 3X_1 + 7Y_1 = 0 \implies X_1 = -\frac{7}{3} \tag{2.123}$$

and $Y_0 = -1 + \frac{7}{3} = \frac{4}{3}$ and $X_0 = -\frac{7}{3}Y_0 = -\frac{28}{9}$.

We therefore get,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{bmatrix} \tag{2.124}$$

Finally, we now have to look at the exponential term. Let's try,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} e^{2t} \tag{2.125}$$

so $\frac{dx}{dt} = 2Xe^{2t}$ and $\frac{dy}{dt} = 2Ye^{2t}$.

From this we find the following,

$$\begin{bmatrix} 2 & 2+1 \\ 3 & 7-2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.126}$$

Therefore,

$2X + 3Y = 0 \quad \rightarrow \quad X = -\frac{3}{2}Y$

$3X + 5Y = 1 \quad \rightarrow \quad (-\frac{9}{2} + 5)Y = 1$.

So we find that $Y = 2$ and $X = -3$..

To find the general solution we just bring all of these together, i.e. CF + PI_1 + PI_2.

$$\begin{bmatrix} x \\ y \end{bmatrix} = X_a \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} e^{t} + X_b \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} e^{3t} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{bmatrix} \tag{2.127}$$

where the first two terms come from the CF and the second two terms come from the two PIs.

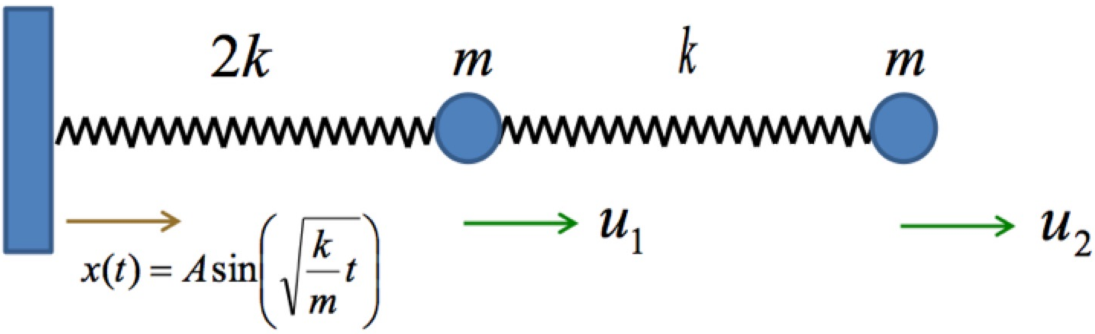

Figure 2.10: The driven horizontal spring.

2.6 Horizontal spring-mass system with a driving term

This brings us on to the penultimate system that we are going to consider in this part of the course.

We now consider two masses moving without friction connected by two springs, of spring constant $2k$ and k, with the spring of $2k$ connected to a wall, and is driven by an external force to have a time-dependent displacement, given by $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right)$. This time-dependent term basically describes an additional oscillatory driving force of the system (see Fig. 2.10).

The equations of motion are therefore,

$$\begin{aligned} m\ddot{u}_1 &= 2k[x(t) - u_1] - k(u_1 - u_2) \\ m\ddot{u}_2 &= k(u_1 - u_2) \end{aligned} \tag{2.128}$$

So as in Sec. 2.5 we first find the complementary function, and consider the homogeneous case with $x(t) = 0$. In this case the equations of motion form the following,

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.129}$$

Let us try the usual form of the solution for an oscillating system,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t} \tag{2.130}$$

Equating the determinant of this matrix to zero, and finding the eigenvalues.

$$\left(-\omega^2 + \frac{3k}{m}\right)\left(-\omega^2 + \frac{k}{m}\right) - \left(\frac{k}{m}\right)^2 = 0$$

$$\omega^2 = \frac{2k}{m} \pm \frac{1}{2}\sqrt{16\left(\frac{k}{m}\right)^2 - 8\left(\frac{k}{m}\right)^2}$$

$$\Rightarrow \omega^2 = \frac{k}{m}(2 \pm \sqrt{2})$$

Substituting these values for ω back into the eigenvector equation we find:

for $\omega^2 = (2 + \sqrt{2})k/m$

$$X(3k - (2 + \sqrt{2})k) = kY \quad \text{therefore} \quad Y = (1 - \sqrt{2})X \tag{2.131}$$

for $\omega^2 = (2 - \sqrt{2})k/m$

$$X(3k - (2 - \sqrt{2})k) = kY \quad \text{therefore} \quad Y = (1 + \sqrt{2})X \tag{2.132}$$

So the ratio of the amplitudes in the CF are

$$\left(\frac{Y}{X}\right)_1 = 1 - \sqrt{2} \quad \text{and} \quad \left(\frac{Y}{X}\right)_2 = 1 + \sqrt{2}$$

Now we have to go back to the original equations of motion and find a solution for the particular integral. Writing the equations of motion in matrix form and replacing $x(t) = A\sin(\sqrt{\frac{k}{m}}t)$ with its equivalent in exponential notation, i.e. $x(t) = A\Re e^{i\sqrt{\frac{k}{m}}t}$, we have

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{Ak}{m} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Re \left[e^{i\sqrt{\frac{k}{m}}t} \right] \tag{2.133}$$

To solve this we try the ansatz,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Re \left(\begin{bmatrix} P \\ Q \end{bmatrix} e^{i\sqrt{\frac{k}{m}}t} \right) \tag{2.134}$$

from which we obtain,

$$\begin{bmatrix} -\frac{k}{m} + \frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \frac{kA}{m} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{2.135}$$

This equation is of the form $\mathbf{MU} = \mathbf{V}$, rearranging to find $\mathbf{U} = \mathbf{M^{-1}V}$. So we need to calculate the inverse of the matrix $\mathbf{M}$. To calculate the inverse, we are required to find the determinant and the transpose of the cofactor matrix (see Eq. 2.24).

The determinant is simply $|\mathbf{M}| = -(\frac{k}{m})^2$, and the adjoint matrix is

$$\mathbf{adj} = \begin{bmatrix} 0 & \frac{k}{m} \\ \frac{k}{m} & \frac{2k}{m} \end{bmatrix} \quad \text{and therefore} \quad \mathbf{M^{-1}} = -\left(\frac{m}{k}\right)^2 \begin{bmatrix} 0 & \frac{k}{m} \\ \frac{k}{m} & \frac{2k}{m} \end{bmatrix} = \left(\frac{m}{k}\right) \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \tag{2.136}$$

Substituting back into Eq. 2.135 we obtain

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \frac{kA}{m}\left(\frac{m}{k}\right) \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{2.137}$$

Therefore,

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ -2A \end{bmatrix} \tag{2.138}$$

So finally we get to the general solution which is the sum of the CF and PI. Expressing in terms of trig functions we have:

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix} \cos\left\{ \left[\frac{k}{m}(2+\sqrt{2})\right]^{\frac{1}{2}} t + \phi_1 \right\} \\ +A_2 \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix} \cos\left\{ \left[\frac{k}{m}(2-\sqrt{2})\right]^{\frac{1}{2}} t + \phi_2 \right\} \\ + \begin{bmatrix} 0 \\ -2A \end{bmatrix} \cos\sqrt{\frac{k}{m}} t \end{aligned} \tag{2.139}$$

2.7 The Forced Coupled Pendulum with a Damping Factor

The last example that we will work through in this section of the course, is one in which we not only have a driving force, but also a damping term. This is therefore more akin to a real-life system where things are not frictionless. In fact such a system forms the basis for many mechanical systems in the real world.

So let us consider the system which is shown in Fig. 2.11. However, in addition to the gravitational force and the force due to the spring, we also apply a driving force to the x mass, as denoted by the $F \cos \alpha t$ term, which provides a cyclic pumping motion. We

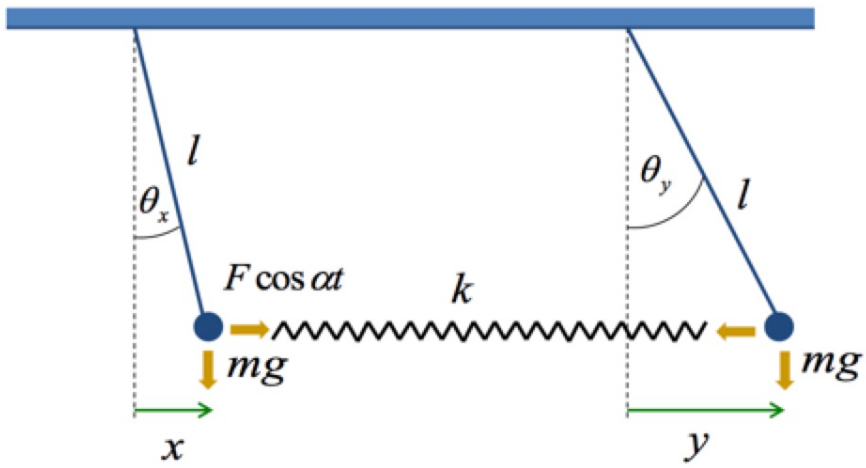

Figure 2.11: The Forced Coupled Pendulum. The coupled pendulum in this case has both a driving term given by $F \cos \alpha t$ and a retarding force or damping term $\gamma \times v$, where v is the velocity of the masses.

also consider a damping force acting on both masses, which is related to the velocity of the masses by $F_{\text{ret}} = \gamma v$.

As before we set up the equations of motion.

$$m\ddot{x} = -\gamma\dot{x} - \frac{mg}{l}x + k(y - x) + F \cos \alpha t \tag{2.140}$$

$$m\ddot{y} = -\gamma\dot{y} - \frac{mg}{l}y - k(y - x) \tag{2.141}$$

These are exactly the same equations of motion as for the coupled pendulum, as you would expect, but with the additional terms due to the damping force (in both expressions), and the driving force (in the expression for the x motion).

This looks like a non-trivial problem, so let us go straight to using the matrix method. The general matrix for the equations of motion is given by

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}\frac{d}{dt} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}\frac{d}{dt} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{F}{m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Re(e^{i\alpha t}) \tag{2.142}$$

We use the fact that $\cos \alpha t = \Re(e^{i\alpha t})$ here, but you will obtain the same solution if you use $\cos \alpha t$.

The right-hand side is not equal to zero, so this is an inhomogeneous matrix as we could infer directly from the equations of motion. Therefore, we need to find the solution to the homogeneous equivalent (the complementary function; CF), and the particular integral.

To find the CF, we write down the homogeneous equations and solve as before;

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}\frac{d}{dt} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}\frac{d}{dt} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.143}$$

Try the following,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Re \begin{bmatrix} X \\ Y \end{bmatrix} e^{i\omega t} \tag{2.144}$$

so that

$$\frac{dx}{dt} = i\omega X e^{i\omega t} \quad \text{and} \quad \frac{d^2x}{dt^2} = -\omega^2 X e^{i\omega t}$$
$$\frac{dy}{dt} = i\omega Y e^{i\omega t} \quad \text{and} \quad \frac{d^2y}{dt^2} = -\omega^2 Y e^{i\omega t}$$

and find the eigenvalues from the following determinant,

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}i\omega & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} + \frac{\gamma}{m}i\omega \end{vmatrix} = 0, \tag{2.145}$$

which leads to,

$$\begin{aligned} &\left(-\omega^2 + i\omega\frac{\gamma}{m} + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \\ \implies &\left(-\omega^2 + i\omega\frac{\gamma}{m} + \frac{g}{l} + \frac{k}{m}\right) = \pm\left(\frac{k}{m}\right) \end{aligned} \tag{2.146}$$

Solve using the equation for solving a quadratic for both $\pm(k/m)$.

For $+(k/m)$,

$$\bar{\omega}_1 = \frac{i\gamma}{2m} \pm \sqrt{\frac{g}{l} - \left(\frac{\gamma}{2m}\right)^2} \tag{2.147}$$

For $-(k/m)$,

$$\bar{\omega}_2 = \frac{i\gamma}{2m} \pm \sqrt{\left(\frac{g}{l} + \frac{2k}{m}\right) - \left(\frac{\gamma}{2m}\right)^2} \tag{2.148}$$

You should notice that these are similar to the eigenvalues in the simple coupled pendulum case, where $\omega_1^2 = g/l$ and $\omega_2^2 = g/l + 2k/m$ (Eq. 2.28). We have the same terms for this system, as you might expect, but also the additional terms that include the damping factor.

So we can rewrite these in terms of these original values for $\omega_{1,2}$ as,

$$\bar{\omega}_{1,2} = \frac{i\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}. \tag{2.149}$$

There is no physical difference between the $\pm$ variants here, so from now on we will just use the positive solution.

We substitute these eigenvalues into the eigenvector equation to find the ratio of the amplitudes in each normal mode, and thus find the CF. With a bit of simple maths, for $\bar{\omega}_1$, we find $X = Y$ and for $\bar{\omega}_2$ we find $X = -Y$.

Remembering Eq. 2.144,

$$x = \Re(Xe^{i\bar{\omega}_{1,2}t}) \quad \text{and} \quad y = \Re(Ye^{i\bar{\omega}_{1,2}t})$$

we find that,

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\left(-\frac{\gamma t}{2m}\right)} \left\{ A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos\left[\left(\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2\right)^{\frac{1}{2}} t + \phi_1\right] + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos\left[\left(\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2\right)^{\frac{1}{2}} t + \phi_2\right] \right\} \tag{2.150}$$

where $\omega_1^2 = g/l$ and $\omega_2^2 = g/l + 2k/m$. As we might expect, we introduce an exponential decay factor, which describes the damping force acting on the two masses.

As a note, you can solve this part with the decoupling method and then use a trial solution of the form $q = \Re(e^{i\omega t})$, and you get the same result.

To obtain the general solution, which includes the driving force term, we need to calculate the PI as well.

Starting from Eqn. 2.142, we now try the ansatz,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Re\left\{ \begin{bmatrix} P \\ Q \end{bmatrix} e^{i\alpha t} \right\} \tag{2.151}$$

which requires us to solve a matrix of the form $\mathbf{MU} = \mathbf{V}$, as we did in Sec. 2.6, so we rearrange to find $\mathbf{U} = \mathbf{M}^{-1}\mathbf{V}$.

Therefore we have,

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \mathbf{M}^{-1} \frac{F}{m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.152}$$

and use Eq. 2.24 to find $\mathbf{M}^{-1}$.

So we first find the determinant of $\mathbf{M}$,

$$\begin{aligned}|\mathbf{M}| &= \left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right)\right]^2 - \left[\frac{k}{m}\right]^2 \\ &= \left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \omega_1^2\right)\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \omega_2^2\right)\end{aligned} \tag{2.153}$$

where $\omega_1^2 = g/l$ and $\omega_2^2 = g/l + 2k/m$.

This will get a bit unwieldy, so it is generally easier to write this in terms of polar coordinates, i.e.

$$|\mathbf{M}| = B_1 e^{-i\theta_1}.B_2 e^{-i\theta_2}, \tag{2.154}$$

where

$$B_{1,2} = \left[(\omega_{1,2}^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right]^{\frac{1}{2}} \quad \text{and} \quad \tan\theta_{1,2} = \frac{-\alpha\gamma}{m(\omega_{1,2}^2 - \alpha^2)}$$

So we now have the determinant, and the next thing to find is the adjoint matrix, which is just the transpose of the cofactor matrix.

$$\text{adj}\mathbf{M} = \begin{bmatrix} -\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) \end{bmatrix} \tag{2.155}$$

We can write the diagonals of the matrix as,

$$\begin{aligned}-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) &= \frac{1}{2}\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right) + \frac{1}{2}\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l} + \frac{2k}{m}\right) \\ &= \frac{1}{2}\left(B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2}\right).\end{aligned} \tag{2.156}$$

Similarly, we can also write

$$\begin{aligned}\frac{k}{m} &= \frac{1}{2}\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l} + \frac{2k}{m}\right) - \frac{1}{2}\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right) \\ &= \frac{1}{2}\left(B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1}\right).\end{aligned} \tag{2.157}$$

Substituting these back to find the adjoint,

$$\text{adj}\mathbf{M} = \frac{1}{2}\begin{bmatrix} \left(B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2}\right) & \left(B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1}\right) \\ \left(B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1}\right) & \left(B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2}\right). \end{bmatrix} \tag{2.158}$$

Therefore,

$$\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|}\text{adj}\mathbf{M} = \frac{e^{i(\theta_1+\theta_2)}}{2B_1B_2}\begin{bmatrix}\left(B_1e^{-i\theta_1}+B_2e^{-i\theta_2}\right) & \left(B_2e^{-i\theta_2}-B_1e^{-i\theta_1}\right)\\ \left(B_2e^{-i\theta_2}-B_1e^{-i\theta_1}\right) & \left(B_1e^{-i\theta_1}+B_2e^{-i\theta_2}\right)\end{bmatrix}$$
$$= \frac{1}{2B_1B_2}\begin{bmatrix}\left(B_1e^{i\theta_2}+B_2e^{i\theta_1}\right) & \left(B_2e^{i\theta_1}-B_1e^{i\theta_2}\right)\\ \left(B_2e^{i\theta_1}-B_1e^{i\theta_2}\right) & \left(B_1e^{i\theta_2}+B_2e^{i\theta_1}\right)\end{bmatrix} \quad (2.159)$$

Recall,

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{F}{m}\Re\left(\mathbf{M}^{-1}\begin{bmatrix}1\\0\end{bmatrix}e^{i\alpha t}\right), \quad (2.160)$$

therefore

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{F}{2mB_1B_2}\begin{bmatrix}B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1)\\ B_2\cos(\alpha t+\theta_1)-B_1\cos(\alpha t+\theta_2)\end{bmatrix} \quad (2.161)$$

with

$$B_{1,2} = \left[(\omega_{1,2}^2-\alpha^2)^2+\left(\frac{\alpha\gamma}{m}\right)^2\right]^{\frac{1}{2}} \qquad \text{and} \qquad \tan\theta_{1,2} = \frac{-\alpha\gamma}{m(\omega_{1,2}^2-\alpha^2)}$$

So finally, the general solution is the sum of the CF and the PI;

$$\begin{bmatrix}x\\y\end{bmatrix} = e^{\left(-\frac{\gamma t}{2m}\right)}\left\{A_1\begin{bmatrix}1\\1\end{bmatrix}\cos\left[\left(\omega_1^2-\left(\frac{\gamma}{2m}\right)^2\right)^{\frac{1}{2}}t+\phi_1\right]\right\}$$
$$+e^{\left(-\frac{\gamma t}{2m}\right)}\left\{A_2\begin{bmatrix}1\\-1\end{bmatrix}\cos\left[\left(\omega_2^2-\left(\frac{\gamma}{2m}\right)^2\right)^{\frac{1}{2}}t+\phi_2\right]\right\} \quad (2.162)$$
$$+\frac{F}{2mB_1B_2}\begin{bmatrix}B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1)\\ B_2\cos(\alpha t+\theta_1)-B_1\cos(\alpha t+\theta_2)\end{bmatrix}.$$

The complementary function is the “transient” solution determined by the external conditions, whereas the particular integral gives the “steady state” solution determined by the driving force.

Chapter 3

Normal modes II - towards the continuous limit

In the last few sections we have built the foundations for the study of waves in general. For the remainder of the course we will focus on various types of wave, and the general applicability of the equations which govern waves across many parts of physics.

We will look at two general forms of waves; first we will consider the non-dispersive system, which means that the speed at which the wave travels is independent of the wavelength and frequency. Later, we will look at dispersive waves, where the speed at which the wave travels does depend on the frequency, this leads to the introduction of the concept of *group velocity*.

