Quantum Club

Entrance test 2022

1. You drop a metal ball from an aerostat and hear a bang *t* seconds later. Find the height *d* of the aerostat in terms of time *t*, speed of sound *c* and free-fall acceleration *g*. Neglect air resistance.

   Let t_0 be the time it takes the rock to fall. Then $d = gt_0^2/2$ and $t = t_0 + d/c$. Hence

   $$gt_0^2/2c + t_0 - t = 0 \quad \Rightarrow \quad t_0 = \frac{-c+\sqrt{c^2+2gct}}{g} \quad \Rightarrow \quad d = \frac{c^2+gct-c\sqrt{c^2+2gct}}{g}$$

   Follow-up question: consider the limiting case of $gt >> c$. Intuitively, what answer would you expect? Can you show that your calculation is consistent with your expectation?

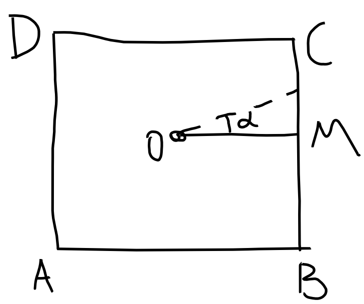
   The largest term in the numerator is *gct*, hence the answer is *ct*.

2. A criminal is in the centre of a square swimming pool, and a police officer, initially at the pool's corner, is trying to catch the criminal. The officer cannot swim but can run three times faster than the criminal can swim. Can the criminal get out of the pool and run away from the officer? Once out of the pool, the criminal can run faster than the officer.

   Yes – see below.

   Follow-up question: write the inequality for the allowed range of angles α at which the criminal can swim (you don't have to solve it).

   $$\frac{3}{2} + \frac{\tan\alpha}{2} > 3\left(\frac{1}{2}\frac{1}{\cos\alpha}\right) \quad \Rightarrow \quad 3\cos\alpha + \sin\alpha > 3 \quad \Rightarrow \quad 0 < \tan\alpha < \frac{3}{4}$$



3. Calculate $\sin^3 x - \cos^3 x$ if $\sin x - \cos x = u$.

   $\sin^3 x - \cos^3 x = (\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x) = u(1 + \sin x \cos x)$.

   To find $\sin x \cos x$, notice that $u^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x \quad \Rightarrow \quad \sin x \cos x = \frac{1-u^2}{2}$.

   Hence $\sin^3 x - \cos^3 x = u\frac{3-u^2}{2}$.

   Follow-up question: what if we replaced "–" by "+" in both expressions?

   $u^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x \quad \Rightarrow \quad \sin x \cos x = \frac{u^2-1}{2}$

   $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = u(1 - \sin x \cos x) = u\frac{3-u^2}{2}$. The same answer.

4. Paper cards numbered 1 to 10 are placed on a table in a row, in a random order. What is the probability that cards 1 and 2 are situated next to each other?

   Let us calculate the total number of volume permutations such that volumes 1 and 2 stand next to each other.

   If volume 1 is in position 1, volume 2 must be in position 2.

   If volume 1 is in position 2, volume 2 must be in position 1 or 3.

   …

   If volume 1 is in position 10, volume 2 must be in position 9.

   Total: 18. This needs to be multiplied by 8!, the number of permutations of the other 8 volumes.

   Answer: 18×8!/10! = 18/(9×10)=1/5.

   Follow-up question: what if we additionally required that the cards 1 and 2 be in the correct order?

   Answer: 1/10.

5. Solve the equation $4\sin x + \sqrt{3}\sin 2x = 2\cos 2x \sin x$.

   $4\sin x + 2\sqrt{3}\sin x \cos x = (4\cos^2 x - 2)\sin x$

   One of the solutions is $\sin x = 0 \quad \Rightarrow \quad x = m\pi.$ Otherwise:

   $4\cos^2 x - 2\sqrt{3}\cos x - 6 = 0$

   $\cos x = \dfrac{\sqrt{3} \pm \sqrt{27}}{4} = -\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad x = \dfrac{5\pi}{6} + 2\pi m \text{ or } \dfrac{7\pi}{6} + 2\pi m$

   No follow-up question.

6. What is the probability that a random 6-digit phone number (a sequence of digits between 0 and 9) has at least two identical digits next to each other?

   Let us count the numbers that do not have two identical digits next to each other. The first digit can be chosen arbitrarily out of 10, subsequent ones arbitrarily out of 9 (must not be equal to the previous digit). Hence this number is 10×9^5. The probability that the number has at least two identical digits next to each other is then $1 - \dfrac{10\times 9^5}{10^6} \approx 0.41$.

   Follow-up question: what if we removed the condition “next to each other” in the question?

   $1 - \dfrac{10\times 9\times 8\times 7\times 6\times 5}{10^6} \approx 0.85$

7. Find the complete set of all positive integer numbers such that each of them is 13 times greater than the sum of its digits.

   The number must be either 2- or 3-digit. If it is 2-digit:

   $10a + b = 13(a+b) \quad \Rightarrow \quad 3a + 12b = 0.$ Impossible.

   If it is 3-digit:

   $100a + 10b + c = 13(a+b+c) \quad \Rightarrow \quad 87a = 3b + 12c.$

   Digit a must be 1, because otherwise equality cannot be reached for $b, c < 10$. Then

   $3b + 12c = 87 \quad \Rightarrow \quad b + 4c = 29.$

   Possible options: $b = 1$, $c = 7$ (117), $b = 5$, $c = 6$ (156) and $b = 9$, $c = 5$ (195).

   No follow-up question.