Appendix S5
Solutions to Chapter 5 exercises

Solution to Exercise 5.1.

a)

$$|H\rangle\langle H| \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

b)

$$\begin{aligned}
&(\psi_H |H\rangle + \psi_V |V\rangle)(\psi_H^* \langle H| + \psi_V^* \langle V|) \\
&\quad = |\psi_H|^2 |H\rangle\langle H| + \psi_H \psi_V^* |H\rangle\langle V| + \psi_H^* \psi_V |V\rangle\langle H| + |\psi_V|^2 |V\rangle\langle V| \\
&\quad = |\psi_H|^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \psi_H \psi_V^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \psi_H^* \psi_V \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + |\psi_V|^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\
&\quad \simeq \begin{pmatrix} |\psi_H|^2 & \psi_H \psi_V^* \\ \psi_H^* \psi_V & |\psi_V|^2 \end{pmatrix}.
\end{aligned}$$

c)

$$\begin{aligned}
&\frac{1}{2} |{+45^\circ}\rangle\langle{+45^\circ}| + \frac{1}{2} |{-45^\circ}\rangle\langle{-45^\circ}| \\
&\quad = \frac{1}{2} \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle) \frac{1}{\sqrt{2}} (\langle H| + \langle V|) + \frac{1}{2} \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle) \frac{1}{\sqrt{2}} (\langle H| - \langle V|) \\
&\quad = \frac{1}{4} (|H\rangle\langle H| + |H\rangle\langle V| + |V\rangle\langle H| + |V\rangle\langle V|) + \frac{1}{4} (|H\rangle\langle H| - |H\rangle\langle V| - |V\rangle\langle H| + |V\rangle\langle V|) \\
&\quad = \frac{1}{2} (|H\rangle\langle H| + |V\rangle\langle V|) \\
&\quad \simeq \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

d)

$$\frac{1}{2}\frac{1}{\sqrt{2}}(|H\rangle+|V\rangle)\frac{1}{\sqrt{2}}(\langle H|+\langle V|)+\frac{1}{4}|H\rangle\langle H|+\frac{1}{4}|V\rangle\langle V|$$
$$=\frac{1}{2}|H\rangle\langle H|+\frac{1}{4}|H\rangle\langle V|+\frac{1}{4}|V\rangle\langle H|+\frac{1}{2}|V\rangle\langle V|$$
$$\simeq\begin{pmatrix}1/2 & 1/4\\ 1/4 & 1/2\end{pmatrix}.$$

Solution to Exercise 5.2. For each component $|\psi_i\rangle$ of ensemble (5.1), the probability to observe $|v_m\rangle$ is $\mathrm{pr}_{m|i} = |\langle v_m|\,\psi_i\rangle|^2 = \langle v_m|\,\psi_i\rangle\langle\psi_i|\,v_m\rangle$. Since each $|\psi_i\rangle$ occurs with probability p_i, the probability to observe $|v_m\rangle$ in ensemble (5.1) is

$$pr_m = \sum_i p_i \mathrm{pr}_{m|i} = \sum_i p_i \langle v_m|\,\psi_i\rangle\langle\psi_i|\,v_m\rangle = \langle v_m|\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)|v_m\rangle = \langle v_m|\,\hat{\rho}\,|\,v_m\rangle\,.$$

Here we used Eq. (B.6) for the sum of conditional probabilities.

Solution to Exercise 5.3. Writing the density matrix in the canonical basis as $\hat{\rho}\simeq\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}$, we find, using Eq. (5.2):

a)

$$\mathrm{pr}_H = \langle H|\,\hat{\rho}\,|\,H\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \rho_{HH};$$
$$\mathrm{pr}_V = \langle H|\,\hat{\rho}\,|\,H\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \rho_{VV};$$

b)

$$\mathrm{pr}_+ = \langle +|\,\hat{\rho}\,|\,+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(\rho_{HH}+\rho_{HV}+\rho_{VH}+\rho_{VV});$$
$$\mathrm{pr}_- = \langle -|\,\hat{\rho}\,|\,-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}(\rho_{HH}-\rho_{HV}-\rho_{VH}+\rho_{VV});$$

c)

$$\mathrm{pr}_R = \langle R|\,\hat{\rho}\,|\,R\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}(\rho_{HH}+i\rho_{HV}-i\rho_{VH}+\rho_{VV});$$
$$\mathrm{pr}_L = \langle L|\,\hat{\rho}\,|\,L\rangle = \frac{1}{2}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{2}(\rho_{HH}-i\rho_{HV}+i\rho_{VH}+\rho_{VV}).$$

Solution to Exercise 5.4. As defined in Sec. 1.8, an unnormalized state $|\psi_i\rangle$ corresponds to the physical state $|\varphi_i\rangle = |\psi_i\rangle / \| \, |\psi_i\rangle \, \|$ existing with the probability $p_i = \| \, |\psi_i\rangle \, \|^2$. Using the density operator definition (5.1), we find

$$\hat{\rho} = \sum_i p_i \, |\varphi_i\rangle\langle\varphi_i| = \sum_i \| \, |\psi_i\rangle \, \|^2 \frac{|\psi_i\rangle\langle\psi_i|}{\| \, |\psi_i\rangle \, \|^2} = \sum_i |\psi_i\rangle\langle\psi_i| \, .$$