3.1 N-coupled oscillators

From our work looking at normal modes, we know that we can describe a system of coupled oscillators by the linear superposition of N normal modes, where N in the coupled pendulum case was the number of masses on pendula for example.

In this section we will look what happens when we extend this superposition to the continuous limit, i.e. for a single piece of wire or string. We will first look at the case for N masses on a string, where N can be large, then we look at a string that can be considered as being the summation of very small bits of string of length dl and a mass that relates to these very small parts of the string.

Fig. 3.1 shows a diagram of a stretched string, where we split the string up into small

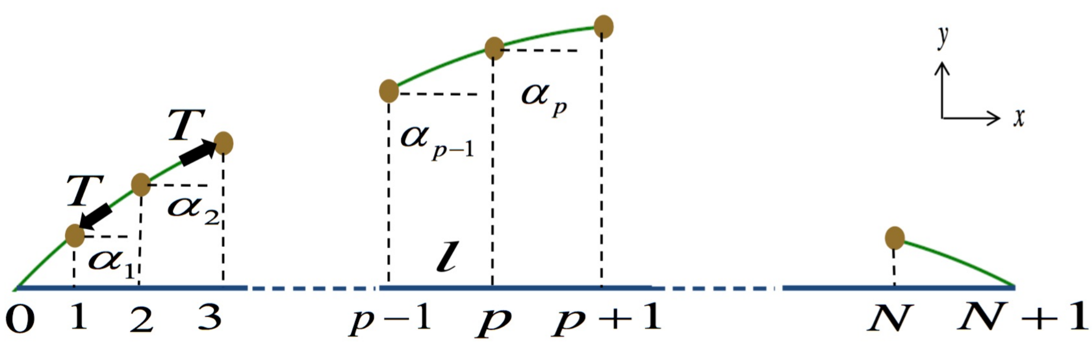

Figure 3.1: A simple string that has been stretched by a very small amount at its centre.

mass elements. In this example, we assume that all of the angles of the stretched string from the horizontal are small.

As before, we write down the equations of motion for each element of the string, but this time we use the tension T in the stretched string around point p:

In the vertical direction we have a net force given by,

$$F_{p_y} = m\ddot{y}_p = -T\sin\alpha_{p-1} + T\sin\alpha_p \tag{3.1}$$

For the horizontal direction we have

$$F_{p_x} = -T\cos\alpha_{p-1} + T\cos\alpha_p. \tag{3.2}$$

As we assume α is small, then $\sin\alpha_i \approx \tan\alpha_i \approx \alpha_i$. We also know that, $\cos\alpha_i \approx 1-\frac{\alpha_i^2}{2}$, and use the trig identities $\cos^2\theta + \sin^2\theta = 1$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Therefore, in the horizontal direction we have zero net force as $\alpha_{p-1} \approx \alpha_p$, as we would expect if we only stretch the string in the vertical direction at its centre.

For the vertical direction, Eq. 3.1 becomes

$$F_p = m\ddot{y}_p = -\frac{T}{l}(y_p - y_{p-1}) + \frac{T}{l}(y_{p+1} - y_p). \tag{3.3}$$

If we define $\omega_0^2 = T/ml$ then this becomes

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0. \tag{3.4}$$

3.1.1 Special cases

Before we move on to the general solution to this equation, let us first look at a few specific, special cases. Namely, the most simple ones with $N = 1$ and $N = 2$.

So if we have $N = 1$ then the stretch string just looks like the system shown in Fig. 3.2.

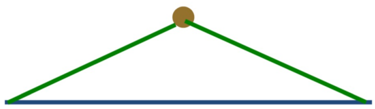

Figure 3.2: A stretched string with $N = 1$.

Eq.3.4 becomes

$$\ddot{y}_1 + 2\omega_0^2 y_1 = 0, \tag{3.5}$$

as $y_{p+1} = y_{p-1}$ in this case, and the equation is the usual form for SHM, i.e. acceleration proportional to the negative of the displacement. Therefore, we can immediately state that the angular frequency of the oscillation:

$$\omega = \sqrt{2}\omega_0 \quad \text{with} \quad \omega_0^2 = \frac{T}{ml} \quad \text{for} \quad N = 1$$

Now let us look at the $N = 2$ case. By just drawing what the possibilities are in this case (Fig. 3.3), we know that we should have two solutions.

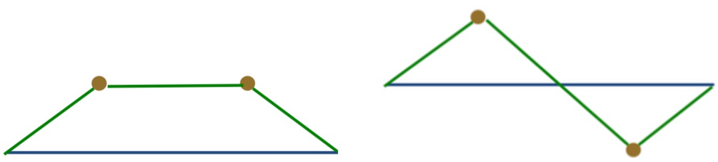

Figure 3.3: Two possibilities for the starting points for a stretched string with $N = 2$. Obviously you can have the inverse of these but that is just the same as inverting the direction of y.

Writing 3.4 for $N = 2$ we get,

$$\begin{aligned} \ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 y_2 &= 0 \\ \ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2 y_1 &= 0. \end{aligned} \tag{3.6}$$

Let us define the normal coordinates, as we did in Sec. 2.1, i.e.

$$q_1 = \frac{1}{\sqrt{2}}(y_1 + y_2) \quad \text{and} \quad q_2 = \frac{1}{\sqrt{2}}(y_1 - y_2).$$

Adding and subtracting Eqns. 3.6, and substituting for q_1 and q_2, we find,

$$\ddot{q}_1 + \omega_0^2 q_1 = 0 \Longrightarrow \omega = \omega_0$$

$$\ddot{q}_2 + 3\omega_0^2 q_2 = 0 \Longrightarrow \omega = \sqrt{3}\omega_0.$$

Therefore, we have two possible normal modes, as you would expect for $N = 2$. This is simply analogous to the coupled pendula and N-spring systems that we looked at in previous lectures.

So now that we have discussed these two special cases, let us move on and try to find a general solution to the system, where N can be anything.

3.1.2 General case

From Eq. 3.4 we have

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} - y_{p-1}) = 0.$$

This is a second order differential equation and we expect an oscillatory solution, so let us consider a solution of the form $y_p = A_p \cos \omega t$.

Note that here we have not included a phase offset term, which we represented by ϕ in previous lectures. By omitting this term here we are just imposing the fact that all the masses start at rest. Obviously we could reintroduce this term later.

So substituting the trial solution into Eq. 3.4, and dividing through by $\cos(\omega t)$, gives N equations:

$$\begin{aligned} (-\omega^2 + 2\omega_0^2)A_p - \omega_0^2(A_{p+1} + A_{p-1}) = 0 \quad \text{with} \quad p = 1, 2, 3...., N \\ \Longrightarrow \frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} \end{aligned} \tag{3.7}$$

But we know that since ω is the same for all masses, i.e. it doesn't matter where we are on the string, then the right-hand side cannot depend on p. So if the RHS does not depend on p, neither can the left hand side. We also know that if the string is fixed at both ends, then for $p = 0$ and $p = N+1$, $A_p = 0$.

So we need to look for forms of A_p which satisfy these criteria.

Let us try a description of A_p of the form, $A_p = C\sin p\theta$, so the left-hand side of Eq. 3.7 becomes,

$$\frac{A_{p+1}+A_{p-1}}{A_p} = \frac{C\left[\sin(p+1)\theta + \sin(p-1)\theta\right]}{C\sin(p\theta)} = \frac{2C\sin(p\theta)\cos\theta}{C\sin(p\theta)} = 2\cos\theta, \tag{3.8}$$

which satisfies the criteria for the equation to be independent of p.

Now for the second criteria of when $p = 0$ or $p = N+1$, then $A_p = 0$, we just have to require that $(N+1)\theta = n\pi$.

So combining all of this we have

$$\boxed{A_p = C\sin\left(\frac{pn\pi}{N+1}\right) \qquad \text{and} \qquad \frac{A_{p+1}+A_{p-1}}{A_p} = 2\cos\left(\frac{n\pi}{N+1}\right)} \tag{3.9}$$

Substituting this back into Eq. 3.7,

$$\omega^2 = 2\omega_0^2\left[1-\cos\left(\frac{n\pi}{N+1}\right)\right] \tag{3.10}$$

and using the trig identity: $1-\cos 2\theta = 2\sin^2\theta$,

$$\Longrightarrow \omega = 2\omega_0\sin\left(\frac{n\pi}{2(N+1)}\right). \tag{3.11}$$

So now we can write the general solution for the displacement in y. Combining Eq. 3.4 with Eq. 3.11 and reintroducing the phase offset such that $y_p = A_p\cos(\omega t + \phi_p)$, we obtain

$$\boxed{\begin{gathered} y_{pn}(t) = C_n\sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t + \phi_n) \quad \text{with} \quad \omega_n = 2\omega_0\sin\left(\frac{n\pi}{2(N+1)}\right) \\ \text{where} \quad \omega_0 = \sqrt{\frac{T}{ml}} \end{gathered}} \tag{3.12}$$

Although the value of n can be greater than N, this would just generate duplicate solutions, and there are always N normal modes in total. Fig. 3.4 shows the 5 normal modes for the case of $N = 5$. One thing to note is that all the masses are displaced in such a way that they fall on an underlying sine curve, as you would expect given the solution that we have just found.

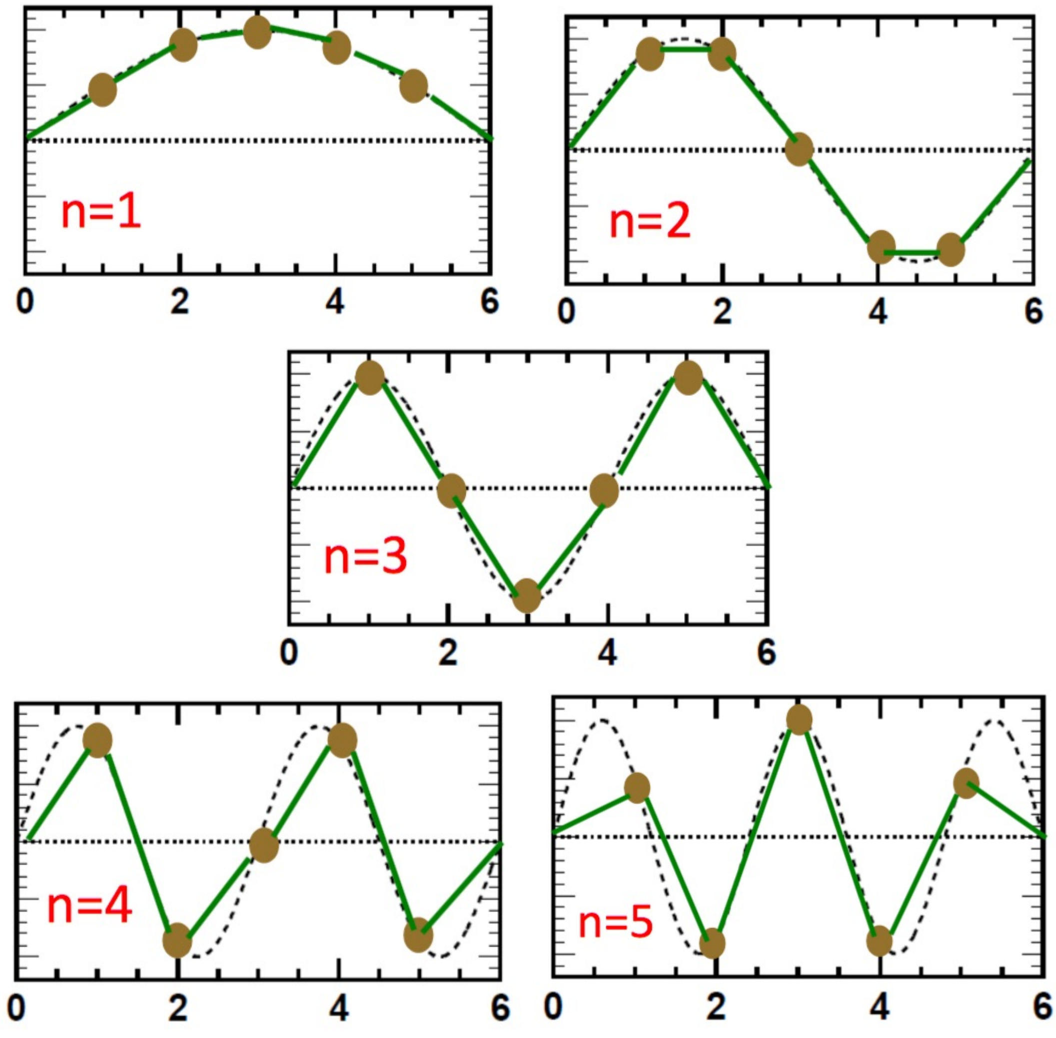

Figure 3.4: Normal modes for the case of $N = 5$ with snapshots taken at $t = 0$. With $N = 5$ there are five normal modes and the subsequent motion of the string can be described by the superposition of these modes, with the initial conditions dictating which normal modes are active in a system. Also, note that the displacement of the masses on each string (filled red circles) all fall on a sine curve.

3.1.3 N very large

Obviously the system we considered in the last section does not really represent what happens to a continuous, uniform piece of string. We generally do not have masses along the string that represent the number of displacement points. But we can use this as a starting point for considering a real string, i.e. we can assume that N is very large and describe the masses in terms of the linear density of the string and assume that the string has a uniform density per unit length.

So let us consider a string-mass system which has a length L and a total mass M. We can consider this string as being made up of a series of small elements of length l and mass m, such that

$L = (N+1)l$ and $M = Nm$, and define the linear density of the string as $\rho = m/l$.

Eq. 3.11 describes the frequency of the oscillation for each normal mode. If we just consider the mode numbers n which are small in comparison to N, which would be the case if N is very large as we are assuming, then we essentially remove the sine dependence and find,

$$\begin{aligned}\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) &= 2\sqrt{\frac{T}{ml}}\sin\left(\frac{n\pi}{2(N+1)}\right)\\ \Longrightarrow \omega_n &\approx 2\sqrt{\frac{T}{m/l}}\left(\frac{n\pi}{2(N+1)l}\right).\end{aligned} \tag{3.13}$$

As we have defined the linear density above, this then becomes,

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}. \tag{3.14}$$

This means that all of the normal frequencies are integer multiples of the lowest frequency given when $n = 1$, i.e.

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\rho}}.$$

This is the fundamental frequency, or sometimes just referred to as the *fundamental*. The fundamental may be created by vibration over the full length of a string or air column. The fundamental is one of the harmonics in music.

So we now know that the normal frequencies are just given by integer multiples of the lowest frequency mode. These are the other harmonics of a vibrating system.

We now consider the displacement of the string for the same limit of n small compared to N. Starting from Eq. 3.12, we have

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t + \phi_n),$$

but as the elements of the string, each of length l, become smaller and smaller, we approach a continuous variable along the $x-$axis, which we can define as $x = pl$, such that

$$y_n(x,t) = C_n \sin\left(\frac{xn\pi}{L}\right)\cos(\omega_n t + \phi_n), \tag{3.15}$$

resulting in a sinusoidal wave in both x and t, i.e. we have derived the equation for the complete motion of the string, at least for when $n << N$. From this we can calculate the vertical (y) displacement of the string at any time t and at any point along the axis of the string.

Let us now look what happens when we move to $n = N$. From Eq. 3.14 we know that this must give us the highest frequency mode of the oscillations,

$$\omega_N = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \approx 2\omega_0. \tag{3.16}$$

If we now consider the ratio of the displacements of successive elements of the string for the $n = N$ mode using Eq. 3.12, i.e.

$$\frac{y_p}{y_{p+1}} = \frac{\sin\left(\frac{pN\pi}{N+1}\right)}{\sin\left(\frac{(p+1)N\pi}{N+1}\right)} \approx \frac{\sin(p\pi)}{\sin(p\pi+\pi)} \approx -1. \tag{3.17}$$

So every successive element is approximately displaced equally but in the opposite direction to the previous element. If we lost the approximations, and calculated a more rigorous solution then what we would find is that we obtain adjacent positive and negative displacements (Fig. 3.5) where the amplitude of the string is the maximum at the centre.

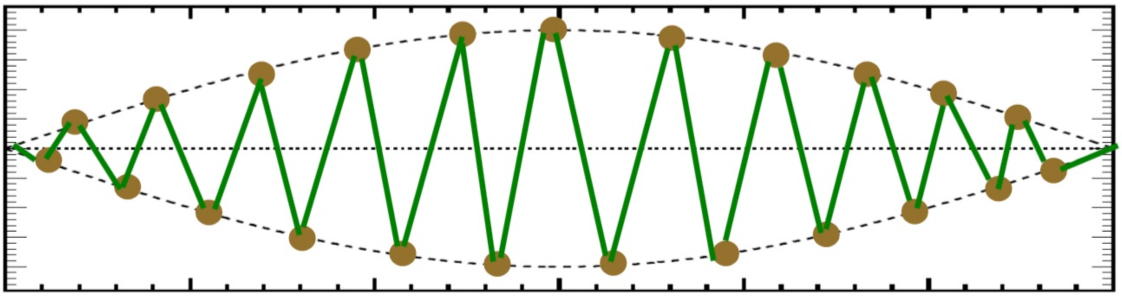

Figure 3.5: Illustration of the adjacent displacement of the string for the $n = N$ mode, resulting in the highest frequency mode.

So referring back to Eq. 3.1, we now know that for $n = N$, $y_{p-1} \approx -y_p \approx y_{p+1}$, therefore

$$F_p = \ddot{y}_p = -\frac{T}{lm}(y_p - y_{p-1}) + \frac{T}{lm}(y_{p+1} - y_p) = -\frac{T}{lm}(2y_p) - \frac{T}{lm}(2y_p) = -4y_p\frac{T}{lm} \quad (3.18)$$

We defined $\omega_0 = T/ml$, therefore $\ddot{y} \approx -4y\omega_0^2$, from which we find the result $\omega_N \approx 2\omega_0$.

3.1.4 Longitudinal Oscillations

In the previous section we considered only transverse motion of a string. Obviously waves can also exist as longitudinal waves, e.g. sound waves and oscillations on a spring. Here we will quickly demonstrate that the same equations that we found in the previous section also hold for longitudinal waves.

So rather than the string with N-mass elements, let us consider a horizontal spring system with N masses between similar springs with spring constant k (Fig. 3.6).

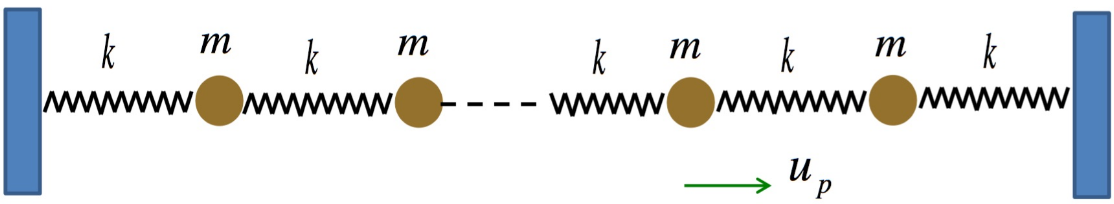

Figure 3.6: Horizontal spring-mass system with N-masses.

