Quantum Club

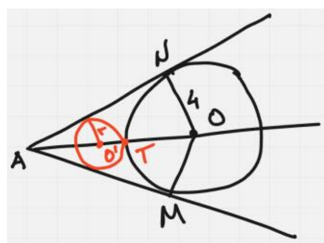
Entrance test 2024

1. Point E lies on side BC of a parallelogram ABCD such that BE/EC = 2/3. F is the intersection point of lines DE and AC. Find the ratio of the areas of triangle AFD and parallelogram ABCD.
Triangles AFD and CFE are similar with side ratio 5:3. The sum of their heights equals the height of parallelogram ABCD, hence the height of triangle AFD is 5/8*h*, where *h* is the height of ABCD. The bases of AFD and ABCD are the same, hence their areas' ratio is 1/2 × 5/8 = 5/16.

2. Find all positive integers *m, n, p* such that *mnp* = *m* + *n* + *p*.
Let p be the largest number of the three. Consider the following options for *m* and *n*:
    - If *m* = 1 and *n* = 1, we have *mnp* = *p* and *m* + *n* + *p* = *p* + 2. The equation cannot be satisfied.
    - If *m* = 1 and *n* = 2, we have *mnp* = 2*p* and *m* + *n* + *p* = *p* + 3. The equation can be satisfied if *p* = 3.
    - If $m \geq 2$ and $n \geq 2$, we write *m* = 2 + *i* and *n* = 2 + *j*, where $i,\ j \geq 0$. Then *mnp* = *p* (4 + *i* + *j* + *ij*) > 3*p* ≥ *m* + *n* + *p* (because p is the largest number of the three).

    We conclude that the only option for *m*, *n*, *p* is 1, 2, 3 (in any order).

3. Two tangents of a circle of radius 4 intersect at an angle of magnitude 2 arcsin (3/5). Another circle is tangent to both sides of that angle as well as the first circle. Find its radius.



The second circle can be either at the "inner or "outer" side of the first one. Let us specialize to the former case (see the diagram). Because $\sin\angle OAN = \frac{3}{5}$ and *ON* = 4 (radius of the first circle), we have $OA = ON\Big/\frac{3}{5} = \frac{20}{3}$. Now consider the second circle, centred at O'. Let its radius be *r*. Then $O'A = r\Big/\frac{3}{5} = \frac{5}{3}r$. Hence $OA = O'A + O'T + OT = \frac{5}{3}r + r + 4. = \frac{8}{3}r + 4$. So $\frac{8}{3}r + 4 = \frac{20}{3}$ and *r* = 1. If the second circle is at the "outer" side, we have due to similarity $\frac{r}{4} = \frac{4}{r'}$, where r' is the radius of that circle. We conclude $r' = 16$.

4. A three-digit number is randomly composed of digits 0,3,4,5,6 and 9 such that all digits in the number are different (numbers starting with a zero, e.g. 045, are not considered 3-digit). What is the probability that the number is divisible by 45?
Let us first find the total quantity of possible numbers. You can choose one of the digits 3,4,5,6 and 9 (5 options) for the first position. For the second position there are 6 – 1 = 5 candidates because the digit must be different from that in the first position (and zero is allowed). For the third position, there are 6 – 2 = 4 candidates. So there are 5×5×4 = 100 possible numbers.

For a number to be divisible by 45, it must be divisible by 5 and 9. So it must end with either 0 or 5 and the sum of its digits must be divisible by 9. If the last digit is 0, the sum of the first two digits can be either 9 or 18. This is satisfied by 450, 540, 360 and 630. If the last digit is 5, the sum of the first two digits can be either 4 or 13. This is satisfied by 405, 945 and 495. So there are a total of 7 numbers divisible by 45, and the probability of this number occurring is 7/100.

5. Solve the inequality sin 2x + tan $x \geq 2$.

Let us rewrite the equation as

$$2\frac{\tan x}{\tan^2 x+1}+\tan x \geq 2$$

and denote tan x = y. We have

$$3y+y^3 \geq 2y^2+2$$
$$y^3-2y^2+3y-2 \geq 0$$

The polynomial in the left-hand side can be factored:

$$(y-1)\left(y^2-y+2\right) \geq 0$$

The second factor y^2-y+2 is always positive, so the inequality becomes simply $y-1 \geq 0$ or $\tan x \geq 1$. Answer: $\frac{\pi}{4}+\pi n \leq x < \frac{\pi}{2}+\pi n$.

6. A bottle contains salt dissolved in water. One fourth of the contents is taken from the bottle and heated so that part of the water evaporates and the concentration of salt doubles. The resulting solution is then re-mixed with the contents of the bottle. As a result, the concentration of salt becomes 2% higher compared to the initial solution. Find the concentration of salt in the initial solution.

The concentration is the ratio of the masses of salt and water in the solution. The mass of salt is the same in the initial and final solutions, because salt does not evaporate. But the volume of water decreases by one eight (one half of one fourth of the water evaporates). Hence the concentration increases by a factor of 8/7. If the initial concentration is C, we have 8/7 C = C + 0.02, so C = 14%.

7. Solve the equation $\tan 2x+\frac{1}{\sin x}=\frac{1}{\tan x}+\frac{1}{\sin 5x}$.

Let us transform the initial equation:

$$\tan 2x-\frac{1}{\tan x}=\frac{1}{\sin 5x}-\frac{1}{\sin x}$$
$$\frac{\sin 2x}{\cos 2x}-\frac{\cos x}{\sin x}=\frac{\sin x-\sin 5x}{\sin x \sin 5x}$$

The left-hand side becomes $\frac{\sin 2x \sin x-\cos 2x \cos x}{\cos 2x \sin x}=\frac{-\cos 3x}{\cos 2x \sin x}$.

And in the right-hand side $\sin x-\sin 5x=\sin(3x-2x)-\sin(3x+2x)=-2\cos 3x \sin 2x$

The equation becomes

$$\frac{-\cos 3x}{\cos 2x \sin x}=\frac{-2\cos 3x \sin 2x}{\sin x \sin 5x},$$

which is equivalent to

$\sin 5x=2\sin 2x \cos 2x$ or $\cos 3x=0$.

Noticing that $2\sin 2x \cos 2x=\sin 4x$, we rewrite the first equation as

$\sin 5x - \sin 4x = 0$

$\sin(\tfrac{9}{2}x + \tfrac{1}{2}x) - \sin(\tfrac{9}{2}x - \tfrac{1}{2}x) = 0$

$2\cos(\tfrac{9}{2}x)\sin(\tfrac{1}{2}x) = 0$

So we have three options: $\cos(\tfrac{9}{2}x) = 0$, $\sin(\tfrac{1}{2}x) = 0$ or $\cos(3x) = 0$. At the same time, we must make sure the denominator in the initial equation does not turn to zero, so $\sin x \neq 0, \sin 5x \neq 0, \cos 2x \neq 0$. Note thar the first of these restrictions can be absorbed into the second one, because, whenever $\sin x = 0$ we also have $\sin 5x = 0$. Also, whenever $\sin(\tfrac{1}{2}x) = 0$, we also have $\sin x = 0$, so the second solution option is void.

Answer: $x = \frac{2}{9}\left(\frac{\pi}{2} + \pi n\right)$ or $x = \frac{1}{3}\left(\frac{\pi}{2} + \pi n\right)$, excluding $x = \frac{1}{5}(\pi m)$ and $x = \frac{1}{2}\left(\frac{\pi}{2} + \pi m\right)$.