Solution to Exercise 5.5.

$$\frac{1}{2}|H\rangle\langle H| + \frac{1}{2}|V\rangle\langle V| \simeq \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \simeq \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$\frac{1}{2}|R\rangle\langle R| + \frac{1}{2}|L\rangle\langle L| \simeq \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix} + \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$\begin{aligned}\frac{1}{2}|\theta\rangle\langle\theta| + \frac{1}{2}|\pi/2+\theta\rangle\langle\pi/2+\theta| &\simeq \frac{1}{2}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\begin{pmatrix}-\sin\theta & \cos\theta\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Solution to Exercise 5.6. Suppose the ensemble (5.1) represents some pure state $|\psi\rangle$ (i.e. it is equal to $|\psi\rangle\langle\psi|$). Let us measure this ensemble in an orthonormal basis containing $|\psi\rangle$ as one of its elements. The probability to observe $|\psi\rangle$ then equals

$$\mathrm{pr}_\psi = \langle\psi|\, \hat{\rho} \,|\psi\rangle = \sum_i p_i |\langle\psi|\psi_i\rangle|^2.$$

For all i, $|\psi_i\rangle$ are normalized, so $|\langle\psi|\psi_i\rangle|^2 \le 1$ thanks to the Cauchy-Schwarz inequality. Furthermore, because not all $|\psi_i\rangle$ are equal, this inequality is strict ($|\langle\psi|\psi_i\rangle|^2 \le 1$) for at least one i. Hence

$$\sum_i p_i |\langle\psi|\psi_i\rangle|^2 < \sum_i p_i = 1,$$

which means that $\mathrm{pr}_\psi < 1$. This contradicts our initial assumption.

Solution to Exercise 5.7. Using the result of Ex. 5.6, we see that states (a) and (b) are pure while (c) and (d) are not.

Solution to Exercise 5.8. For any measurement basis $\{|v_m\rangle\}$, we have $\mathrm{pr}_m = \langle v_m| \, (\hat{\mathbf{1}}/N) \, |v_m\rangle = 1/N$.

Solution to Exercise 5.9. As we found when solving Ex. 5.5, all these states have the same density matrix $\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, which corresponds to the fully mixed state.

Solution to Exercise 5.10. Using the result of Ex. 4.27, we find

$$|m_x=1\rangle\langle m_x=1| \simeq \frac{1}{4}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & \sqrt{2} & 1\\ \sqrt{2} & 2 & \sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix};$$

$$|m_x=0\rangle\langle m_x=0| \simeq \frac{1}{2}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{pmatrix};$$

$$|m_x=-1\rangle\langle m_x=-1| \simeq \frac{1}{4}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & -\sqrt{2} & 1\\ -\sqrt{2} & 2 & -\sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}.$$

The mixture of these three states has matrix

$$\frac{1}{3}(|m_x=1\rangle\langle m_x=1| + |m_x=0\rangle\langle m_x=0| + |m_x=-1\rangle\langle m_x=-1|) = \frac{1}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

which corresponds to the fully mixed state.

Solution to Exercise 5.11. This result follows from Ex. 5.2. However, it can also be proven mathematically. Using the definition (5.1) of the density matrix, we find for its diagonal elements in the basis $\{|v_m\rangle\}$:

$$\rho_{mm} = \langle v_m|\,\hat{\rho}\,|v_m\rangle = \sum_i p_i |\langle v_m|\,\psi_i\rangle|^2. \tag{S5.1}$$

Because $\forall i\ p_i \geq 0$, we have $\rho_{mm} \geq 0$. Furthermore, because each $|\psi_i\rangle$ is normalized, $\sum_i |\langle v_m|\,\psi_i\rangle|^2 = 1$. Hence

$$\sum_m \rho_{mm} = \sum_i p_i \left(\sum_m |\langle v_m|\,\psi_i\rangle|^2\right) = \sum_i p_i = 1.$$

Solution to Exercise 5.12.

a) The off-diagonal element
$$\rho_{mn} = \langle v_m|\,\hat{\rho}\,|v_n\rangle = \sum_i p_i \langle v_m|\,\psi_i\rangle\langle\psi_i|\,v_n\rangle$$
can be seen as an inner product $(\vec{a},\vec{b})$ of the vectors[1]
$$\vec{a} = (\sqrt{p_1}\langle v_m|\,\psi_1\rangle, \sqrt{p_2}\langle v_m|\,\psi_2\rangle, \ldots)$$
and
$$\vec{b} = (\sqrt{p_1}\langle v_n|\,\psi_1\rangle, \sqrt{p_2}\langle v_n|\,\psi_2\rangle, \ldots).$$
The diagonal elements $\rho_{mm} = \sum_i p_i |\langle v_m|\,\psi_i\rangle|^2$ and $\rho_{nn} = \sum_i p_i |\langle v_n|\,\psi_i\rangle|^2$ are then equal to the absolute values $|a|^2$ and $|b|^2$ of these vectors, squared. By applying the Cauchy-Schwarz inequality, we obtain the desired result.