If u_p is the displacement from equilibrium of mass p, then the equation of motion of each mass is given by,

$$\begin{aligned} m\ddot{u}_p &= k(u_{p+1} - u_p) + k(u_{p-1} - u_p) \\ \Longrightarrow \ddot{u}_p + 2\omega_0^2 u_p - \omega_0^2(u_{p+1} + u_{p-1}) &= 0 \qquad \text{with} \quad \omega_0^2 = \frac{k}{m}. \end{aligned} \tag{3.19}$$

This is exactly the same form as the equation of motion as given in Eq. 3.4 for the transerve oscillations of a string made up of N masses. Therefore, the solutions will be exactly the same.

Chapter 4

Waves I

In the last sections we have shown how we can build up a view of the motion of an oscillating system by considering mass elements that oscillate along either a string or a spring-mass system. It is then the superposition of all of the normal modes in these systems that describes the overall motion of the system. The motion in these systems is then dependent on the initial conditions, and how many normal modes are active at the start of the oscillations.

In this section we move away from the discretised analysis of mass elements in a string or spring system, and progress towards a completely continuous description of waves.

4.1 The wave equation

4.1.1 The Stretched String

Consider a segment of string of constant linear density ρ that is stretched under tension T, as shown in Fig. 4.1.

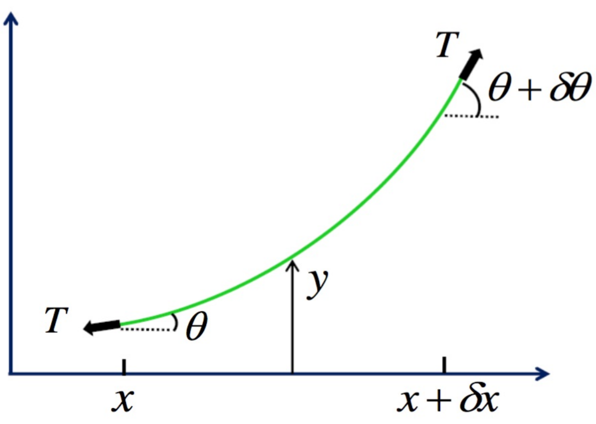

Figure 4.1: Zoom in of a segment of a stretched string.

If we consider the net force acting on the string in both the vertical and horizontal directions we find,

$$\begin{aligned} F_y &= T\sin(\theta+\delta\theta) - T\sin\theta \\ F_x &= T\cos(\theta+\delta\theta) - T\cos\theta. \end{aligned} \tag{4.1}$$

If we assume that $\delta\theta$ is small, then

$$\begin{aligned} F_y &\approx T\delta\theta \\ F_x &\approx 0, \end{aligned} \tag{4.2}$$

and expressing the force in terms of the linear density and the acceleration, we obtain

$$F_y = ma_y = (\rho\delta x)a_y = (\rho\delta x)\frac{\partial^2 y}{\partial t^2} = T\delta\theta \tag{4.3}$$

Note that here we have used the partial derivative, rather than the $\ddot{y}$, which implies a normal derivative (d^2y/dt^2). This is because we know that the amplitude of the displacement in the y-direction is dependent on both the time t *and* the distance along the x-axis. A lot of the work that follows will use partial differential equations.

Fig. 4.1 also helps us to describe the relationship between the vertical displacement and the position along the horizontal x axis.

For example, we can easily see that

$$\begin{aligned} \tan\theta = \frac{\partial y}{\partial x} \quad \text{and} \quad \frac{\partial \tan\theta}{\partial\theta} &= \sec^2\theta = \frac{\partial^2 y}{\partial x\partial\theta} \\ \text{therefore} \quad \sec^2\theta\frac{\partial\theta}{\partial x} &= \frac{\partial^2 y}{\partial x^2}. \end{aligned} \tag{4.4}$$

As before, if θ and $\delta\theta$ are small, then $\sec\theta \approx 1$ and

$$\sec^2\theta\frac{\partial\theta}{\partial x} \approx \frac{\delta\theta}{\delta x}. \tag{4.5}$$

Therefore,

$$\frac{\delta\theta}{\delta x} \approx \frac{\partial^2 y}{\partial x^2}$$

and using Eq. 4.3,

$$\Longrightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \tag{4.6}$$

This is a form of the wave equation. If we now look at the dimensions of $\frac{\rho}{T}$ we find that it has units of $velocity^{-2}$ and as we go on we actually do see that the parameters that sit in front the second derivative, do indeed define the velocity of the travelling waves on a string.

> This is the wave equation,
>
> $$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}} \tag{4.7}$$

4.2 d'Alembert's solution to the wave equation

The wave equation provides us with a general equation for the propagation of waves. In this section we will look at a few solutions to the wave equation.

The wave equation links together the displacements of a wave in the y direction with the time and also the displacement along the perpendicular x axis. Therefore, we need to look for solutions that link together the dependence on both x and t.

d'Alembert was a French mathematician and music theorist. Given the relevance of the wave equation to music, then his work on waves is probably unsurprising.

In d'Alembert's solution, the displacement in y is defined as a function of two new variables, u and v, such that

$$\begin{aligned} u &= x - ct \\ v &= x + ct \end{aligned} \tag{4.8}$$

In order to link these solutions to the wave equation, we just need differentiate each with respect to x and t. Using the chain rule to get the first derivatives,

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \qquad \frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} \\ &= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} & &= -c\frac{\partial y}{\partial u} + c\frac{\partial y}{\partial v} \end{aligned} \tag{4.9}$$

and the second derivatives,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}\right) \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}\right)$$

$$= \left(\frac{\partial^2 u}{\partial x \partial u} + \frac{\partial^2 v}{\partial x \partial v}\right)\frac{\partial y}{\partial x} \qquad = \left(\frac{\partial^2 u}{\partial t \partial u} + \frac{\partial^2 v}{\partial t \partial v}\right)\frac{\partial y}{\partial t}$$

using the equation for the first derivative (Eq. 4.9),

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) \qquad \frac{\partial^2 y}{\partial t^2} = \left(\frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v}\right)\left(-c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\right)$$

Rearranging and substituting in the fact that (Eq. 4.8),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 1 \quad \text{and} \quad -\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = c \tag{4.10}$$

we find,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \qquad \frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right). \tag{4.11}$$

Substituting this in to the wave equation, we find that

$$\frac{\partial^2 y}{\partial u \partial v} = 0.$$

We see that y is separable into functions of u and v,

$$\implies y(u,v) = f(u) + g(v)$$

So the general solution to the wave equation is,

$$y(x,t) = f(x - ct) + g(x + ct) \tag{4.12}$$

Here, f and g are any functions of $(x - ct)$ and $(x + ct)$, with the values determined by the initial conditions.

4.2.1 Interpretation of d'Alembert's solution

In the last section we found that the general solution to the wave equation (Eq. 4.7) is provided by the linear combination of a function in $(x - ct)$ and $(x + ct)$.

If we just focus on the $f(x - ct)$ part of the solution, i.e.

$$y(x,t) = f(x - ct), \tag{4.13}$$

then at time $t = t_1$ we have $y(x, t_1) = f(x - ct_1)$ and at time t_2, $y(x, t_2) = f(x - ct_2)$. We can rewrite this as

$$y(x, t_2) = f(x + ct_1 - ct_2 - ct_1)$$
$$= f([x - c(t_2 - t_1)] - ct_1).$$

Looking at this equation in terms of what is happening physically, we can see that the displacement at time t_2 and position x is equal to the displacement at time t_1 displaced by a distance $c(t_2 - t_1)$ to the left of x.

Therefore, the $f(x - ct)$ solution describes a wave travelling to the right with velocity c (see Fig. 4.2).

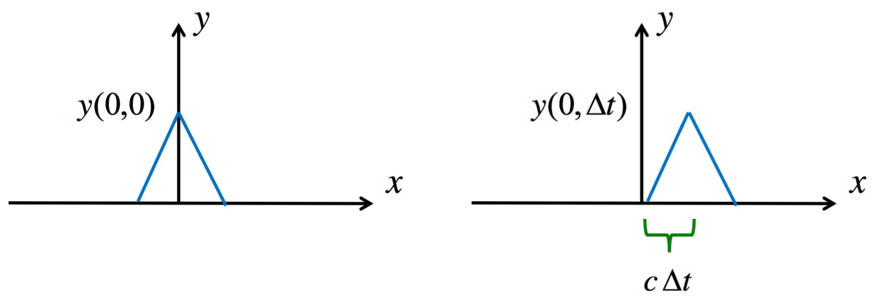

Figure 4.2: Motion of a triangle waveform for a solution with $y(x, t) = f(x - ct)$. The wave moves to the right with velocity c, therefore in time Δt the wave moves a distance $c\Delta t$ to the right.

Similarly, $g(x + ct)$ describes wave travelling to the left with velocity c (see Fig. 4.3).

4.2.2 d'Alembert's solution with boundary conditions

In order to obtain a particular solution for any given system that is described by the wave equation we have to consider the initial or boundary conditions of that system. Here we will consider implementing boundary conditions for d'Alembert's solution to the wave equation.

So we have,

$$y(x, t) = f(x - ct) + g(x + ct).$$

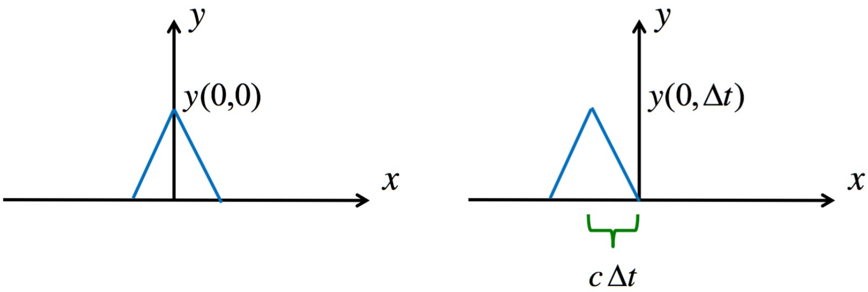

Figure 4.3: Motion of a triangle waveform for a solution with $y(x,t) = f(x+ct)$. The wave moves to the left with velocity c, therefore in time Δt the wave moves a distance $c\Delta t$ to the left.

At time $t = 0$ the wave has an initial displacement $U(x)$ and velocity $V(x)$, such that

$$y(x,0) = U(x) = f(x) + g(x) \tag{4.14}$$

Having one of the boundary conditions for the velocity should immediately alert us to taking the derivative of the y displacement with respect to t. Therefore,

$$\begin{aligned} \frac{\partial y(x,0)}{\partial t} = V(x) = \frac{\partial(x-ct)}{\partial t}\frac{df}{d(x-ct)} + \frac{\partial(x+ct)}{\partial t}\frac{dg}{d(x+ct)} \\ = -c\frac{df(x)}{dx} + c\frac{dg(x)}{dx}. \end{aligned} \tag{4.15}$$

Integrating this we find,

$$\int \frac{df(x)}{dx} - \frac{dg(x)}{dx} = f(x) - g(x) = -\frac{1}{c}\int_b^x V(x).dx. \tag{4.16}$$

Combining with Eq. 4.14,

$$\begin{aligned} g(x) &= \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x).dx \\ f(x) &= \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x).dx \end{aligned} \tag{4.17}$$

Therefore the solution would be,

$$\begin{aligned} y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} V(x).dx - \int_b^{x-ct} V(x).dx\right] \\ = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x).dx \end{aligned} \tag{4.18}$$

Let us now consider a rectangular waveform, of length $2a$ released from rest, therefore $V(x) = 0$. Then Eq. 4.18 is now,

$$y(x, t) = [U(x - ct) + U(x + ct)] \tag{4.19}$$

If we follow the motion of the wave with time, then we see the wave traverses along the x axis as shown in Fig. 4.4 below,

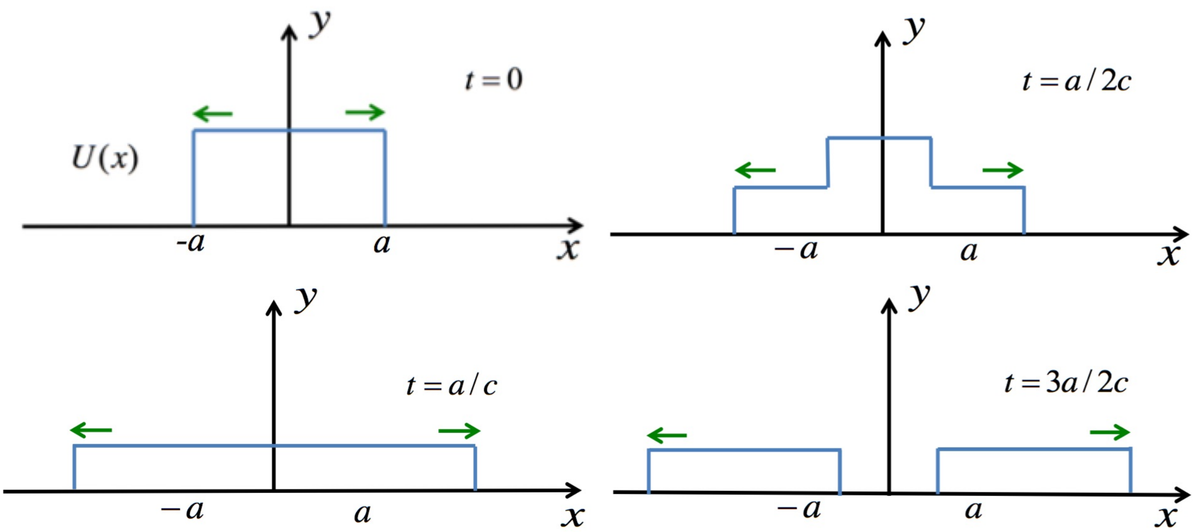

Figure 4.4: The motion of a square wave over $t = 0$ through to $t = 3a/2c$. One can see that the original square wave has components moving in both directions, which is a result of the $-ct$ and $+ct$ components in the d'Alembert solution.

4.3 Sinusoidal waves

By looking at the solutions found using d'Alembert's method and also using the separation of variables we can see that some relations naturally fall out. So we have

$$y(x, t) = A\cos(kx - \omega t) + B\cos(kx + \omega t),$$

with constants k, ω, along with the usual amplitude constants A and B. We could equally replace the cosines with sines, unless we are comparing one wave with another and thus the relative phase becomes important. We also then find that

- the speed of the wave is a constant value and is linearly related by $c = \omega/k$. It must also be equal to whatever constant appears in the wave equation e.g. $\sqrt{T/\rho}$. If this relation was not true than we get dispersive waves (see Sec. 5.2), where $\omega \neq ck$.

- the frequency $f = 1/T = \omega/2\pi$, where ω is the *angular frequency*.
- the wavelength $\lambda = 2\pi/k$, where k is the *wave number* or *wave vector* is used to indicate the direction of the wave.

4.4 Phase Differences

When we are comparing two or more waves then it is not sufficient to describe such waves with just their kx and ωt terms, as this does not provide enough information to relate where one wave is to another at any given time. An extra term is needed that describes the offset of one wave with respect to another at time $t = 0$. This is the *phase difference*, usually denoted by ϕ.

Let us consider two waves, such that

$$
\begin{aligned}
y_1(x,t) &= A\cos(kx - \omega t), \\
y_2(x,t) &= A\cos(kx - \omega t + \phi).
\end{aligned}
\tag{4.20}
$$

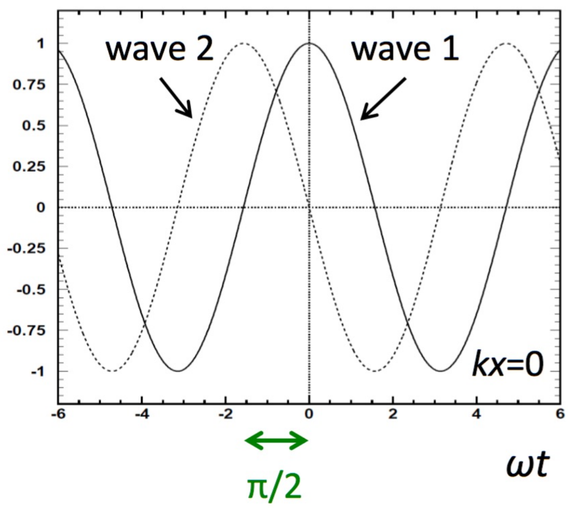

Figure 4.5: Example of Wave 2 leading Wave 1 by $\pi/2$, i.e. the phase difference $\phi = -\pi/2$, along the ωt axis.

In this case the second wave leads the first wave if $\phi < 0$, and lags the first wave if $\phi > 0$. This is demonstrated in Fig. 4.4 where we show the two waves along the ωt axis, and Wave 2 reaches it maxima and minima by $\pi/2$ earlier than Wave 1.

As for all the waves that we have considered so far, this can also be expressed in complex notation,

$$y_2(x,t) = \Re\left[Ae^{i(kx-\omega t+\phi)}\right], \tag{4.21}$$

and we could also subsume the phase into the amplitude,

$$y_2(x,t) = \Re\left[Ae^{i(kx-\omega t)}\right], \quad \text{with} \quad A = |A|e^{i\phi}. \tag{4.22}$$

Chapter 5

Waves II

5.1 Standing Waves - I

A standing, or stationary wave, is a wave in which each point along the axis of the wave has a constant amplitude. The locations at which the amplitude from the axis is at a minimum are called nodes, and the points along the axis where the amplitude is maximum are called antinodes.

We can think of a stranding wave as the combination of two waves moving in opposite directions, which essentially work to cancel out in a way that results in the waves just oscillating in amplitude around fixed points along the axis. To show this, let us take two waves of equal amplitude and frequency that are moving in opposite directions.

$$y(x,t) = A\sin(kx - \omega t) + A\sin(kx + \omega t) = 2A\sin kx \cos \omega t, \tag{5.1}$$

which we can also define in terms of the wavelength and time period,

$$y(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{T}\right). \tag{5.2}$$

Thus we have a wave that is factorised in space- and time-dependent parts, where every point on the string is moving with a certain time dependence, but the amplitude of the oscillation is dependent on the displacement along the string.

If the string is fixed at both ends, then it is also important that we have a sine function rather than cosine in the x-dependent part, as the cosine would not satisfy the boundary condition of $y(0,t) = 0$ for all values of t. However, it wouldn't matter if we had sine or cosine for the time dependent part as we can always turn one into the other with a phase shift ϕ.

