Quantum Club

Entrance test 2021

Moscow 06 November 2021

Problems

1. Find $\log_{45}375$ if $\log_3 5 = a$.

   $$\log_{45}375 = \frac{\ln 375}{\ln 45} = \frac{\ln 3 + \ln 5 + \ln 5 + \ln 5}{\ln 3 + \ln 3 + \ln 5} = \frac{1 + 3(\ln 5/\ln 3)}{2 + (\ln 5/\ln 3)} = \frac{1+3a}{2+a}.$$

2. For which x does the function $y = \sin 2x - 2\sin^2 x$ reach its maximum?

   $y = \sin 2x - 2\sin^2 x = \sin 2x + \cos 2x - 1 = \sin\left(2x + \frac{\pi}{4}\right) - 1$. The maxima are reached for

   $2x + \frac{\pi}{4} = \frac{\pi}{2} + 2\pi m$, i.e. $x = \frac{\pi}{8} + \pi m$

3. Two vectors of length 1 are randomly chosen in a plane. Find the expectation value of the squared length of their sum.

   Use cosine law: $\left|\vec{a} + \vec{b}\right|^2 = a^2 + b^2 + 2ab\cos\theta = 2 + 2\cos\theta$, where θ is the angle between the two vectors. The expectation of $\cos\theta$ for a random pair of vectors is zero, hence the answer is 2.

4. A 3-digit number is chosen randomly. What is the probability that all of the following conditions are fulfilled:
   - the number contains the digits 0,1,4,6,7 and 8 only;
   - all three digits are different;
   - the number is divisible by 3.

   Note that, e.g., 045 is not considered a 3-digit number.

   Let us find out how many numbers satisfy these criteria. In order for the number to be divisible by 3, in must consist of one of the following triplets: (0,1,8), (0,4,8), (0,7,8), (1,4,7), (1,6,8), (4,6,8), (6,7,8). The first three triplets contain zero, so each of them can give rise to 4 numbers (e.g. 108, 180, 810, 801). Each of the last four triplets can generate 6 numbers. Hence we have a total of 36 numbers. Given that there is a total of 900 3-digit numbers, we have $p = 36/900 = 1/25$.

5. Calculate $\mathrm{tg}\left[\frac{1}{4}\left(\arccos\frac{3}{5}\right)\right]$.

   Let $\frac{1}{4}\left(\arccos\frac{3}{5}\right) = x$. Then

   $$\cos x = \sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\frac{1+\sqrt{\frac{1+\cos 4x}{2}}}{2}} = \sqrt{\frac{1+\sqrt{\frac{1+3/5}{2}}}{2}} = \sqrt{\frac{1+\sqrt{4/5}}{2}}.$$

   Therefore $\sin x = \sqrt{\frac{1-\sqrt{4/5}}{2}}$ and $\mathrm{tg}\, x = \sqrt{\frac{1-\sqrt{4/5}}{1+\sqrt{4/5}}} = \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} = \sqrt{5} - 2$.

6. Show that $\sqrt{2\sin\frac{3}{4}} < \frac{6}{5}$ without using calculators or tables.

$$\sin^2\frac{3}{4} < \sin^2\frac{\pi}{4} = \frac{1}{2} < \left(\frac{36}{50}\right)^2 = \left(\frac{6}{5\sqrt{2}}\right)^4.$$

7. Find the locus of points z on the complex plane such that $\arg\frac{z-1}{z+1} = \frac{\pi}{2}$.

$\arg\frac{z-1}{z+1} = \arg(z-1) - \arg(z+1)$. In other words, the angle between the vectors from the points 1 and -1 to the point z is 90°. The corresponding locus is a semicircle above the diameter [-1,1].