[1] Of course, these vectors represent just sets of numbers, not quantum states.

b) For a pure state $|\psi\rangle$, off-diagonal elements equal $\rho_{mn} = \langle v_m|\ \psi\rangle\langle\psi|\ v_n\rangle$, while the diagonal ones are $\rho_{mm} = |\langle v_m|\ \psi\rangle|^2$ and $\rho_{nn} = |\langle v_n|\ \psi_i\rangle|^2$. Substituting these expressions into inequality (5.3), we see that its left- and right-hand sides become equal to each other.
To prove the converse statement, suppose $\hat{\rho}$ is a mixed ensemble of states including at least two unequal elements, which we denote $|\psi_1\rangle$ and $|\psi_2\rangle$. The decompositions of these elements into the basis $\{|v_i\rangle\}$ must be different, which means that there exists a pair of basis elements $|v_m\rangle$ and $|v_n\rangle$ such that

$$\frac{\langle v_m|\ \psi_1\rangle}{\langle v_m|\ \psi_2\rangle} \neq \frac{\langle v_n|\ \psi_1\rangle}{\langle v_n|\ \psi_2\rangle}.$$

This implies, in turn, that vectors $\vec{a}$ and $\vec{b}$ are non-collinear, so the Cauchy-Schwarz inequality cannot saturate (Ex. A.26).

Solution to Exercise 5.15. Using the density matrix definition (5.1) we write for any of its elements

$$\rho_{mk} = \sum_i p_i \langle v_m|\ \psi_i\rangle\langle\psi_i|\ v_k\rangle = \sum_i p_i \langle v_k|\ \psi_i\rangle^* \langle\psi_i|\ v_m\rangle^* = \rho^*_{mk},$$

so the density operator is Hermitian.

Solution to Exercise 5.16. The possibility of spectral decomposition (5.4) follows from the fact that the density operator is Hermitian [see Ex. A.60]. The results $\sum_i q_i = 1$ and $q_i > 0$ follow from the diagonal elements' being the probabilities of measurement outcomes for the orthogonal basis in which the density matrix is written (Ex. 5.2).

Solution to Exercise 5.17.

a) $|H\rangle\langle H|$ (pure state);
b) $(x|H\rangle + y|V\rangle)(x^*\langle H| + y^*\langle V|)$ (pure state);
c) $\frac{1}{2}|{+45^\circ}\rangle\langle{+45^\circ}| + \frac{1}{2}|{-45^\circ}\rangle\langle{-45^\circ}| = \frac{1}{2}(|H\rangle\langle H| + |V\rangle\langle V|)$ (fully mixed state);
d) Solving the characteristic equation, we find the eigenvalues $3/4$ and $1/4$ and the corresponding eigenstates $|+\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$. Therefore the density operator is

$$\frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-| \simeq \begin{pmatrix}1/2 & 1/4\\ 1/4 & 1/2\end{pmatrix}$$

.

Solution to Exercise 5.18. A pure state $|\psi\rangle$'s density matrix is diagonal in any orthonormal basis that contains $|\psi\rangle$ as one of its elements. In that basis,

$$|\psi\rangle\langle\psi| \simeq \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

with the only nonzero element of the matrix corresponding to the basis element $|\psi\rangle$.

Solution to Exercise 5.19. According to Ex. 5.16, all eigenvalues of the (Hermitian) density operator are non-negative, which means that the density operator is also non-negative as per Ex. A.72.

Solution to Exercise 5.20.

a) In the Fock basis:

$$(a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|) \simeq \begin{pmatrix} aa^* & ab^* & 0 & \dots \\ ba^* & bb^* & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

b) In the position basis, using wavefunctions $\psi_0(X)$ and $\psi_1(X)$ of the first two Fock states, given by Eqs. (3.107) and (3.108), respectively, we find:

$$\begin{aligned} \rho(X,X') &= [a\psi_0(X) + b\psi_1(X)][a^*\psi_0^*(X') + b^*\psi_1^*(X')] \\ &= \pi^{-1/2} e^{-(X^2+X'^2)/2}[a + bX\sqrt{2}][a^* + b^*X'\sqrt{2}]. \end{aligned}$$

Solution to Exercise 5.21.

a) When diagonalized, all elements of unitary operator have absolute value 1 according to Ex. A.83. On the other hand, as we found in Ex. 5.16, the diagonal elements of the density operator are positive and add up to 1. These two conditions are incompatible for any Hilbert space of dimension greater than one.

b) If $\hat{\rho} = |\psi\rangle\langle\psi|$ is a pure state, then $\hat{\rho}^2 = |\psi\rangle\langle\psi|\,\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \hat{\rho}$. To prove the converse statement, let $\hat{\rho} = \sum_i q_i |v_i\rangle\langle v_i|$ be the spectral decomposition of $\hat{\rho}$. Then $\hat{\rho}^2 = \sum_i q_i^2 |v_i\rangle\langle v_i|$. Equality $\hat{\rho} = \hat{\rho}^2$ implies that either $q_i = 0$ or $q_i = 1$ for any i. Because $\sum_i q_i = 1$ for a normalized state, only one of the q_i's is equal to 1 while others are 0. This means that $\hat{\rho}$ is a pure state.