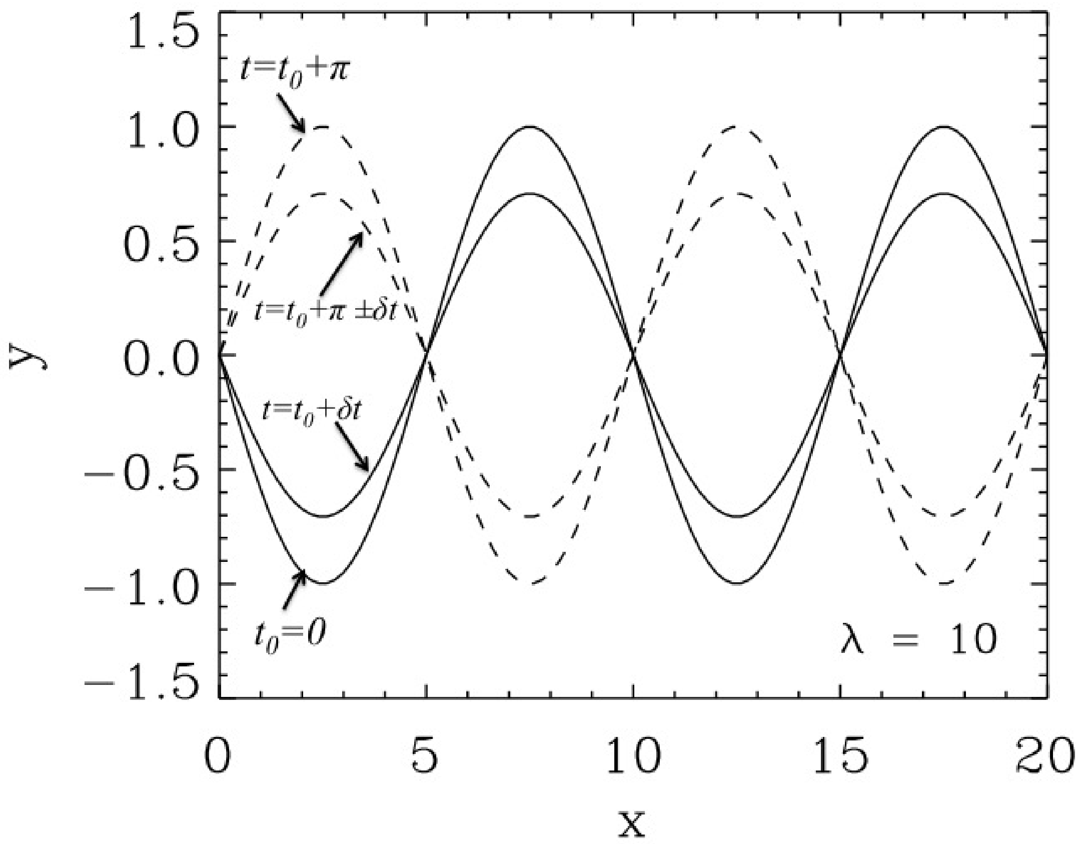

Figure 5.1: Standing wave with two full wavelengths shown. The solid curves are for $t_0 = 0$ and $t = t_0 + \delta t$ and the dashed curved shows where the waves would be at a time $t = t_0 + \pi$ and $t = t_0 + \pi \pm \delta t$ later. For this standing wave the nodes are the stationary points where the wave crosses the $y = 0$ axis.

In order to obtain the form of the wave shown in Fig. 5.1, then it is clear that we are assuming that the string is fixed at both ends. In this situation, each end of the string must therefore be a node. For a string of length L, Eq. 5.2 along with the boundary condition of $y(0,t) = y(L,t) = 0$, results in the fact that

$$\sin\left(\frac{2\pi L}{\lambda}\right) = 0, \tag{5.3}$$

this is satisfied when $2\pi L/\lambda = n\pi$, therefore

$$\lambda_n = \frac{2L}{n}, \tag{5.4}$$

where λ_n is the wavelength of the n^{th} mode. In Fig. 5.1 the excited mode shown is for $n = 4$, i.e. two whole wavelengths.

We can check if this is consistent with the previous description of a lumpy string, recalling that the velocity of the wave is just $c = \sqrt{T/\rho}$ and the relation between angular frequency and wavelength is just $\lambda = 2\pi c/\omega$, then

$$\omega_n = n\frac{\pi}{L}\sqrt{\frac{T}{\rho}}. \tag{5.5}$$

We already obtained this result in Sec. 3.1, when discussing the "lumpy string" with N large.

We therefore obtain stationary points along the x-direction, these are the *nodes* with $y = 0$ and they occur every $\lambda/2$ wavelengths. Between these nodes, i.e. the peaks, are the *anti-nodes*. All points on the string have the same phase, or are multiples of π, in terms of how the oscillations move in time. For example, all the points are at rest at the same time, when the string is at a maximum displacement from the equilibrium position, and they all pass through the origin or equilibrium position at the same time. These waves are therefore called *standing waves*, as opposed to travelling waves.

We will return to standing waves when discussing how waves are reflected and transmitted across boundaries.

5.2 Dispersion

The waves that we have considered thus far have all been *dispersionless*, i.e. waves for which the speed is independent of ω and k. All of these dispersionless waves, whether they are transverse or longitudinal, obey the wave equation of the form,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{5.6}$$

where the velocity c, just depends on the type of system that we are considering. For the stretched string we just found that the wave velocity is $c = \sqrt{T/\rho}$, i.e. the velocity at which the waves travel just depends on the properties of the string and does not have any dependence on the frequency (or wavelength) of the wave. But this is an idealised system!

In most systems the velocity of a wave does have a dependence on ω and λ, this dependence is called *dispersion*. A well know example of dispersion is when one shines white light through a prism. The light in the prism, which has a refractive index n, has a velocity $c_m = c/n$. In the case of the prism, the refractive index, and hence the velocity of the wave, varies depending on the wavelength of the light. This is why the light is bent at different angles depending on the wavelength of the light.

Figure 5.2: The dispersion of white light through a prism.

Let us now go back to the example of the lumpy string (Sec. 3.1), we found that the angular frequency of the nth normal mode is given by,

$$\omega = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right), \tag{5.7}$$

where $\omega_0 = \sqrt{T/ml}$, and $\lambda_n = 2L/n$, $k_n \equiv 2\pi/\lambda_n = n\pi/L$.

Fig. 5.3 shows how ω_n, normalised to ω_0 varies with k_n. Remembering that the velocity of the wave is just given by the gradient of this line, we immediately see that the velocity cannot be constant as the gradient is getting shallower as we move to higher k_n. This is dispersion. Note also that there is a *cut-off* frequency - a maximum frequency above which it is not possible to excite system/transmit waves - this is a property often found in a dispersive system, and is $2\omega_0$ as we found before.

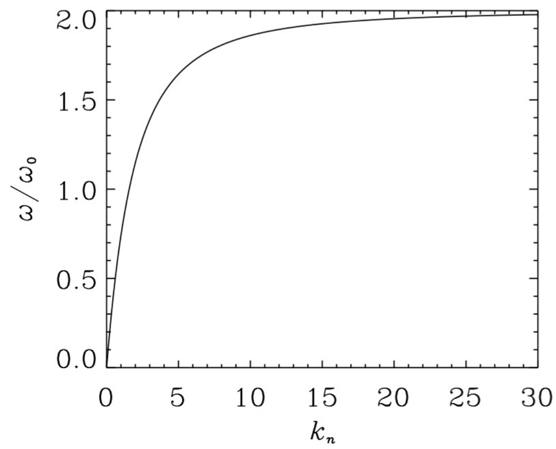

Figure 5.3: The dependence of the angular frequency ω_n/ω_0 on the wave number k_n of the nth normal mode.

Using the relations between k_n, n, L and λ, we can rewrite Eq. 5.7 as

$$\omega(k) = 2\omega_0 \sin\left(\frac{kl}{2}\right). \tag{5.8}$$

This is obviously different to the usual relation of $\omega = ck$ that we found for a continuous string. What is the velocity of a wave with wavenumber k? The velocity is actually still given by ω/k, so we have

$$c(k) = \frac{\omega}{k} = \frac{2\omega_0 \sin(kl/2)}{k}. \tag{5.9}$$

If we take this equation to the limit where l, the individual length elements of the lumpy string, is very small, then we have a continuous string, so hopefully Eq. 5.9 reduces to $c = \sqrt{T/\rho}$, the result we found for a transverse wave on a continuous string.

$$c(k) = \frac{2\omega_0 \sin(kl/2)}{k} \approx \frac{2\omega_0(kl/2)}{k} = \omega_0 l = \sqrt{\frac{T}{m/l}} \equiv \sqrt{\frac{T}{\rho}}, \tag{5.10}$$

where ρ is the density per unit length, so it does give us the result for a continuous string.

However, when l is not small, or more specifically, when $l \ll \lambda$ is not applicable, then the velocity of the wave, given by ω/k is no longer independent of k, so the lumpy string has dispersion.

Fig. 5.4 shows the behaviour ω with respect to k_n again, but this time the $x-$axis is altered to the parameter in the sine function in Eq. 5.7 or 5.9, divided by π.

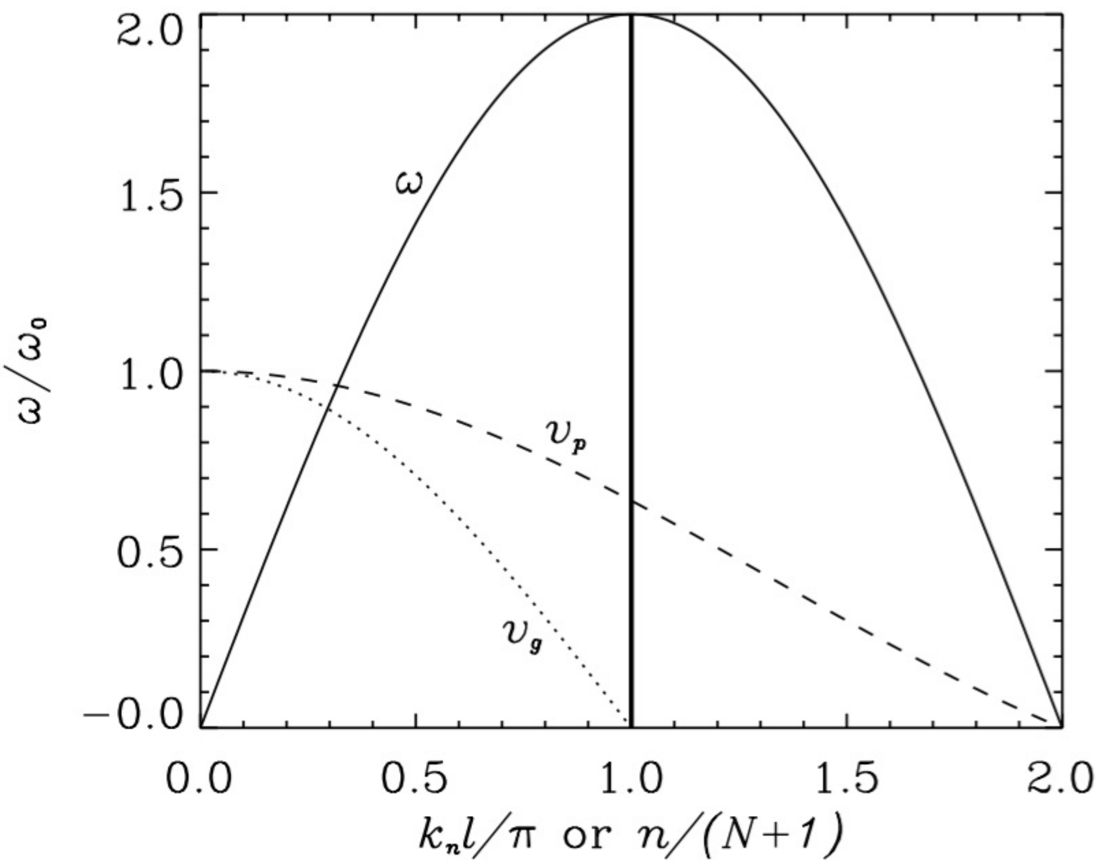

Figure 5.4: The dependence of the angular frequency ω_n/ω_0 (solid line) on the wave number k_n of the nth normal mode. Also shown are the phase velocity (v_p; dashed line) normalised to a maximum velocity of unity, and the group velocity (v_g; dotted line) again normalised to a maximum velocity of unity.

It is clear that the maximum angular frequency occurs at $k_n = \pi/l$, with minima occurring at $k_n = 0$ and $2\pi/l$. If this figure were extended we would find that the function is completely oscillatory to $k_n \to \infty$. However, the curve beyond $k_n = \pi/l$ is just a repeat of the curve to the left of $k_n = \pi/l$, albeit reversing in gradient and sign. This is obvious if we think about the relation between k_n, n, l and N. We know that n is just the number of segments of length l on a lumpy string that makes up a total length L which runs from $x = 0 \to x = N + 1$. So having $n > N$ does not really make much sense. However, let us just explore this briefly. Consider the case where we have a second wave of wave number $k_2 = 2\pi/l - k_1$, which we can express as

$$
\begin{aligned}
A\cos(k_2 x - \omega t) &= A\cos\left[\left(\frac{2\pi}{l} - k_1\right)x - \omega t\right] \\
&= A\cos(2n\pi - k_1(nl) - \omega t) \quad \text{as} \quad x = nl \\
&= A\cos(-k_1 x - \omega t).
\end{aligned} \tag{5.11}
$$

Therefore, a wave travelling to the right with $k_2 = 2\pi/l - k_1$ and frequency ω gives the same position for a point along the wave as a wave travelling to the left with wavenumber k_1 and frequency ω. Although the waves from each of k_1 and k_2 have the same displacements along the $y-$axis, the waves themselves obviously look different, as their wavelength is defined as $\lambda_n = 2\pi/k_n$. For example, if $k_1 = \pi/2l$, then $k_2 = 2\pi/l - k_1 = 3\pi/2l$, i.e. $k_2 = 3k_1$, and therefore has a factor of 3 shorter wavelength. However, the two waves would have the same displacement at $x = nl$(where $l = 1$ here). The fact that ω is the same for both waves, but the k is different means that the two waves move with different velocity, remembering $c = \omega/k$, so as $k_2 = 3k_1$, wave 2 moves a factor of 3 more slowly than wave 1.

Given that the velocity of the wave decreases with increasing k_n then it is unsurprising that eventually the velocity of the waves with very high k tend towards zero.

5.3 Information transfer and Wave Packets

Regardless of whether a system is dispersionless or has dispersion, the phase velocity v_p, is always given by ω/k. The phase velocity therefore describes the speed of a single sinusoidal travelling wave. However, in order to transmit information, then waves need to be modulated somehow, for example the pitch of a note can be changed by modulating the frequency of a wave, the volume of a note or noise can be increased by increasing the amplitude of a wave, which is a general form of pulse modulation in which the wave is essentially switched off and on. By implementing such modulation to a wave form, then information can be transmitted. Fig. 5.5 shows the various forms of modulation.

A simple example for transmitting information is if we combine two waves of different frequency and/or wavelength. Consider two waves which differ by $2\delta\omega$ and $2\delta k$ in angular

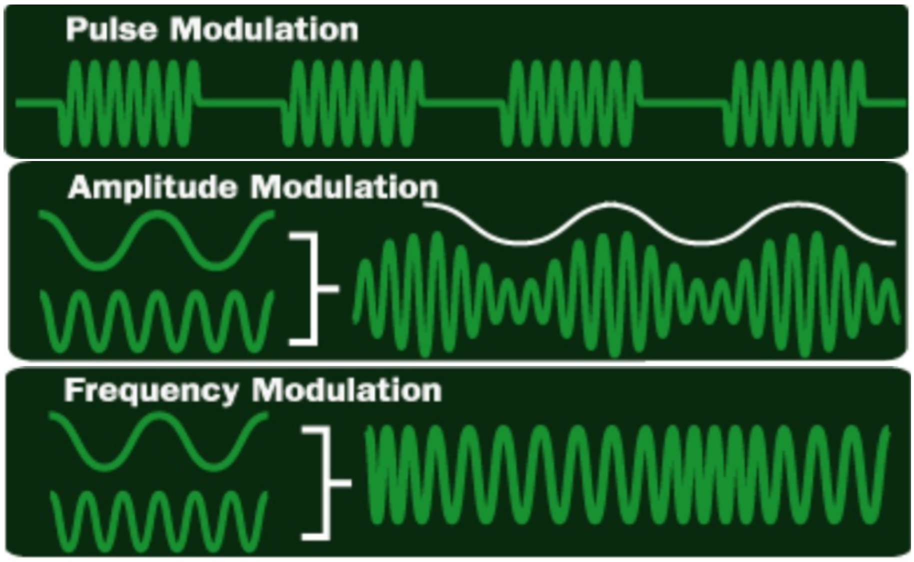

Figure 5.5: Various wave modulation strategies for transmitting information.

frequency and wave-number respectively:

$$\begin{aligned} y_1 &= A\sin\left[(k+\delta k)x - (\omega+\delta\omega)t\right] \\ y_2 &= A\sin\left[(k-\delta k)x - (\omega-\delta\omega)t\right]. \end{aligned} \tag{5.12}$$

Therefore,

$$y = y_1 + y_2 = 2A\cos(\delta kx - \delta\omega t)\sin(kx - \omega t) \tag{5.13}$$

Fig. 5.6 shows what such a combination of waves looks like, and with just two waves there is a wave with high frequency within a low-frequency wave. This is not really a "wave-packet", to form a modulated wave that resorts in a wave form with sharper edges would require many more individual waves.

However, it is worth noting that the velocity of the *wave packet* is $\delta\omega/\delta k$.

5.4 Group Velocity

The velocity of the wave packet is known as the *group velocity.* In almost all cases, this is the speed at which information in transmitted. In a dispersive medium the group velocity is not the same at the velocity of the individual waves, which is the *phase velocity.* We noted in the last section that the wave packet that was produced by combining two waves, moved with a velocity of $\delta\omega/\delta k$, and indeed the group velocity is defined as the differential of the angular frequency with respect to the wave number.

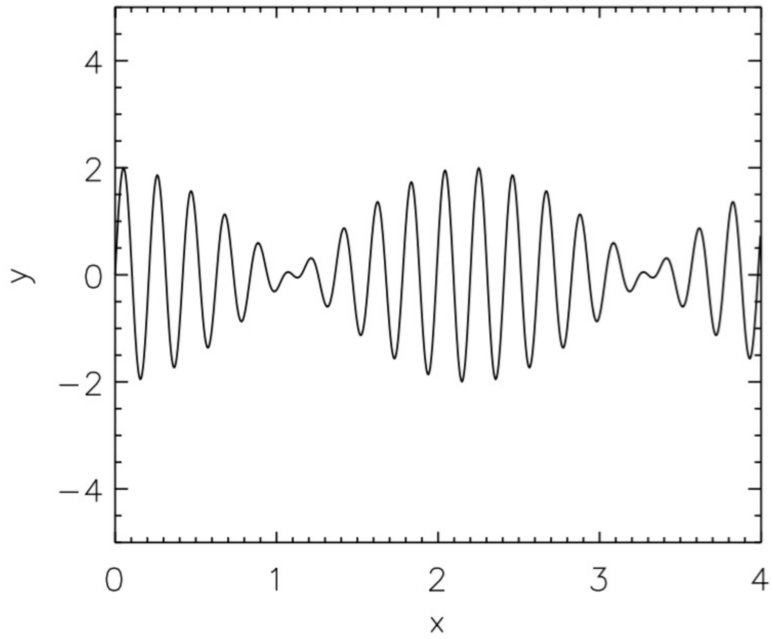

Figure 5.6: Combination of two sine waves, showing a low-frequency wave enveloping a higher frequency wave.

$$\text{Group Velocity} \quad v_g = \frac{d\omega}{dk} \qquad\qquad \text{Phase Velocity} \quad v_p = \frac{\omega}{k} \tag{5.14}$$

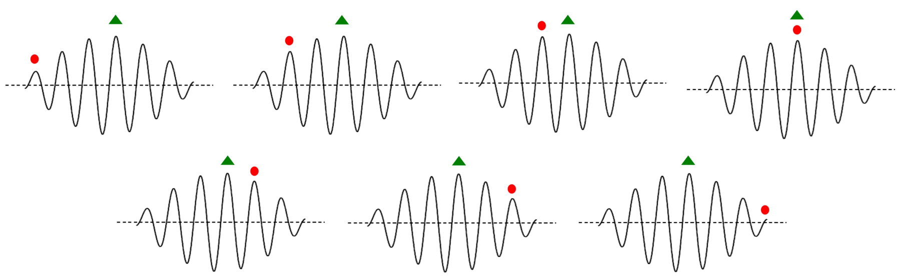

Figure 5.7: A travelling wave packet. The triangle marks the top of the wave packet which moves with the group velocity. The circle indicates a component of a wave crest which enters the wave packet, moves through it, and leaves with the phase velocity.