Solution to Exercise 5.22.

a) Let us decompose each element $\hat{\rho}_i$ of the ensemble in a form analogous to (5.1):

$$\hat{\rho}_i = \sum_j p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}| \tag{S5.2}$$

with $\sum_j p_{ij} = 1$. Then the ensemble in which each $\hat{\rho}_i$ occurs with the probability p_i is equivalent to a mixture of pure states $|\psi_{ij}\rangle$ that occur, respectively, with probabilities $p_i p_{ij}$. This ensemble is described by the density operator

$$\hat{\rho} = \sum_{ij} p_i p_{ij} \left|\psi_{ij}\right\rangle\left\langle\psi_{ij}\right|,$$

which is the same as Eq. (5.6).

b) Let one of the components of $\hat{\rho}$ be non-pure. Then its density matrix in any basis must contain at least two nonzero diagonal elements. Because all diagonal elements are non-negative for each $\hat{\rho}_i$ [Ex. 5.11(a)], the matrix of $\hat{\rho}$ must then also contain at least two nonzero diagonal elements. But, as we found in Ex. 5.18, if $\hat{\rho}$ were pure, there would exist a basis in which its density matrix contained only one element. We have arrived at a contradiction.

Solution to Exercise 5.23.

a) Using the representation of the density operator as a statistical ensemble (5.1), and using the Schrödinger equation (1.31) we have

$$\begin{aligned}
\dot{\hat{\rho}} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right| \\
&= \sum_i p_i (\left|\dot{\psi}_i\right\rangle\left\langle\psi_i\right| + \left|\psi_i\right\rangle\left\langle\dot{\psi}_i\right|) \\
&= \sum_i p_i \left(-\frac{i}{\hbar}\hat{H}\left|\psi_i\right\rangle\left\langle\psi_i\right| + \left|\psi_i\right\rangle\left\langle\psi_i\right|\frac{i}{\hbar}\hat{H}\right) \\
&= -\frac{i}{\hbar}(\hat{H}\hat{\rho} - \hat{\rho}\hat{H}) \\
&= -\frac{i}{\hbar}[\hat{H},\hat{\rho}].
\end{aligned}$$

b) Using Eqs. (1.29) and (1.30), we find

$$\begin{aligned}
\hat{\rho}(t) &= \sum_i p_i \left|\psi_i(t)\right\rangle\left\langle\psi_i(t)\right| \\
&= \sum_i p_i \hat{U}(t)\left|\psi_i(0)\right\rangle\left\langle\psi_i(0)\right|\hat{U}^\dagger(t) \\
&= e^{-\frac{i}{\hbar}\hat{H}t}\hat{\rho}(0)e^{\frac{i}{\hbar}\hat{H}t},
\end{aligned}$$

where $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ is the evolution operator.

Solution to Exercise 5.24.

a) Since $\left|\psi(0)\right\rangle = (\left|E_1\right\rangle + \left|E_2\right\rangle)/\sqrt{2}$, and because $\left|E_1\right\rangle$ and $\left|E_2\right\rangle$ are eigenstates of the Hamiltonian, we have

$$\left|\psi(t)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(0)\right\rangle = (e^{-\frac{i}{\hbar}E_1 t}\left|E_1\right\rangle + e^{-\frac{i}{\hbar}E_2 t}\left|E_2\right\rangle)/\sqrt{2},$$

and thus

$$\hat{\rho}(t) = \left|\psi(t)\right\rangle\left\langle\psi(t)\right| = \frac{1}{2}(\left|E_1\right\rangle\left\langle E_1\right| + e^{-\frac{i}{\hbar}(E_1-E_2)t}\left|E_1\right\rangle\left\langle E_2\right| + e^{-\frac{i}{\hbar}(E_2-E_1)t}\left|E_2\right\rangle\left\langle E_1\right| + \left|E_2\right\rangle\left\langle E_2\right|).$$

b) Using Eq. (5.8),

$$\begin{aligned}\hat\rho(t) &= e^{-\frac{i}{\hbar}\hat H t}\hat\rho(0)e^{\frac{i}{\hbar}\hat H t}\\ &= \frac{1}{2}(e^{-\frac{i}{\hbar}E_1 t}|E_1\rangle\langle E_1|e^{\frac{i}{\hbar}E_1 t}+e^{-\frac{i}{\hbar}E_2 t}|E_2\rangle\langle E_2|e^{\frac{i}{\hbar}E_2 t})\\ &= \frac{1}{2}(|E_1\rangle\langle E_1|+|E_2\rangle\langle E_2|)\\ &= \hat\rho(0).\end{aligned}$$

Solution to Exercise 5.25.

a) The Hamiltonian is $\hat H = -\frac{1}{2}\Omega_L\sigma_x$, where $\Omega_L = \gamma B$ is the Larmor frequency. The evolution operator for this Hamiltonian has been found in Exercise 4.62(c). Specializing to $\theta_0 = \pi/2$, we have

$$e^{-\frac{i}{\hbar}\hat H t} \simeq \begin{pmatrix}\cos(\Omega_L t/2) & i\sin(\Omega_L t/2)\\ i\sin(\Omega_L t/2) & \cos(\Omega_L t/2)\end{pmatrix},$$

Hence

$$\begin{aligned}e^{-\frac{i}{\hbar}\hat H t}|\uparrow\rangle &\simeq \begin{pmatrix}\cos(\Omega_L t/2)\\ i\sin(\Omega_L t/2)\end{pmatrix};\\ e^{-\frac{i}{\hbar}\hat H t}|\downarrow\rangle &\simeq \begin{pmatrix}i\sin(\Omega_L t/2)\\ \cos(\Omega_L t/2)\end{pmatrix}\end{aligned}$$

Accordingly,

$$\begin{aligned}\hat\rho(t) &= \frac{3}{4}e^{-\frac{i}{\hbar}\hat H t}|\uparrow\rangle\,\text{Adjoint}\left(e^{-\frac{i}{\hbar}\hat H t}|\uparrow\rangle\right)+\frac{1}{4}e^{-\frac{i}{\hbar}\hat H t}|\downarrow\rangle\,\text{Adjoint}\left(e^{-\frac{i}{\hbar}\hat H t}|\downarrow\rangle\right)\\ &\simeq \begin{pmatrix}\frac{1}{4}+\frac{1}{2}\cos^2(\Omega_L t/2) & -\frac{i}{2}\cos(\Omega_L t/2)\sin(\Omega_L t/2)\\ \frac{i}{2}\cos(\Omega_L t/2)\sin(\Omega_L t/2) & \frac{3}{4}-\frac{1}{2}\cos^2(\Omega_L t/2)\end{pmatrix}.\end{aligned}$$

b) The initial density matrix is

$$\hat\rho(0) = 3/4\,|\uparrow\rangle\langle\uparrow| + 1/4\,|\downarrow\rangle\langle\downarrow| \simeq \begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}.$$

Applying the evolution operator directly to the density matrix according to Eq. (5.8), we obtain the same result:

$$
\begin{aligned}
\hat{\rho}(t) &= e^{-\frac{i}{\hbar}\hat{H}t}\hat{\rho}(0)e^{\frac{i}{\hbar}\hat{H}t} \\
&\simeq \begin{pmatrix} \cos(\Omega_L t/2) & i\sin(\Omega_L t/2) \\ i\sin(\Omega_L t/2) & \cos(\Omega_L t/2) \end{pmatrix} \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \cos(\Omega_L t/2) & -i\sin(\Omega_L t/2) \\ -i\sin(\Omega_L t/2) & \cos(\Omega_L t/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\Omega_L t/2) & i\sin(\Omega_L t/2) \\ i\sin(\Omega_L t/2) & \cos(\Omega_L t/2) \end{pmatrix} \begin{pmatrix} \frac{3}{4}\cos(\Omega_L t/2) & -\frac{3}{4}i\sin(\Omega_L t/2) \\ -\frac{1}{4}i\sin(\Omega_L t/2) & \frac{1}{4}\cos(\Omega_L t/2) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}+\frac{1}{2}\cos^2(\Omega_L t/2) & -\frac{i}{2}\cos(\Omega_L t/2)\sin(\Omega_L t/2) \\ \frac{i}{2}\cos(\Omega_L t/2)\sin(\Omega_L t/2) & \frac{3}{4}-\frac{1}{2}\cos^2(\Omega_L t/2) \end{pmatrix}.
\end{aligned}
$$

c) We write Eq. (5.7) for the density matrix $\hat{\rho} \simeq \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}$ and the Hamiltonian $\hat{H} \simeq -\frac{\Omega_L}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] \simeq i\frac{\Omega_L}{2}\begin{pmatrix} \rho_{\downarrow\uparrow}-\rho_{\uparrow\downarrow} & \rho_{\downarrow\downarrow}-\rho_{\uparrow\uparrow} \\ \rho_{\uparrow\uparrow}-\rho_{\downarrow\downarrow} & \rho_{\uparrow\downarrow}-\rho_{\downarrow\uparrow} \end{pmatrix}.
$$

This translates into a system of differential equations

$$
\begin{cases} \dot{\rho}_{\uparrow\uparrow} = i\frac{\Omega_L}{2}(\rho_{\downarrow\uparrow}-\rho_{\uparrow\downarrow}) \\ \dot{\rho}_{\uparrow\downarrow} = i\frac{\Omega_L}{2}(\rho_{\downarrow\downarrow}-\rho_{\uparrow\uparrow}) \\ \dot{\rho}_{\downarrow\uparrow} = i\frac{\Omega_L}{2}(\rho_{\uparrow\uparrow}-\rho_{\downarrow\downarrow}) \\ \dot{\rho}_{\downarrow\downarrow} = i\frac{\Omega_L}{2}(\rho_{\uparrow\downarrow}-\rho_{\downarrow\uparrow}) \end{cases}. \tag{S5.3}
$$