Another consequence of dispersion is that the wave packet does not retain its shape perfectly, it actually spreads out as it carries the information. For example, a perfect unmodulated sine wave would gradually stretch out to a similar shape as the wave packet shown in Fig. 5.6.

5.5 Another way of deriving group velocity

The group velocity is the speed at which a wave packet travels. If this wave packet consists of many individual wave components, with many different frequencies, then in order for these individual waves to constructively interfere to form the wave packet with a well defined bump then the phases of the individual wave must be equal at the bump. If they weren't we would not get the bump in the first place. So if these individual waves are described by $y_i(x,t) = \omega_i t - k_i x + \phi_i$, and that the peak of the wave packet is situated at $x = 0$ and $t = 0$, by invoking the fact that the phases at the peak are all equal or very similar, then ϕ is independent of k. All we now need to do is calculate at what other points along x and t are the phases roughly equal, so that they constructively interfere to give the peak of the wave packet. So if we want the phase to be independent of k, i.e. $d\phi/dk = 0$, and we implicitly assume that ω is a function of k, i.e, $\omega(k)$, we can set

$$\frac{d(\omega t - kx + \phi)}{dk} = 0, \tag{5.15}$$

which leads to

$$\frac{d\omega}{dk}t - x = 0, \tag{5.16}$$

$$\frac{x}{t} = v_g = \frac{d\omega}{dk} \tag{5.17}$$

It is also worth noting that the phase velocity of a single travelling wave can be found by insisting that the phase of the wave is independent of time, such that

$$\frac{d(\omega t - kx + \phi)}{dt} = 0, \tag{5.18}$$

which leads to

$$\omega - k\frac{dx}{dt} = 0, \tag{5.19}$$

$$\frac{dx}{dt} = v_p = \frac{\omega}{k}. \tag{5.20}$$

5.5.1 Dispersion and the spreading of a wave packet

Another consequence of dispersion is that a wave packet will not retain its shape perfectly, but will spread out (Fig. 5.8). This is due to the fact that in wave packet that is made up of a large number of individual sinusoids, each with wave number k and frequency ω. These individual waves obey the phase velocity relation of $v = \omega/k$, as such the individual waves with higher ω move faster through the wave packet than those waves with lower ω. As such the wave packet spreads out, losing its original shape.

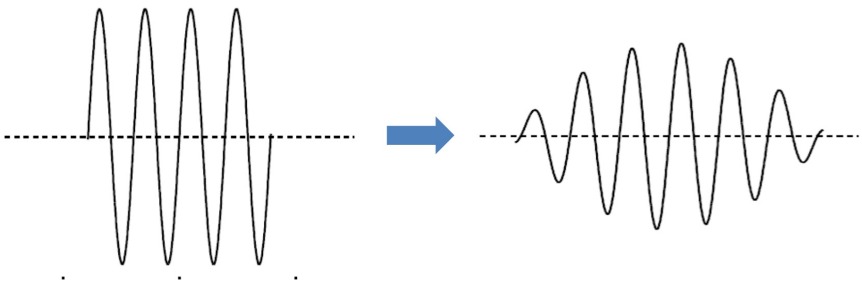

Figure 5.8: A travelling wave packet. After a period of time the wave packet loses its original shape and spreads out.

5.6 Faster than the speed of light?

An obvious comment that one could make about the group velocity is that if $v_g = \delta\omega/\delta k = (\omega_2 - \omega_1)/(k_2 - k_1)$, and $\omega_1 \neq \omega_2$ and $k_1 \sim k_2$ but $k_1 \neq k_2$ then v_g is large. Therefore, if k_1 was very close to k_2, then v_g could be arbitrarily large, and exceed c. Obviously, this cannot be true. The key is that to communicate information, it is a *change* in the wave that matters, and with a bit of maths and relativity (beyond the scope of this course), it can be shown that the *leading edge* of this change can never travel faster than c.

5.7 Uses for dispersion

The dispersion properties of waves can be used in many real-world situations, and we give a few examples of these in the following sections.

5.7.1 Distance to a storm at sea

If we have a dispersion relation of the form $\omega \approx \sqrt{gk}$, such that the phase velocity of the wave is $v_p = \sqrt{(g\lambda)/(2\pi)}$, so longer wavelengths travel more quickly.

We know that the period of a wave is given by:

$$\tau = \frac{\lambda}{v_p} = \left(\frac{2\pi\lambda}{g}\right)^{\frac{1}{2}}, \tag{5.21}$$

and the time of arrival for a crest of a wave if given by:

$$t = t_0 + \frac{L}{v_p} = t_0 + L\frac{\tau}{\lambda}. \tag{5.22}$$

where L is the distance at which the wave was generated. We can eliminate the dependence on the wavelength, finding that

$$\tau = \frac{2\pi L}{g(t - t_0)}. \tag{5.23}$$

Therefore, the rate of decrease in the wave period is related to the distance, and is given by

$$-\frac{d\tau}{dt} = \frac{2\pi L}{g(t - t_0)^2} = \frac{g\tau^2}{2\pi L} \tag{5.24}$$

So if the period and rate of the decrease in the period can be measured, then the distance to the source of the wave can be determined. If the source was a storm for example, it would provide a measurement of the distance to a storm.

5.7.2 Pulsars and the interstellar medium

Pulsars are the spinning remnants of collapsed massive stars. They are roughly the size of a city ($\sim$ 10km in diameter) but have the mass similar to our own Sun ($M \sim 10^{30}$kg). The magnetospheres of these spinning "neutron stars" results in a pulse of radiation at radio frequencies. However, due to their small size, pulsars are relatively weak radio sources. Therefore, the largest radio telescopes in the world are usually needed to observe them. Pulsars emit their largest intensity at low radio frequencies around 400 MHz. In particular at such frequencies, however, the pulses suffer from propagation effects when they travel to Earth through the interstellar medium.

The most obvious effect is dispersion. By interacting with the free electrons in the interstellar medium, pulses at lower frequencies are delayed. In other words, pulses emitted at higher frequencies arrive earlier than those emitted at lower frequencies. This effect is shown in Fig. 5.9 which shows pulses at different, adjacent radio frequencies.

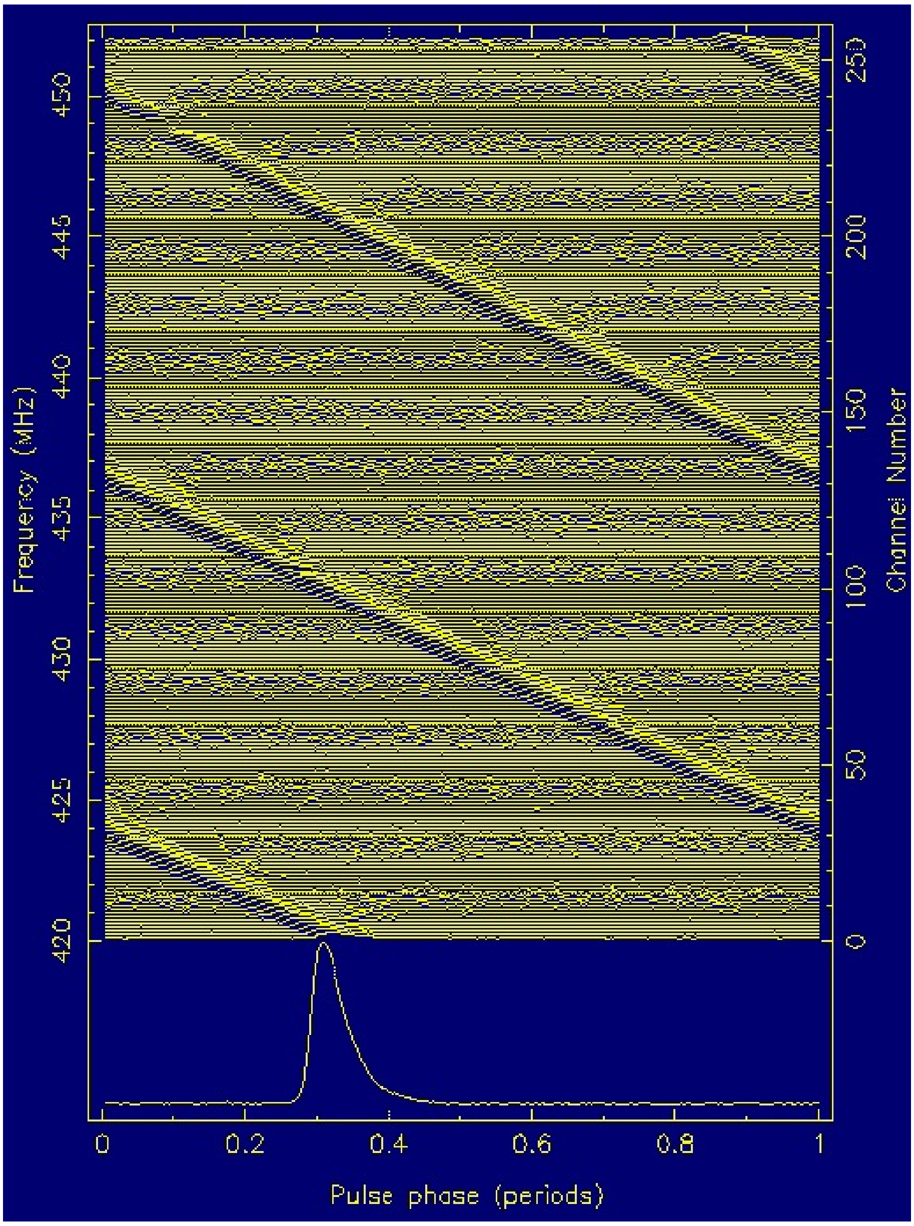

Figure 5.9: Data showing the dispersion effect in the progressive delay of pulses to higher frequencies. Taken from http://www.jb.man.ac.uk/distance/frontiers/pulsars/.

The bottom of Fig. 5.9 shows the pulse profile obtained after delaying the high frequency pulses until the lowest frequency arrived before summing up all frequency channels.

This process to correct for dispersion is called de-dispersion. If the delay would not have been accounted for, the summed pulse would have been blurred and smeared. If the delay is too big, the pulses may become undetectable.

5.7.3 Fast Radio Bursts and the intergalactic medium

A phenomenon that has recently been discovered in the Universe are Fast Radio Bursts (FRBs; the first one was discovered by Lorimer et al., 2007, Science, 318, 777)). These are high energy events that are detected as a radio *pulse* which lasts only a few milli-seconds. As with pulsars, FRBs also exhibit dispersion where the time delay as a function of frequency is consistent with the wave propagating through an ionised plasma. Just over 10 FRBs have been detected thus far.

We don't know what FRBs are but the dispersion gives us some indication about how far away they are. The very high level of dispersion suggests that the waves travel through much more ionised plasma than a typical pulsar. This could be due to a number of reasons, but probably the most obvious and compelling is that they originate at extragalactic distances. Indeed their distribution over the sky also indicates that they may be cosmological in origin, rather than from our own galaxy.

One of the key tools we use to measure the nature and evolution of Drak Energy is to use Type Ia supernovae as standard candles, where we know the intrinsic luminosity of a supernova of this type, and therefore by measuring the flux we receive at Earth we can essentially use the inverse square law to calculate the distance. If we can then measure its redshift, this tells us how much the Universe has expanded in this time. We can therefore relate distance to redshift which in turn can tell us how quickly the Universe is expanding as a function of distance. As is now old news, it was found that the Universe was accelerating and the Nobel Prize for physics was given for this discovery in 2011.

Some authors have recently suggested that FRBs could be used as cosmological probes in a similar way, if we can measure their redshifts. The way this would work (your lecturer remains sceptical) is that if you know the ionised content of the Universe on average (which we more or less do), then the amount of dispersion is directly linked to the actual distance

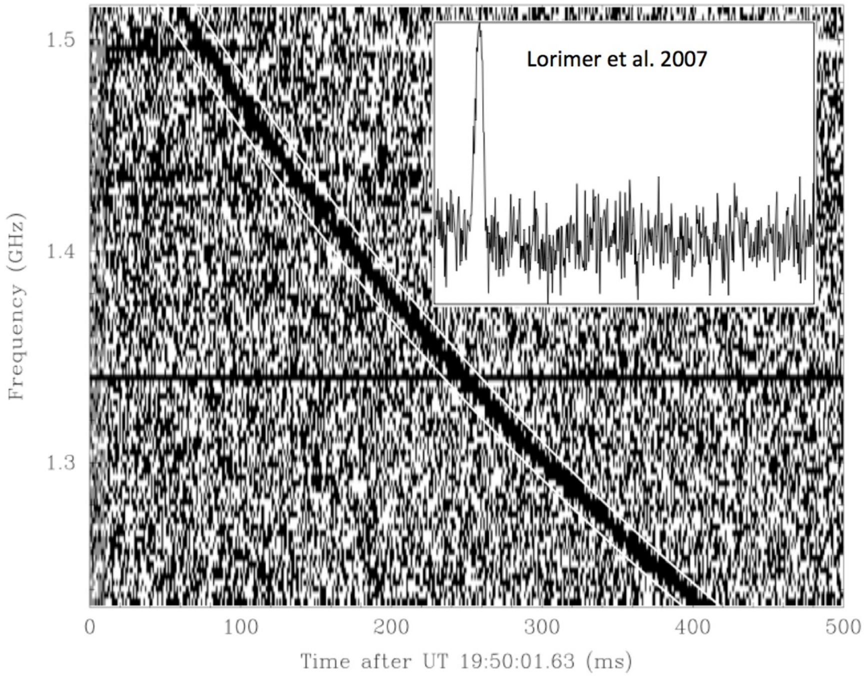

Figure 5.10: The frequency dependent arrival time of the very short pulse of emission from the first FRB to be discovered (Lorimer et al. 2007). The inset shows the pulse after adding together all of the frequency channels and correcting for the dispersion.

of the FRB. So we could have another connection between distance and redshift.

Chapter 6

Waves III

6.1 Energy stored in a mechanical wave

A vibrating string, such as that described in Sec. 4.1.1 must carry energy, but how much? In this section we will look at the relation between kinetic energy and potential energy as the string vibrates.

Consider a small segment of string (Fig. 6.1) of linear density ρ between x and $x+dx$, displaced in the y direction. Assuming the displacement is small then we can calculate the kinetic energy density (the K.E. per unit length) and the potential energy density.

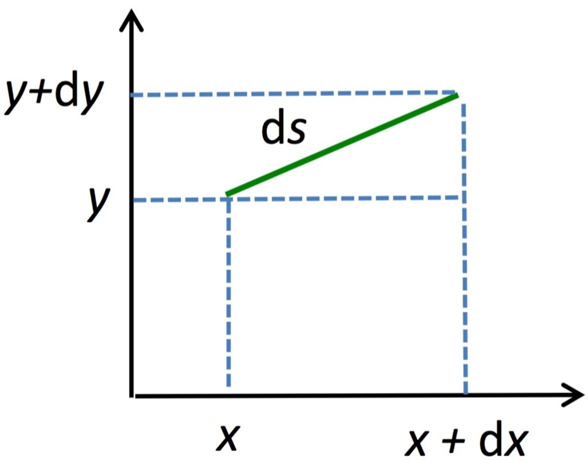

Figure 6.1: Zoom in of a segment of a stretched string.

6.1.1 Kinetic Energy

The kinetic energy is just given by,

$$K = \frac{1}{2}\left(\frac{m}{l}.dx\right)u_y^2 = \frac{1}{2}(\rho.dx)u_y^2 = \frac{1}{2}\rho.dx\left(\frac{\partial y}{\partial t}\right)^2. \quad (6.1)$$

Therefore, the kinetic energy density is,

$$KE \text{ density} = \frac{dK}{dx} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2. \quad (6.2)$$

6.1.2 Potential Energy

The potential energy, U, is equivalent to the work done by deformation,

$$U = T(ds - dx) \quad (6.3)$$

and

$$(ds)^2 = (dx)^2 + (dy)^2 = (dx)^2\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right) \quad (6.4)$$

Using a Binomial series expansion,

$$ds \approx dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + ...\right) \quad (6.5)$$

Therefore,

$$U = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2.dx \quad (6.6)$$

and

$$PE \text{ density} = \frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \quad (6.7)$$

We know that the solutions to the wave equation take the form

$$y(x, y) = f(x \pm ct),$$

therefore

$$\frac{\partial y}{\partial x} = f'(x \pm ct), \quad \text{and} \quad \frac{\partial y}{\partial t} = \pm cf'(x \pm ct).$$

Therefore,

$$\frac{dK}{dx} = \frac{1}{2}\rho c^2 \left[f'(x \pm ct)\right]^2 \quad \text{and} \quad \frac{dU}{dx} = \frac{1}{2}T \left[f'(x \pm ct)\right]^2. \tag{6.8}$$

We also know that $c = \sqrt{T/\rho}$, therefore we find that the kinetic energy density is equal to potential energy density. This is one manifestation of the *Virial Theorem.*

If we substitute a solution for the wave equation into these equations for the KE and PE density of the form $y = A\sin(kx - \omega t)$, we can evaluate the energy over n wavelengths.

$$K = \frac{1}{2}\rho \int_x^{x+n\lambda} A^2\omega^2 \cos^2(kx - \omega t).dx \qquad U = \frac{1}{2}T \int_x^{x+n\lambda} A^2k^2 \cos^2(kx - \omega t).dx$$

$$K = \frac{1}{2}\rho A^2\omega^2 \int_x^{x+n\lambda} \frac{1}{2}\left(1 + \cos[2(kx - \omega t)]\right).dx \qquad U = \frac{1}{2}TA^2k^2 \int_x^{x+n\lambda} \frac{1}{2}\left(1 + \cos[2(kx - \omega t)]\right).dx$$

$$K = \frac{1}{2}\rho A^2\omega^2 \frac{n\lambda}{2} \qquad U = \frac{1}{2}TA^2k^2 \frac{n\lambda}{2} \tag{6.9}$$

As $c = \sqrt{T/\rho} = \omega/k \Rightarrow \rho\omega^2 = Tk^2$, therefore these expressions for the kinetic and potential energy are equal.

Therefore the total energy per unit length $= \frac{1}{2}\rho A^2\omega^2$.

Now we have the energy it is trivial to evaluate the energy flow per unit time, which is equivalent to the power to generate the wave.

$$P = \text{Energy/wavelength} \times \text{distance travelled/time}$$
$$P = \frac{1}{2}\rho A^2\omega^2 v = \frac{1}{2}Tk^2A^2\frac{\omega}{k} = \frac{1}{2}T\omega kA^2 \tag{6.10}$$

6.2 Solving the wave equation by separation of variables

In Section 4.2 we solved the wave equation using d'Alembert's solution, where we split the form of the wave into waves travelling in positive and negative directions. We can also look for solutions that have a separable form in the displacement, x and time, t, i.e.,

$$y(x,t) = X(x)T(t) \tag{6.11}$$

Substituting this solution into the wave equation (Eq. 4.7), we find

$$\begin{aligned} T(t)\frac{d^2X(x)}{dx^2} &= \frac{1}{c^2}X(x)\frac{d^2T(t)}{dt^2} \\ \implies \frac{\ddot{X}}{X} &= \frac{1}{c^2}\frac{\ddot{T}}{T} \end{aligned} \tag{6.12}$$

This equation can only be satisfied if both sides equal a constant, the so-called separation constant.

$$T(t)\frac{d^2X(x)}{dx^2} = \frac{1}{c^2}X(x)\frac{d^2T(t)}{dt^2} = C \tag{6.13}$$

We can consider general assumptions about the constants, i.e. *positive*, *negative* or *zero*, to see what this means physically.