It can be simplified by setting $x = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}$ and $y = \rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow}$. By subtracting the fourth equation from the first, and the third from the second, we find

$$
\begin{cases} \dot{x} = -i\Omega_L y \\ \dot{y} = -i\Omega_L x \end{cases}.
$$

The general solution of this system is as follows:

$$
\begin{cases} x = A\cos\Omega_L t + B\sin\Omega_L t \\ y = -iA\sin\Omega_L t + iB\cos\Omega_L t \end{cases}.
$$

We find from the initial density matrix that $\rho_{\uparrow\downarrow}(0) = \rho_{\downarrow\uparrow}(0) = 0$ and hence $B = 0$. Subsequently, using $\rho_{\uparrow\uparrow}(0) - \rho_{\downarrow\downarrow}(0) = \frac{1}{2}$, we obtain $A = \frac{1}{2}$, and hence $x = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow} = \frac{1}{2}\cos\Omega_L t$. Combining this with with the rule $\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow} = 1$ [from Ex. 5.11(b)], we find

$$
\begin{aligned}
\rho_{\uparrow\uparrow} &= \frac{1}{2} + \frac{1}{4}\cos\Omega_L t; \\
\rho_{\downarrow\downarrow} &= \frac{1}{2} - \frac{1}{4}\cos\Omega_L t.
\end{aligned}
$$

Next, we look for the off-diagonal elements of $\hat{\rho}(t)$. Because $\dot{\rho}_{\downarrow\uparrow} = i\frac{\Omega_L}{2}(\rho_{\downarrow\downarrow} - \rho_{\uparrow\uparrow})$ [from Eq. (S5.3)], we find

$$\rho_{\downarrow\uparrow} = \frac{i}{4}\sin\Omega_L t + C$$

with $C = 0$ from the initial density matrix. Finally,

$$\rho_{\uparrow\downarrow} = \rho_{\downarrow\uparrow}^* = -\frac{i}{4}\sin\Omega_L t.$$

To summarize, the density matrix is

$$\hat{\rho}(t) \simeq \begin{pmatrix} \frac{1}{2} + \frac{1}{4}\cos\Omega_L t & -\frac{i}{4}\sin\Omega_L t \\ \frac{i}{4}\sin\Omega_L t & \frac{1}{2} - \frac{1}{4}\cos\Omega_L t \end{pmatrix}. \tag{S5.4}$$

Using the trigonometric identities $\cos(\Omega_L t/2)\sin(\Omega_L t/2) = \frac{1}{2}\sin\Omega_L t$ and $\cos^2(\Omega_L t/2) = \frac{1}{2} + \frac{1}{2}\cos\Omega_L t$, we find our result to be identical to that of parts (a) and (b).

Solution to Exercise 5.26. Let $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ be two different bases in $\mathbb{V}$. Then the trace of the matrix in basis $\{|v_i\rangle\}$ is

$$\mathrm{Tr}(\hat{A}) = \sum_i A_{ii} = \sum_i \langle v_i|\hat{A}|v_i\rangle \tag{S5.5}$$

Inserting identity operators, we see

$$\begin{aligned}
\mathrm{Tr}(\hat{A}) &= \sum_i \langle v_i|\hat{\mathbf{1}}\hat{A}\hat{\mathbf{1}}|v_i\rangle \\
&= \sum_{i,j,k} \langle v_i|w_j\rangle\langle w_j|\hat{A}|w_k\rangle\langle w_k|v_i\rangle \\
&= \sum_{i,j,k} \langle w_k|v_i\rangle\langle v_i|w_j\rangle\langle w_j|\hat{A}|w_k\rangle \\
&= \sum_{j,k} \langle w_k|\left(\sum_i |v_i\rangle\langle v_i|\right)|w_j\rangle\langle w_j|\hat{A}|w_k\rangle \\
&= \sum_{j,k} \langle w_k|\hat{\mathbf{1}}|w_j\rangle\langle w_j|\hat{A}|w_k\rangle \\
&= \sum_{j,k} \delta_{jk}\langle w_j|\hat{A}|w_k\rangle \\
&= \sum_j \langle w_j|\hat{A}|w_j\rangle,
\end{aligned}$$

hence the trace is basis-independent.

Solution to Exercise 5.27. This statement is the same as that of Ex. 5.11(b).

Solution to Exercise 5.29.

a) This follows from Ex. 5.28.
b) This follows from part (a) if we denote $\hat{A}_1 \dots \hat{A}_{k-1} = \hat{A}$ and $\hat{A}_k = \hat{B}$.

Solution to Exercise 5.30. For Pauli matrices, we have

$$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

and

$$\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}.$$

In the first case the trace equals $2i$, in the second case $-2i$.