6.2.1 Negative C

So let us first consider the case where C is negative, such that $C = -k^2$.

$$\frac{d^2X(x)}{dx^2} = -k^2X \quad \text{and} \quad \frac{d^2T(t)}{dt^2} = -(ck)^2T.$$

These have a familiar form, in this case the 2nd derivative of the displacement and the time terms are equal to the negative of the displacement and time respectively, i.e. they are SHM equations. We therefore can write down the solutions we know for such a equations, namely the sine waves.

$$X(x) = A\cos kx + B\sin kx \quad \text{and} \quad T(t) = D\cos ckt + E\sin ckt \tag{6.14}$$

Plugging this back into Eq. 6.11 we obtain,

$$y(x,t) = X(x)T(t) = (A\cos kx + B\sin kx)(D\cos ckt + E\sin ckt)$$

$$= AD\cos kx\cos ckt + BE\sin kx\sin ckt + AE\cos kx\sin ckt + BD\sin kx\sin ckt$$

which using some trig identities just reduces to,

$$\begin{gathered} y(x,t) = F\sin(kx+ckt) + G\cos(kx-ckt) \\ y(x,t) = F\sin(kx+\omega t) + G\cos(kx-\omega t) \quad \text{with} \quad \omega = ck. \end{gathered} \tag{6.15}$$

This is exactly the same as d'Alembert's solution where we have a positive and negative components of a travelling wave, and the functions f and g are just the sine and cosine terms.

6.2.2 Positive C

If C is positive we can express in terms of the exponential in exactly the same way as we could have done for the $-k^2$ case. But as we do not require the second derivative to be equal to the negative of a constant then we do not require the complex exponential form.

$$y(x,t) = (Ae^{kx} + Be^{-kx})(De^{kct} + Ee^{-kct}) \tag{6.16}$$

6.2.3 C=0

For $C = 0$ then there is no variation in the second derivative, and we find a linear form, such that

$$y(x,t) = (A + Bx)(D + Et). \tag{6.17}$$

We have only considered three broad examples for the separation constant C, but an infinite number exist in reality, and this purely depends on the physical situation. In this course we are generally interested in oscillatory solutions, i.e. C is negative. But there are obviously many different negative solutions which give different values for k.

6.3 String with Fixed Ends: Superposition of modes

If we now consider a string with fixed ends once again, and use a solution to the wave equation of the form derived from the separation of variables, i.e

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct), \tag{6.18}$$

along with the boundary conditions:

i) String initially at rest, i.e. $\partial y/\partial t = 0$ for all $x \Longrightarrow D = 0$

ii) $y(0,t) = 0 \Longrightarrow A = 0$

iii) $y(L,t) = 0 \Longrightarrow kl = n\pi$, where n is any integer. Each value of n corresponds to a normal mode. For this continuous system n can go to infinity.

Let us now look at a specific example of a string in which the initial conditions mean that more than one mode is excited.

So from the principle of superposition, the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \tag{6.19}$$

i.e. sum of all possible solutions with coefficients F_n given by the initial displacement, which is the boundary condition we have not yet invoked.

If $h(x)$ describes a pattern for the initial displacement of a finite string then,

$$y(x,0) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L} = h(x). \tag{6.20}$$

If $h(x)$ is just a single $n = 5$ normal mode then we find

$$h(x) = \sin\frac{5\pi x}{L} \quad \text{for } F_5 = 1 \text{ and } F_n = 0 \text{ when } n \neq 5 \tag{6.21}$$

therefore,

$$y(x,t) = \sin\frac{5\pi x}{L}\cos\frac{5\pi vt}{L}. \tag{6.22}$$

However, when there is more than one mode active, e.g. when $F_1 = 1$, $F_2 = 0.5$ and $F_n = 0$

when $n \neq 1$ or 2, we obtain

$$y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi v t}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi v t}{L}. \tag{6.23}$$

In contrast to the case with just a single normal mode, the subsequent motion of the case with > 1 mode active is not equal to the initial displacement multiplied by a time-dependent amplitude. This is because the shorter waves move faster, resulting in the shape of the wave varying with time.

Note: Even if the initial displacement takes the most simple form (i.e. a plucked string at the centre), it can be expressed as a sum of normal modes. You will see more of this in Year 2 when considering Fourier Series.

6.4 Energies of normal modes for string with fixed ends

Let us now consider the energy associated with each normal mode for the finite string solution discussed in the previous section.

The general solution for the motion of a string fixed at both ends, is given by

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L} \tag{6.24}$$

So as before, we can calculate the kinetic energy in the fixed string for the nth normal mode:

$$\begin{aligned} K_n &= \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 .dx \\ &= \frac{1}{2}\rho F_n^2\left(\frac{n\pi v}{L}\right)^2 \sin^2\frac{n\pi v t}{L}\int_0^L \sin^2\frac{n\pi x}{L}.dx \\ &= \frac{\rho(F_n n\pi v)^2}{4L}\sin^2\frac{n\pi v t}{L} \end{aligned} \tag{6.25}$$

and the potential energy in the fixed string:

$$\begin{aligned} U_n &= \int_0^L \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 .dx \\ &= \frac{1}{2}TF_n^2\left(\frac{n\pi}{L}\right)^2 \cos^2\frac{n\pi v t}{L}\int_0^L \cos^2\frac{n\pi x}{L}.dx \\ &= \frac{T(F_n n\pi)^2}{4L}\cos^2\frac{n\pi v t}{L} \end{aligned} \tag{6.26}$$

The total energy in each normal mode is given by $E_n = K_n + U_n$, and since, $v = \sqrt{T/\rho}$, then

$$\begin{aligned} E_n = K_n + U_n &= \frac{\rho L F_n^2}{4} v^2 \left(\frac{n\pi}{L}\right)^2 \\ &= \frac{\rho L F_n^2 \omega_n^2}{4} \quad \text{as} \quad \omega_n = \frac{n\pi v}{L} \end{aligned} \tag{6.27}$$

6.5 Total energy in a fixed string

We can now determine the total energy in string fixed at both ends by just generalising the calculation in Sec. 6.4. So for a system with initial arbitrary displacement,

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}$$

the partial derivative with respect to x and t have the form,

$$\begin{aligned} \left(\frac{\partial y_n}{\partial t}\right)^2 =& (\alpha c F_n)^2 \sin^2\alpha x \sin^2\alpha ct + (\beta c F_m)^2 \sin^2\beta x \sin^2\beta ct \\ & + 2\alpha\beta c^2 F_n F_m \sin\alpha x \sin\beta x \sin\alpha ct \sin\beta ct \\ \left(\frac{\partial y_n}{\partial x}\right)^2 =& (\alpha F_n)^2 \cos^2\alpha x \cos^2\alpha ct + (\beta F_m)^2 \cos^2\beta x \cos^2\beta ct \\ & + 2\alpha\beta F_n F_m \cos\alpha x \cos\beta x \cos\alpha ct \cos\beta ct \end{aligned} \tag{6.28}$$

where $\alpha = \dfrac{n\pi}{L}$ and $\beta = \dfrac{m\pi}{L}$.

This is simple extension of exercise for individual normal modes, but with additional terms

$$E = \sum_{n=1}^{\infty} E_n + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\alpha x \sin\beta x \quad \text{or} \quad \cos\alpha x \cos\beta x \quad \text{terms} \tag{6.29}$$

Now if $n \neq m$,

$$\begin{aligned} \int_0^L \sin\alpha x \sin\beta x &= \left[\frac{\sin[(\alpha-\beta)x]}{2(\alpha-\beta)} - \frac{\sin[(\alpha+\beta)x]}{2(\alpha+\beta)}\right]_0^L = 0 \\ \int_0^L \cos\alpha x \cos\beta x &= \left[\frac{\sin[(\alpha-\beta)x]}{2(\alpha-\beta)} + \frac{\sin[(\alpha+\beta)x]}{2(\alpha+\beta)}\right]_0^L = 0 \end{aligned} \tag{6.30}$$

therefore the cross-terms all cancel, and we are left with,

$$E_{\text{tot}} = \sum_{n}^{\infty} E_n, \tag{6.31}$$

i.e. the total energy in the system is the sum of the energies in each normal mode, as we found in the first set of lectures for the coupled pendulum and spring-mass systems.

6.6 Reflection & Transmission of waves

Let us now consider what happens to a wave travelling along a string which no longer has a single uniform density, but has a step change in density at $x = 0$, with the string essentially extending from $-\infty < x < 0$ with a density of ρ_1 and for $0 < x < \infty$ with a density of ρ_2.

If the wave travels from the left-hand of the string towards the right, then we can write,

$$y(x,t) = A \sin(\omega t - k_1 x). \tag{6.32}$$

for $x < 0$. In this case k_1 contains the relevant information about the density of the string to the left of the boundary. Remembering that $k = \omega/c$, and that $c_{1,2} = \sqrt{T/\rho_{1,2}}$, thus $k_{1,2} \propto \sqrt{\rho_{1,2}}$.

We also know that although the density is no longer uniform, the tension in the string is uniform throughout, otherwise there would be a non-zero horizontal acceleration somewhere.

The wave moves to the right along the string towards $x = 0$, at $x = 0$ two things could happen, (i) the wave could be reflected resulting in a wave travelling to the left, and (ii) the wave could be transmitted across the boundary and continue moving to the right. Therefore, for the wave to the left of $x = 0$ we can write it as the sum of the incident and reflected waves,

$$y(x,t) = A \sin(\omega t - k_1 x) + A' \sin(\omega t + k_1 x). \tag{6.33}$$

For the transmitted wave, we just have the component moving to the right at $x > 0$:

$$y(x,t) = A'' \sin(\omega t - k_2 x) \tag{6.34}$$

where we now have k_2 which contains the information about the density of the string at $x > 0$.

6.6.1 Boundary Conditions

We can now apply some boundary conditions to determine how the amplitude of the transmitted and reflected waves depends on the density of the string.

We know that the string is continuous across the boundary, so that

$$y_1(0,t) = y_2(0,t)$$

We also know that the tension throughout the string is also constant, implying that the vertical tension to the left of the boundary is balanced by the vertical component of the tension to the right of the boundary. Therefore, from Eq. 4.2,

$$F_y = T.\frac{\delta y}{\delta x} = \tan \delta\theta \approx T.\delta\theta,$$

therefore,

$$\frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t). \tag{6.35}$$

So applying these boundary conditions at $x = 0$ along with the fact that the string is continuous, we find

$$\begin{aligned} A \sin \omega t + A' \sin \omega t &= A'' \sin \omega t \\ \Rightarrow A + A' &= A'', \end{aligned} \tag{6.36}$$

and that we have balanced vertical tension,

$$\begin{aligned} -k_1 A \cos \omega t + k_1 A' \cos \omega t &= -k_2 A'' \cos \omega t \\ \Rightarrow k_1(A - A') &= k_2 A''. \end{aligned} \tag{6.37}$$

We can rewrite these equations in terms of *reflection* and *transmission coefficients*, which are just the ratios of the amplitudes of the reflected and transmitted waves to the incident wave respectively.

$$r \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2} \tag{6.38}$$

$$t \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}. \tag{6.39}$$

6.6.2 Particular cases

Given these reflection and transmission coefficients, we can consider some specific cases,

- $k_1 = k_2$

  $r = 0$, $t = 1$ as you would expect, the string is just a single uniform density and there is no reflection, only transmission

- $k_1 < k_2$

  A' is negative and we can write down the equation for the reflected wave as $-|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$, i.e. there is a phase change at the boundary as we move from a less dense to a more dense string.

- $k_1 > k_2$

  the A' is positive, i.e. we don't get the phase change in this case where $\rho_1 > \rho_2$.

- $k_2 \to \infty$ (or $\rho_2 \to \infty$)

  in this case $r = \frac{A'}{A} \to -1$, i.e. full reflection with a phase change and no transmitted wave. This is unsurprising as it is just the same as the second string being immovable, i.e. having the string attached to a brick wall at $x = 0$.

6.7 Power flow at a boundary

We now have the reflection and transmission coefficients,

$$r \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2} \tag{6.40}$$

$$t \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}. \tag{6.41}$$

In Sec. 6.1 we showed that the power to generate a wave, was given by

$$P = \frac{1}{2} T\omega k A^2 \tag{6.42}$$

So the ratios of the reflected to incident power, R_r and the transmitted to incident power, R_t, are given by

$$R_r = \frac{\frac{1}{2}k_1 T\omega A'^2}{\frac{1}{2}k_1 T\omega A^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad R_t = \frac{\frac{1}{2}k_2 T\omega A''^2}{\frac{1}{2}k_1 T\omega A^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2} \tag{6.43}$$

Therefore,

$$R_r + R_t = \frac{(k_1^2 + k_2^2 - 2k_1 k_2) + (4k_1 k_2)}{(k_1 + k_2)^2} = 1, \tag{6.44}$$

as expected, there is no power loss in the system.

6.8 Impedance

Impedence is a general term in physics that describes the opposition of a material to a time varying current (in an electrical circuit) or indeed any wave-carrying system.

A general definition is that it is a measure of resistance to an alternating effect, and is equivalent to the ratio of a *push variable* (i.e. voltage or pressure) to a *flow variable* (i.e. current or particle velocity).

6.8.1 Impedence along a stretched string

One of the key assumptions that we made in the previous sections was that the tension in the string is uniform throughout the string. What happens if we relax this condition? What does this actually mean anyway?

First of all, let us consider how we might be able to alter the tension either side of $x = 0$, given that this implies that the nearly massless atom within the string at $x = 0$, would experience $\sim \infty$ acceleration!

We can get around this by joining the two halves of the string via a massless ring,which encircles a fixed frictionless pole (Fig. 6.2). The pole that sits at the boundary now balances the horizontal components of the tensions, so that the net horizontal force on the ring is zero. This obviously has to be the case as the ring must remain on the pole and can only move vertically.

However, in this case the net vertical force on the ring must also be zero, otherwise it would have infinite acceleration (as it is massless). This zero vertical component of the force means that $T_1 \sin\theta_1 = T_2 \sin\theta_2$. This can be written as,

$$T_1 \frac{\partial y_1(x=0,t)}{\partial x} = T_2 \frac{\partial y_2(x=0,t)}{\partial x} \tag{6.45}$$

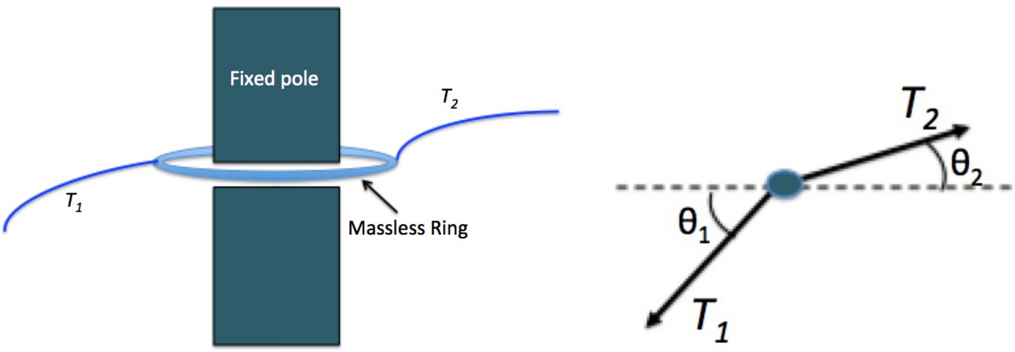

Figure 6.2: (*left*) Set up to imitate a system with non-uniform tension across the $x = 0$ boundary.(*right*) Forces acting on the ring.

In the previous examples, the vertical component of T_1 was equal to the vertical component of T_2, but now these vertical components to the tensions can be different. So we have the same form as Eq. 6.35, but now with the additional tension terms which no longer cancel out. Therefore, implementing the same form of the wave solution, i.e. $y(x,t) = A\sin(\omega t - kx)$, for incident, reflected and transmitted waves, we arrive at a very similar result, but with the tension in the string on either side of the massless ring also included, i.e.

$$A \sin \omega t + A' \sin \omega t = A'' \sin \omega t$$
$$\Rightarrow (A + A') = A'' \tag{6.46}$$

and differentiating with respect to x,

$$-k_1 T_1 A \cos \omega t + k_1 T_1 A' \cos \omega t = -k_2 T_2 A'' \cos \omega t$$
$$\Rightarrow k_1 T_1 (A - A') = k_2 T_2 A''. \tag{6.47}$$

Therefore, the new coefficients of reflection and transmission becomes slightly modified,

$$r \equiv \frac{A'}{A} = \frac{k_1 T_1 - k_2 T_2}{k_1 T_1 + k_2 T_2} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$
$$t \equiv \frac{A''}{A} = \frac{2 k_1 T_1}{k_1 T_1 + k_2 T_2} = \frac{2 Z_1}{Z_1 + Z_2} \tag{6.48}$$

where $Z_1 = k_1 T_1$ and $Z_2 = k_2 T_2$, or more correctly,

$$Z_1 = T_1 / v_1$$
$$Z_2 = T_2 / v_2 \tag{6.49}$$

where $v_{1,2}$ are the wave velocities. Note that replacing $k_{1,2}$ with $1/v_{1,2}$ just means that we have assumed $\omega_1 = \omega_2$, which it must across the massless ring (remembering $v = \omega / k$).

6.8.2 Physical meaning of impedence

Although we can describe impedence in the way we have above, what does it actually mean for this system?

The force acting on the right-hand side of the massless ring is just given by,

$$F_y = T_2 \frac{\partial y_2(x = 0, t)}{\partial x} \tag{6.50}$$

Substituting in a solution of the form $y_2(x,t) = A \sin[\omega(t - x/v_2)]$ (which is equivalent to $y_2(x,t) = A \sin(\omega t - kx)$, using the normal relation between k, ω and v).

The partial derivatives with respect to t and x,

$$\begin{aligned}\frac{\partial y_2}{\partial x} &= -\frac{A\omega}{v_2}\cos[\omega(t - x/v_2)] \\ \frac{\partial y_2}{\partial t} &= A\omega\cos[\omega(t - x/v_2)]\end{aligned} \tag{6.51}$$

therefore,

$$\frac{\partial y_2}{\partial x} = -\frac{1}{v_2}\frac{\partial y_2}{\partial t}. \tag{6.52}$$

Then we find,

$$F_y = -\frac{T_2}{v_2}\frac{\partial y_2(x=0,t)}{\partial t} = -\frac{T_2}{v_2}v_y \equiv -\gamma v_y, \tag{6.53}$$

where v_y is the transverse velocity of the ring at $x = 0$, and γ is defined as T_2/v_2. So we have a force that is proportional to the negative of the transverse velocity. Therefore, it acts exactly like a *damping force*! This means that from the perspective of the left string, the right string acts like a resistance that is being dragged against.

6.9 Reflection from a mass at the boundary

Now let us consider a slightly more complicated system, rather than having a massless ring around a frictionless pole at the boundary, we now have an object of mass M. Either side of this mass we have semi-infinite strings of linear density ρ_1 to the left and ρ_2 to the right, as shown in Fig. 6.3.

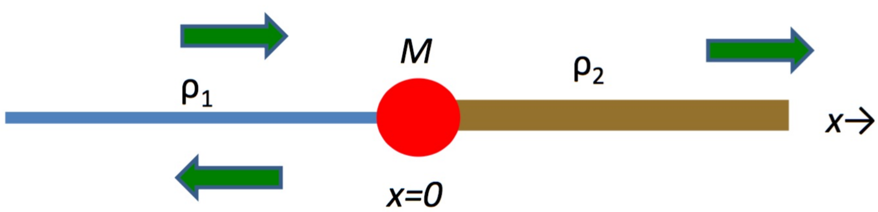

Figure 6.3: System of two strings of density ρ_1 and ρ_2 attached to an object of mass M at $x = 0$.

We can solve this system as we did in the previous examples, but in this case the boundary conditions are different.

We have the usual boundary condition that the system is continous and that at $x = 0$ the y displacement must be the same for the left and right side of the mass, i.e. $y_1(0,t) = y_2(0,t)$.

However, if we consider the forces at the boundary, we now have to consider the transverse acceleration of the objects which has finite mass, i.e.

$$-T\frac{\partial y_1(0,t)}{\partial x} + T\frac{\partial y_2(0,t)}{\partial x} = M\frac{\partial^2 y_1(0,t)}{\partial t^2} = M\frac{\partial^2 y_2(0,t)}{\partial t^2}. \tag{6.54}$$

As we will have to consider second derivatives, let us express the wave in terms of exponentials rather than sines and cosines. Then for a wave travelling from left to right (Fig. 6.3) with reflection and transmission at a boundary, we have

$$y_1(x,t) = \Re\left\{Ae^{i(\omega t - k_1 x)} + A'e^{i(\omega t + k_1 x)}\right\} \tag{6.55}$$

and

$$y_2(x,t) = \Re\left\{A''e^{i(\omega t - k_2 x)}\right\} \tag{6.56}$$

As before, with a continuous system we have $A + A' = A''$, but from Eq. 6.54 we have

$$\begin{aligned} ik_1TA - ik_1TA' - ik_2TA'' &= -\omega^2 M(A + A') = -\omega^2 MA'' \\ \Longrightarrow ik_1(A - A') &= \left(ik_2 - \frac{\omega^2 M}{T}\right) A'' \end{aligned} \tag{6.57}$$

From these we find,

$$r \equiv \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = Re^{i\theta} \tag{6.58}$$

$$t \equiv \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = Te^{i\phi} \tag{6.59}$$

where R and T are real numbers.

θ is the phase shift of the reflected wave and ϕ is the phase shift of the transmitted wave with respect to the incident wave. Therefore,

$$y_1(x,t) = A\cos(\omega t - k_1 x) + RA\cos(\omega t + k_1 x + \theta) \tag{6.60}$$

and

$$y_2(x,t) = TA\cos(\omega t - k_2 x + \phi) \tag{6.61}$$

where,

$$R = \left[\frac{(k_1-k_2)^2T^2+\omega^4M^2}{(k_1+k_2)^2T^2+\omega^4M^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \theta = \tan^{-1}\left[\frac{-\omega^2 M}{(k_1-k_2)T}\right] - \tan^{-1}\left[\frac{\omega^2 M}{(k_1+k_2)T}\right] \tag{6.62}$$

$$T = \left[\frac{4k_1^2T^2}{(k_1+k_2)^2T^2+\omega^4M^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = -\tan^{-1}\left[\frac{\omega^2 M}{(k_1+k_2)T}\right] \tag{6.63}$$

Checking that energy is conserved,

$$|r|^2 + \frac{k_2}{k_1}|t|^2 = R^2 + \frac{k_2}{k_1}T^2 = 1, \tag{6.64}$$

so it is.

6.10 Impedence in transmission lines

Consider a system made of inductors and capacitors, and let this system be continuous such that we can express the inductance per unit length (L') and the capacitance per unit length (C') of a coaxial cable. Assuming that this cable is lossless, i.e. it has zero resistance. Then we can express the voltage change across the capacitor and inductor as:

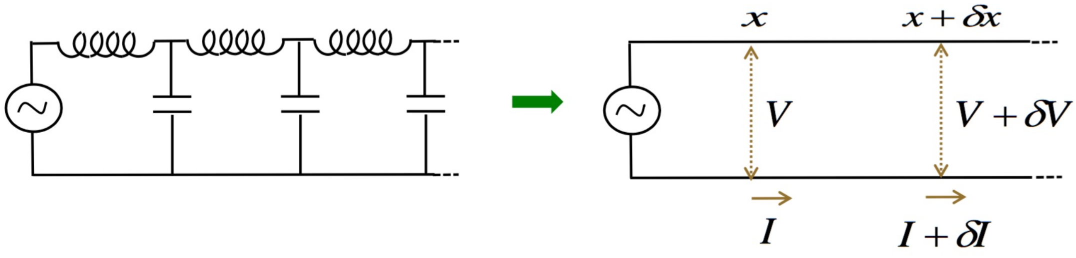

Figure 6.4: Lossless transmission.

Self-inductance of $\delta x = L'\delta x$,

$$\begin{aligned} \delta V &= -(L'\delta x)\frac{\partial I}{\partial t} \\ \Rightarrow \frac{\partial V}{\partial x} &= -L'\frac{\partial I}{\partial t} \end{aligned} \tag{6.65}$$

Capacitance of $\delta x = C'\delta x$,

$$\begin{aligned} \delta V &= -\frac{\delta Q}{C'\delta x} \\ \Rightarrow C'\frac{\partial V}{\partial t} &= -\frac{\partial I}{\partial x} \end{aligned} \tag{6.66}$$

So we have,

$$\frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t} \tag{6.67}$$

$$C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x} \tag{6.68}$$

These are the telegraph equations.

If we now differentiate Eq. 6.67 with respect to t and Eq. 6.68 with respect to x, and combine them, we find

$$\frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2}. \tag{6.69}$$

Likewise, if we differentiate Eq. 6.67 with respect to x and Eq. 6.68 with respect to t, we obtain

$$\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2}. \tag{6.70}$$

These are the wave equation expressed in terms of L' and C', so we know that the solutions have the form

$$V = V_0 \sin(\omega t \pm kx) \qquad I = I_0 \sin(\omega t \pm kx) \tag{6.71}$$

with

$$c = \frac{1}{\sqrt{C'L'}} \tag{6.72}$$

Therefore, if we have

$$\begin{gathered}\frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t} \quad \text{and} \\ V = V_0 \sin(\omega t \pm kx) \quad I = I_0 \sin(\omega t \pm kx)\end{gathered} \tag{6.73}$$

then

$$\pm k V_0 \cos(\omega t \pm kx) = -L' I_0 \omega \cos(\omega t \pm kx), \tag{6.74}$$

then the characteristic impedence is the ratio of the push variable (Voltage) to the flow variable (current),

$$Z = \frac{V_0}{I_0} = \pm\frac{\omega}{k}L' = \pm\frac{1}{\sqrt{C'L'}}L' = \pm\sqrt{\frac{L'}{C'}}, \tag{6.75}$$

where $\sqrt{L'/C'}$ is positive for a forward travelling wave.

6.10.1 Reflection at a terminated line

We can also consider how a wave reflects for a transmission line of characteristic impedence Z_0, terminated at $x = 0$ by an impedence of Z_T.

As usual, we can write down the potential difference and the current as oscillating incident, reflected and transmitted waves, i.e.

$$\begin{aligned} V(x,t) &= Ae^{i[\omega t - kx]} + A'e^{i[\omega t + kx]} \\ Z_0 I(x,t) &= Ae^{i[\omega t - kx]} - A'e^{i[\omega t + kx]}. \end{aligned} \tag{6.76}$$

At $x = 0$ the ratio V/I must be equal to the terminating impedence, such that

$$\frac{V(0,t)}{I(0,t)} = Z_T. \tag{6.77}$$

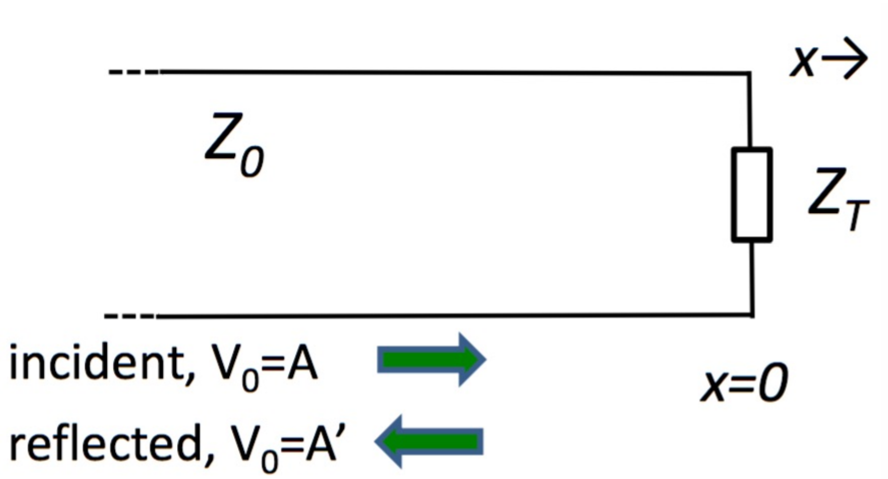

Figure 6.5: A transmission line that is terminated at $x = 0$ by an impedence Z_T.

Therefore, we find

$$\frac{Z_T}{Z_0} = \frac{A + A'}{A - A'}, \tag{6.78}$$

so the reflection coefficient,

$$r = \frac{A'}{A} = \frac{Z_T - Z_0}{Z_T + Z_0}. \tag{6.79}$$

Considering the the following cases;

- $Z_T \to 0$ then $r \to -1$: full reflection with phase shift
- $Z_T = Z_0$ then $r \to 0$: no reflection - matched impedance; all power transmitted to terminating load
- $Z_T \to \infty$ then $r \to 1$: full reflection

These are similar to what we have for the oscillating string with a massless ring encircling a frictionless pole at one end.

The concept of *impedance matching* is significant in many areas, as this ensures that the maximum power is transferred across the boundary (in the case of two strings) or into the terminating load, for the transmission line described above.

6.11 Standing Waves - II

6.11.1 Infinite string with a fixed end

Consider a leftward-moving single sinusoidal wave that is incident on a brick wall at its left end, located at $x = 0$. The most general form of a left ward-moving sinusoidal wave is given by

$$y_i(x,t) = A\cos(kx + \omega t + \phi) \tag{6.80}$$

where $\omega/k = v = \sqrt{T/\rho}$, ϕ is arbitrary and depends only on where the wave is at $t = 0$. The brick wall is equivalent to a system with infinite impedence, i.e. $Z_2 = \infty$, and the reflection coefficient $r = -1$, which gives rise to a reflected wave with amplitude of the same magnitude as the incident wave but with the opposite sign and travelling in the opposite direction, i.e.

$$y_r(x,t) = -A\cos(-kx + \omega t + \phi). \tag{6.81}$$

If we were to observe this system, we would see the summation of these two waves,

$$y(x,t) = A\cos(kx + \omega t + \phi) - A\cos(-kx + \omega t + \phi)$$

which, using trig identities, can be expressed as

$$y(x,t) = 2A\sin(\omega t + \phi)\sin kx \tag{6.82}$$

or,

$$\boxed{y(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\sin\left(\frac{2\pi t}{T} + \phi\right).} \tag{6.83}$$

Rather than invoking the fact that $r = -1$ for a wall, we could always derive this result using the fact $y = 0$ at $x = 0$, for all t and start from the general solution to the wave equation, i.e.

$$y(x,t) = A_1 \sin(-kx + \omega t) + A_2 \cos(-kx + \omega t) + A_3 \sin(kx + \omega t) + A_4 \cos(kx + \omega t)$$

$$\Rightarrow y(x,t) = B_1 \cos kx \cos \omega t + B_2 \sin kx \sin \omega t + B_3 \sin kx \cos \omega t + B_4 \cos kx \sin \omega t$$

at $y(0,t) = 0$ for all t, therefore we should only have the $\sin kx$ terms, i.e.

$$\begin{aligned} y(x,t) &= B_2 \sin kx \sin \omega t + B_3 \sin kx \cos \omega t \\ &= (B_2 \sin \omega t + B_3 \cos \omega t) \sin kx \\ &= B \sin(\omega t + \phi) \sin kx \end{aligned}$$

where if $B_2 = B_3$ then $B = 2B_2 = 2B_3$.

6.11.2 Standing waves with a free end

We can also consider a similar system as discussed in Sec. 6.8, where we fix one end to a massless ring which encircles a frictionless pole at $x = 0$. This ensures that the wave cannot move in the longitudinal direction, but is still free to move in the transverse direction. This is similar to assuming that the string beyond the pole has a density of zero. If we assume that the wave is travelling towards the pole from the left hand side (i.e. along negative x), then we can write

$$y_i(x,t) = A \cos(\omega t - kx + \phi). \tag{6.84}$$

Since the massless ring has zero impedence (remember it was the string on the other side of the ring that provided the impedence in Sec. 6.8), then the reflection coefficient $r = +1$ as $k_2 = 0$. Therefore, we find for the reflected wave we have,

$$y_r(x,t) = r y_i(x,t) = A \cos(\omega t + kx + \phi), \tag{6.85}$$

and therefore the wave we would observe is the summation of the incident and reflected waves,

$$\begin{aligned} y(x,t) = y_i(x,t) + y_r(x,t) &= A \cos(\omega t - kx + \phi) + A \cos(\omega t + kx + \phi) \\ &= 2A \cos(\omega t + \phi) \cos kx \end{aligned} \tag{6.86}$$

As in the case considered before, you can also apply the usual boundary conditions to the general solution to the wave equation and reach the same result. In both of these cases, ω and k can and number and are not necessarily discrete, unlike the case which we will look at next, where we find that only discrete values are allowed.

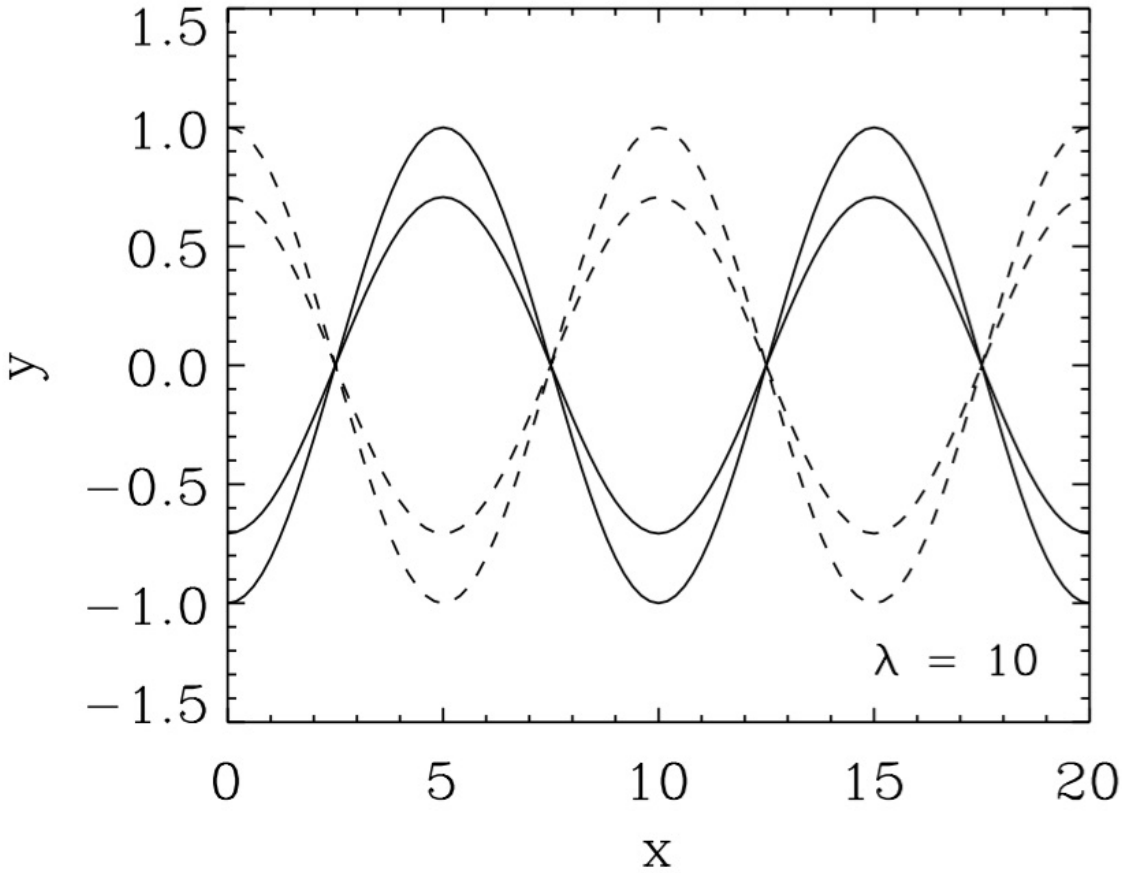

Figure 6.6: Standing wave for a system with a free end, with two full wavelengths shown. As in Fig. 5.1, the solid curves are for $t_0 = 0$ and $t = t_0 = \delta t$ and the dashed curved shows where the waves would be at a time $t = t_0 + \pi$ and $t = t_0 + \pi \pm \delta t$ later. For this standing wave the nodes are the stationary points where the wave crosses the $y = 0$ axis. In this case the end of the string are *anti-nodes*.

6.12 Power in a standing wave

We saw that for a travelling wave, that power is transmitted. A given point on the string does work (which may be positive or negative, depending on the direction of the waves velocity) on the part of the string to its right. And it does the opposite amount of work on the string to its left.

So is there any energy flowing in a standing wave? We know that there is an energy density as the string stretches and moves, this is what we saw in Sec. 6.5, but is any energy transferred along the string?

Given that a standing wave is just the superposition of two waves of equal amplitudes travelling in opposite directions then they should have equal and opposite energy flow. This would result in net energy flow of zero, *on average.*

The power flow in any wave is just given by the rate of work done, or the vertical force multiplied by the transverse velocity, i.e.

$$P(x,t) = \frac{dW}{dt} = F_y \frac{\partial y}{\partial t} = F v_y = \left(-T\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right). \tag{6.87}$$

If our standing wave can be described by

$$y(x,t) = A \sin \omega t \sin kx, \tag{6.88}$$

then

$$P(x,t) = -TA^2(\sin kx \cos kx)(\sin \omega t \cos \omega t). \tag{6.89}$$

This is non-zero for most values of x and t, so energy does flow across a given point. However, at given value of x, the average power over a whole period, is zero. This is because the average of $\sin \omega t \cos \omega t$ over the period is zero. Therefore the *average* power is zero.

6.13 Waves on a finite string

Up until now we have considered only infinite strings which are either free or fixed at one end. In this section we will look at a finite string with both fixed and free ends. We consider a string of length L and with the two ends assigned the values of $x = 0$ and $x = L$. We can think of what the general boundary conditions for such a system are. At a fixed end we know that the displacement in the $y-$direction must be zero at all times, and that the displacement at any free end must result in $\partial y/\partial x = 0$, because the slope must be zero, otherwise we would have a vertical force on a massless end, which in turn would result in infinite acceleration.

6.13.1 Two fixed ends

First, let us consider a system in which the string is fixed at both ends, i.e. at $x = 0$ and $x = L$. Then we have similar boundary conditions to that considered for the infinite string fixed at one end, i.e. the boundary conditions that resulted in Eq. 6.82, but we require not only that $y(0,t) = 0$, but also $y(L,t) = 0$. Therefore, the only way to have $y(L,t) = 0$ for all t is to ensure that $\sin kL = 0$. This implies that kL must be an integer number of π, i.e.

$$k_n = \frac{n\pi}{L},$$

where n is an integer and defines which mode is excited in the string.