Solution to Exercise 5.31. Using Ex. 5.28 and the resolution of identity, we have

$$\begin{aligned} \mathrm{Tr}(\hat{A}\,|\varphi\rangle\langle\psi|) &= \sum_{mn} \left\langle v_n\right| \hat{A} \left| v_m\right\rangle \langle v_m|\,(|\varphi\rangle\langle\psi|)\,|v_n\rangle \\ &= \sum_{mn} \langle\psi|\, v_n\rangle \left\langle v_n\right| \hat{A} \left| v_m\right\rangle \langle v_m|\, \varphi\rangle \\ &= \langle\psi| \left(\sum_n |v_n\rangle\langle v_n|\right) \hat{A} \left(\sum_n |v_n\rangle\langle v_n|\right) |\varphi\rangle \\ &= \left\langle\psi\right| \hat{A} \left|\varphi\right\rangle. \end{aligned}$$

Solution to Exercise 5.32. If $\hat{\rho} = |\psi\rangle\langle\psi|$ is a pure state, then $\hat{\rho}^2 = \hat{\rho}$, so $\mathrm{Tr}(\hat{\rho}^2) = 1$. If the state is non-pure, the diagonalized form of its density matrix $\hat{\rho} \simeq \mathrm{diag}(\rho_{11}, \dots, \rho_{NN})$ has at least two nonzero elements. Because $\mathrm{Tr}\hat{\rho} = 1$, we have $\rho_{ii} < 1$ for any i, and hence $\rho_{ii}^2 < \rho_{ii}$. Therefore

$$\mathrm{Tr}(\hat{\rho}^2) = \sum_i \rho_{ii}^2 < \sum_i \rho_{ii} = \mathrm{Tr}\hat{\rho} = 1.$$

For the inequality $\mathrm{Tr}\hat{\rho}^2 \geq 1/N$, let us consider the scalar product of the vectors $\vec{a} = (\rho_{11}, \dots, \rho_{NN})$ and $\vec{b} = (1, \dots, 1)$. According to the Cauchy-Schwarz inequality,

$$(\vec{a}\cdot\vec{b})^2 \leq (\vec{a}\cdot\vec{a}) \times (\vec{b}\cdot\vec{b}),$$

or

$$\left(\sum_i \rho_{ii}\right)^2 \leq \sum_i \rho_{ii}^2 \times N.$$

The left-hand side of the above inequality is $(\mathrm{Tr}\hat{\rho})^2 = 1$. Therefore $\mathrm{Tr}(\hat{\rho}^2) = \sum_i \rho_{ii}^2 \geq 1/N$, with the inequality saturating for $\vec{a} \propto \vec{b}$, i.e. for the fully mixed state $\hat{\rho} \simeq \mathrm{diag}(1/N, \dots, 1/N)$.

Solution to Exercise 5.33.

a) Let us once again recall that the density matrix is an expression (5.1) for the statistical ensemble of pure states. As we found in Sec. 1.9.1, upon detection of basis element $|v_m\rangle$ in a measurement, each component of the ensemble transforms according to $|\psi_i\rangle \to \hat{\Pi}_m |\psi_i\rangle$. The whole ensemble will therefore transform as follows:
$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \;\to\; \sum_i p_i \hat{\Pi}_m |\psi_i\rangle\langle\psi_i| \hat{\Pi}_m = \hat{\Pi}_m \hat{\rho} \hat{\Pi}_m. \tag{S5.6}$$
Here we used the Hermitian nature of the projection operator.

b) For each component $|\psi_i\rangle$ of the ensemble, the probability of detecting $|v_m\rangle$ is $\mathrm{pr}_{m|i} = |\langle v_m|\,\psi_i\rangle|^2$, so the probability of detecting $|v_m\rangle$ for the full density matrix is, according to the theorem of total probability (see Ex. B.6),
$$\begin{aligned}
\mathrm{pr}_m &= \sum_i p_i \mathrm{pr}_{m|i} \\
&= \sum_i p_i |\langle v_m|\,\psi_i\rangle|^2 \\
&= \sum_i p_i \langle v_m|\,\psi_i\rangle \langle \psi_i|\,v_m\rangle \\
&= \langle v_m|\,\hat{\rho}\,|\,v_m\rangle \\
&\overset{\text{Ex. 5.31}}{=} \mathrm{Tr}\,[\hat{\rho}\,(|v_m\rangle\langle v_m|)] \\
&= \mathrm{Tr}[\hat{\rho}\hat{\Pi}_m] \\
&\overset{\text{Ex. 5.29(a)}}{=} \mathrm{Tr}[\hat{\Pi}_m\hat{\rho}]
\end{aligned}$$

Solution to Exercise 5.34. The projector onto $|{+45^\circ}\rangle$ is
$$\hat{\Pi}_+ = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Accordingly, using the density matrices from Ex. 5.1, we find

a)
$$\mathrm{Tr}[\hat{\Pi}_+\hat{\rho}] = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2},$$
consistent with $|\langle+|\,H\rangle|^2 = \frac{1}{2}$;

b)
$$\mathrm{Tr}[\hat{\Pi}_+\hat{\rho}] = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} |\psi_H|^2 + \psi_H^*\psi_V & \psi_H\psi_V^* + |\psi_V|^2 \\ |\psi_H|^2 + \psi_H^*\psi_V & \psi_H\psi_V^* + |\psi_V|^2 \end{pmatrix} = \frac{1}{2}(|\psi_H|^2 + \psi_H\psi_V^* + \psi_H^*\psi_V + |\psi_V|^2) = \frac{1}{2}|\psi_H + \psi_V|^2,$$
consistent with $|\langle+|(\psi_H|H\rangle + \psi_V|V\rangle)|^2 = |\frac{1}{\sqrt{2}}(\langle H| + \langle V|)(\psi_H|H\rangle + \psi_V|V\rangle)|^2 = \frac{1}{2}|\psi_H + \psi_V|^2$;