The fact that each end must be a node implies that we can only have wavelengths which are related to the length of the string by n, i.e.

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}. \tag{6.90}$$

Therefore, we now have a solution of the form

$$y(x,t) = 2A\sin(\omega t + \phi)\sin\left(\frac{n\pi}{L}\right) = 2A\sin(\omega t + \phi)\sin\left(\frac{n\pi}{L}\right) \tag{6.91}$$

So the allowed wavelengths on the string are all integer divisors of twice the length of the string. This can easily be seen if you consider what the $n = 1$ mode actually is based on previous lectures, i.e. the lowest mode is one in which there are two nodes and a single anti-node halfway between the ends. This unavoidably has half of a full wavelength, where this half wavelength is the length of the string. You can obviously have an $n = 0$ mode as well, but this just means that $\sin(0) = 0$ and the string is just at rest in its equilibrium position.

Now looking at the angular frequency ω, we know that it is related to the velocity of the wave through $\omega/k = \sqrt{T/\rho} = v$, so that $\omega_n = vk_n$, i.e. the frequency of oscillation also has a dependence on n. The frequency is therefore given by,

$$\omega_n = k_n v = \frac{n\pi}{L}v. \tag{6.92}$$

Therefore, the frequency of the oscillations of the string are all integer multiples of the *fundamental frequency*, $\omega_1 = v/2L$.

Combining Eqs. 6.90 and 6.92, we find that $v = \lambda_n/2\pi\omega_n$ as you would expect.

Since the wave equation in Eq. 4.7 is linear, the most general motion of a string with two fixed ends is a linear combination of the solution given in Eq. 6.82, where k can only take a form $k_n = n\pi/L$ and $\omega/k = v$. Therefore the general expression for $y(x,t)$ is the summation over all n, i.e.

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin(\omega_n t + \phi_n) \sin k_n x \tag{6.93}$$

or

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin\left(\frac{n\pi v}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right). \tag{6.94}$$

This is the sum of all possible solutions with the coefficients F_n given by the initial displacement, which is the boundary condition we have yet to invoke. Note that the sine function for the time dependent term could be replaced by a cosine, with the phase difference ϕ_n adjusted accordingly, but this cannot be done for the $x-$dependent sine term.

6.13.2 One fixed end

Now we will look at what happens if one end of a finite string is left completely free. If we take the fixed end to be at $x = 0$ then the boundary conditions are $y(0,t) = 0$ and $\partial y/\partial x|_{x=L} = 0$ for all t. From Eq. 6.82 we find that the slope $(\partial y/\partial x)$ is proportional to $\cos kx$. Therefore, for this to be zero at $x = L$, we require that $kL = n\pi + \pi/2$ for any integer n. Therefore,

$$k_n = \frac{(n+1/2)\pi}{L}. \tag{6.95}$$

The first thing to note here is that now with $n = 0$ we have an excited wave, as $k_0 = \pi/2L$. As $\lambda_n = 2\pi/k$, then $\lambda_0 = 4L$, i.e. the $n = 0$ mode produces a quarter of a wavelength, where the string has length L. This is straightforward to visualise: with one free we have an anti-node, whereas at the fixed end there is a node. In this case, the general solution is again the summation over all possible modes, n, and is given by,

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin(\omega_n t + \phi_n) \cos k_n x \tag{6.96}$$

or

$$y(x,t) = \sum_{n=0}^{\infty} F_n \sin\left(\frac{n\pi v}{L} t + \phi_n\right) \cos\left(\frac{(n+1/2)\pi}{L} x\right) \tag{6.97}$$

6.13.3 Two free ends

Finally, we will look at the case where we have two free ends. In terms of the boundary conditions, we now do not require that $y(x,t) = 0$ at any end of the string, and only require that the gradient of the string $\partial y/\partial x = 0$ at both $x = 0$ and $x = L$, for all t. Therefore Eq. 6.86 provides us with the most general solution for this system, therefore the slope $\partial y/\partial x$ is proportional to $\sin kx$. To ensure that this is zero at $x = L$ and $x = 0$, we require $kL = n\pi$ for any integer n. In this case, we have

$$k_n = \frac{n\pi}{L}, \tag{6.98}$$

which is the same as we found for the case with two fixed ends, and again the possible wavelengths are all integral divisors of $2L$, similarly $\omega_n = n\pi v/L$. So writing down the general solution as the superposition of all the n modes, we find

$$y(x,t) = \sum_{n=0}^{\infty} F_n \cos(\omega_n t + \phi_n) \cos k_n x \tag{6.99}$$

or

$$y(x,t) = \sum_{n=0}^{\infty} F_n \cos\left(\frac{n\pi v}{L}t + \phi_n\right) \cos\left(\frac{n\pi}{L}x\right). \tag{6.100}$$

So in this case the $\cos(k_n x)$ term ensures that we have an anti-node at either end of the string for all n. One things to note about this system is that the equilibrium position of the string does not have to lie at $y = 0$.

Fig. 6.7 shows the possible oscillations for these three different set-ups for a finite string.

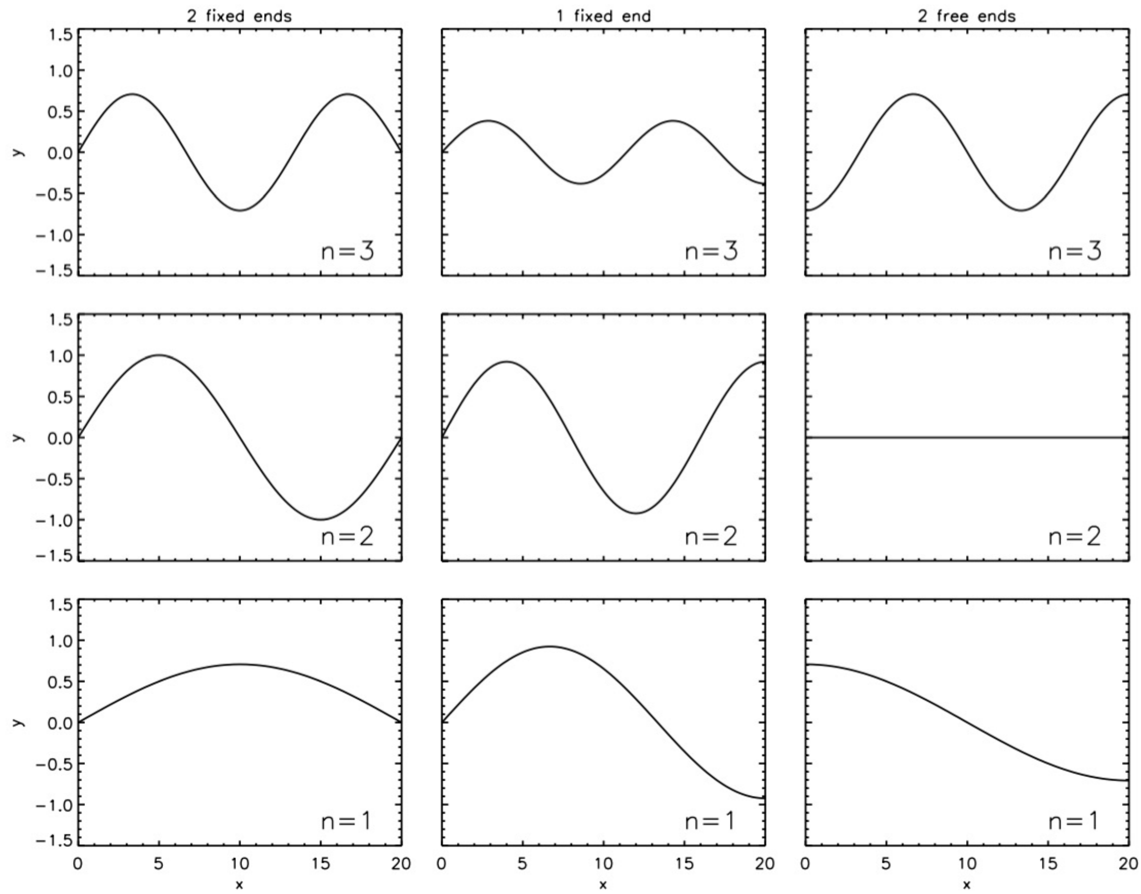

Figure 6.7: Wave pattern for a finite string with two fixed ends (left panels), one fixed end (central panels) and two free ends (right panels). In this case the length of the string is fixed at twice the wavelength for the $n = 1$ mode and the excited modes, defined by n, are shown in each panel. All panels show the wave pattern at $t = \pi$.

6.14 Longitudinal Elastic Waves

Up until now, we have considered mainly transverse waves. In this section two examples are given for longitudinal waves.

6.14.1 Longitudinal waves in a solid bar

Consider a solid bar (Fig. 6.8), initially in equilibrium, in which a disturbance perturbs the position and thickness of a slice of material. The disturbance moves the slice from x to $x + \Psi$ and changes its width by from δx to $\delta x + \delta\Psi$.

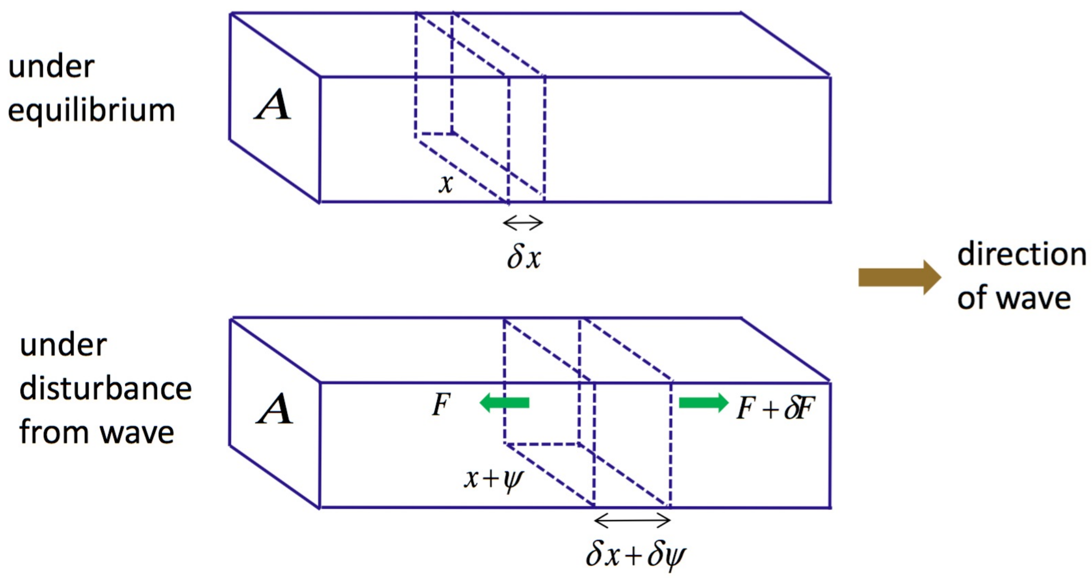

Figure 6.8: Solid bar experiencing a stress force due to the passing of a longitudinal wave that stretches the material.

Denoting the new stress force that stretches the material as F, and the excess force which accelerates the segment to the right, we can write down the force per unit area on the slice using Hooke's law:

$$\frac{F}{A} = Y\left(\frac{\delta\Psi}{\delta x}\right) \tag{6.101}$$

where Y is Young's modulus of the material, and is equal to the tensile stress (F/A) divided by the tensile strain $(\Delta L/L)$. For an infinitely thin slice,

$$F = AY\frac{\partial\Psi}{\partial x}, \tag{6.102}$$

so the excess force which is moving the slice along x, is given by,

$$\delta F = AY\frac{\partial^2 \Psi}{\partial x^2}\delta x. \tag{6.103}$$

The mass of the slice, if it has a density ρ, is given by $A\rho\delta x$, and the acceleration is just $\partial^2\Psi/\partial t^2$, so Newton's 2nd law gives leads to,

$$A\rho\delta x\frac{\partial^2 \Psi}{\partial t^2} = AY\frac{\partial^2 \Psi}{\partial x^2}\delta x. \tag{6.104}$$

Dividing through by δx and A, we therefore find,

$$\frac{\partial^2 \Psi}{\partial x^2} = \left(\frac{\rho}{Y}\right)\frac{\partial^2 \Psi}{\partial t^2}, \quad \text{therefore the velocity } v = \sqrt{\frac{Y}{\rho}} \tag{6.105}$$

As an example, steel has a Young's modulus of $Y = 2 \times 10^{11}$ Nm^{-2} and $\rho = 8000$ kg m^{-13}, which leads to a velocity of $v = 5$ m s^{-1}.

6.14.2 Acoustic Waves in gas

Sound waves are longitudinal waves associated with compression of medium.

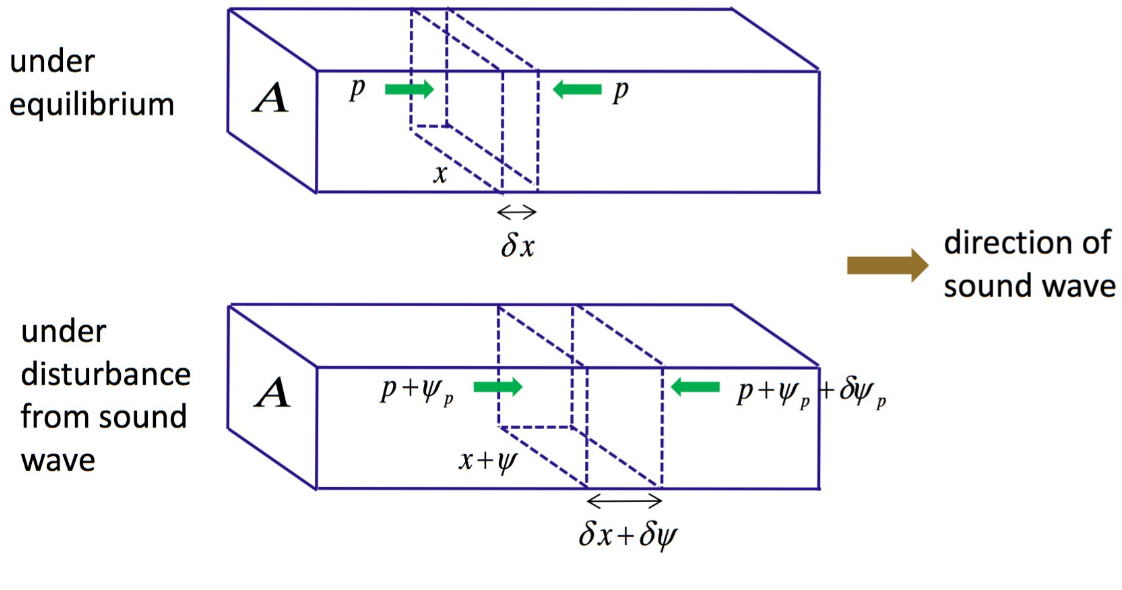

Figure 6.9: Slice of gas experiencing a pressure wave.

Consider a slice of gas (Fig. 6.9), initially at equilibrium, in a tube of cross-sectional area A. The slice is between x and $x + \delta x$. A disturbance moves the slice to $x + \Psi$ and changes its width to $\delta x + \delta\Psi$. The pressure has also changed from p to $p + \Psi_p$, on left-hand side of the slice, and to $p + \Psi_p + \delta\Psi_p$ on the right-hand side.

The slice has had its volume changed by a fractional amount, given by $(A\delta\Psi)/(A\delta x)$, and this happens as a result of a pressure change Ψ_p . The relationship is determined by the elasticity of the gas, i.e. the bulk modulus κ and is analogous to the Young's modulus in the previous example:

$$\frac{\delta\Psi}{\delta x} = -\frac{1}{\kappa}\Psi_p \Rightarrow \Psi_p = -\kappa\frac{\partial\Psi}{\partial x}, \tag{6.106}$$

for an infinitely thin slice.

From this we have the following relations,

$$\delta\Psi_p = -\kappa\frac{\partial^2\Psi}{\partial x^2}\delta x, \tag{6.107}$$

along with,

- Mass of slice $= A\rho\delta x$
- Force on slice in $x-$direction $= A\delta\Psi_p$
- Acceleration of slice $= \frac{\partial^2\Psi}{\partial t^2}$

Combining these we find,

$$\boxed{\frac{\partial^2\Psi}{\partial x^2} = \left(\frac{\rho}{\kappa}\right)\frac{\partial^2\Psi}{\partial t^2}} \tag{6.108}$$

We have obtained a wave equation describing motion of a slice of gas at position Ψ. One might worry that a slice of gas is a rather intangible experimental observable. Instead one can phrase problem in terms of the pressure variations, Ψ_p , which are certainly measurable.

Since, $\Psi_p \propto \frac{\partial \Psi}{\partial x}$ then the pressure difference Ψ_p must also satisfy the wave equation, i.e.

$$\frac{\partial^2 \Psi_p}{\partial x^2} = \left(\frac{\rho}{\kappa}\right) \frac{\partial^2 \Psi_p}{\partial t^2}, \tag{6.109}$$

in both cases the phase velocity of the waves is $v = \sqrt{\kappa/\rho}$.

The characteristic impedance can be defined as

$$Z = \frac{-\kappa \frac{\partial \Psi}{\partial x}}{\frac{\partial \Psi}{\partial t}}, \tag{6.110}$$

so for a forward travelling waves described by $\Psi(x,t) = A\sin(\omega t - kx)$,

$$Z = \frac{\kappa k}{\omega} = \frac{\kappa}{v} = \sqrt{\rho\kappa}. \tag{6.111}$$

6.14.3 Speed of Sound

We have shown thet $v = \sqrt{\kappa/\rho}$, so we can calculate v is we know the bulk modulus and the density of the gas. To calculate κ it is convenient to use the form of the bulk modulus, given by

$$\kappa = -V\frac{\partial p}{\partial V}, \tag{6.112}$$

where p is the pressure and V is the volume of the gas. So we need to specify what else happens to the system when the pressure changes.

Isothermal compression

No temperature change, this is reasonable to assume if the pressure changes are slow enough to allow the tube to exchange heat freely with the surroundings. Therefore for an ideal gas

$$PV = RT \qquad \Longrightarrow \qquad \frac{\partial P}{\partial V} = -\frac{RT}{V^2} \tag{6.113}$$

therefore

$$\kappa = \frac{RT}{V} = p \qquad \Longrightarrow \qquad v = \sqrt{\frac{p}{\rho}} \tag{6.114}$$

Adiabatic compression

Pressure changes occur so rapidly that heat cannot be exchanged from dense to less dense regions. Good approximation to reality.

Adiabatic changes in an ideal gas leads to,

$$pV^\gamma = \text{constant} = k \qquad \text{where } \gamma = \frac{C_p}{C_V} \tag{6.115}$$

i.e. the ratio of the specific heats at constant pressure and volume.

$$\frac{\partial p}{\partial V} = -\frac{k\gamma}{V^{\gamma+1}} = -\frac{pV^\gamma \gamma}{V^\gamma V} = -\frac{\gamma p}{V} \tag{6.116}$$

so in this case $\kappa = \gamma p$, which in turn leads to

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma \frac{RT}{M}} \tag{6.117}$$

i.e. the velocity of the wave is independent of the pressure for an ideal gas. A typical value for air at room temperature is $v \approx 350$ m s^{-1}.