c)
$$\mathrm{Tr}[\hat{\Pi}_+\hat{\rho}] = \frac{1}{4}\mathrm{Tr}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2},$$
consistent with $p_1|\langle+|\,+\rangle|^2 + p_2|\langle+|\,-\rangle|^2 = \frac{1}{2} + 0 = \frac{1}{2}$;

d)

$$\mathrm{Tr}[\hat{\Pi}_+\hat{\rho}] = \mathrm{Tr}\begin{pmatrix} \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} \end{pmatrix} = \frac{3}{4},$$

consistent with $p_1|\langle +|+\rangle|^2 + p_2|\langle +|H\rangle|^2 + p_3|\langle +|V\rangle|^2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4}$.

Solution to Exercise 5.35. We know from Ex. 5.33 that the unnormalized state after the measurement, if the result $|v_m\rangle$ is obtained, is given by

$$\hat{\rho}_m = \hat{\Pi}_m\hat{\rho}\hat{\Pi}_m = \langle v_m|\,\hat{\rho}\,|v_m\rangle\,|v_m\rangle\langle v_m|\,.$$

If the measurement result is unknown, the states $\{\hat{\rho}_m\}$ comprise a statistical ensemble. To find the corresponding density matrices we have to sum over all m's:

$$\hat{\rho}_{\text{after}} = \sum_m \hat{\rho}_m = \sum_m \langle v_m|\,\hat{\rho}\,|v_m\rangle\,|v_m\rangle\langle v_m|\,. \tag{S5.7}$$

Note that we do not explicitly include the probabilities in the sum because the states $\hat{\rho}_m$ are unnormalized, so their probabilities of existence are included in their density matrices (see Ex. 5.4). Expression (5.15) is the matrix of the operator (S5.7) in the basis $\{|v_m\rangle\}$.

Solution to Exercise 5.36. The initial state $|+\rangle$ has the density operator $|+\rangle\langle +|$, which corresponds to the matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in the canonical basis and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the diagonal basis. After the measurement in the canonical basis, the state becomes fully mixed, i.e. $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in both bases. We see that the effect of the measurement on the density matrix written in the canonical basis corresponds to the stripping of the off-diagonal elements. However, if the density matrix is written in the diagonal basis (i.e. not in the measurement basis), the measurement changes the diagonal elements.

Solution to Exercise 5.37. Writing the definition (1.12) of observable operator in the form $\hat{V} = \sum_m v_m |v_m\rangle\langle v_m| = \sum_m v_m\hat{\Pi}_m$, we have

$$\mathrm{Tr}(\hat{V}\hat{\rho}) = \sum_m v_m \mathrm{Tr}(\hat{\Pi}_m\hat{\rho}) \overset{(5.13)}{=} \sum_m v_m \mathrm{pr}_m \overset{(B.1)}{=} \langle V\rangle\,,$$

where $\mathrm{pr}_m = \mathrm{Tr}(\hat{\Pi}_m\hat{\rho})$ is the probability of projecting $\hat{\rho}$ onto the eigenstate $|v_m\rangle$ of $\hat{V}$.

Solution to Exercise 5.38. Using the differential equation (5.7) for the evolution of the density matrix in the Schrödinger picture, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V\rangle = \mathrm{Tr}\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\hat{V}\right) = -\frac{i}{\hbar}\mathrm{Tr}\left([\hat{H},\hat{\rho}]\hat{V}\right) \tag{S5.8}$$

We now use the chain rule for the trace [Ex. 5.29(b)] to write

$$
\begin{aligned}
-\frac{i}{\hbar}\mathrm{Tr}\left([\hat{H},\hat{\rho}]\hat{V}\right) &= -\frac{i}{\hbar}\mathrm{Tr}\left(\hat{H}\hat{\rho}\hat{V}-\hat{\rho}\hat{H}\hat{V}\right)\\
&= -\frac{i}{\hbar}\mathrm{Tr}\left(\hat{\rho}\hat{V}\hat{H}-\hat{\rho}\hat{H}\hat{V}\right)\\
&= \frac{i}{\hbar}\mathrm{Tr}\left(\hat{\rho}[\hat{H},\hat{V}]\right)\\
&\overset{(5.16)}{=} \frac{i}{\hbar}\left\langle[\hat{H},\hat{V}]\right\rangle.
\end{aligned}
$$

Solution to Exercise 5.39.

a) We write $\hat{\rho}_{AB}=\sum_i p_i\left|\Psi_i\right\rangle$ in accordance with the density operator definition (5.1), with $\left|\Psi_i\right\rangle$ being bipartite states (pure, but not necessarily separable). As we found in Chapter 2 [see Eq. (2.22)], Alice's measurement of the state $\left|\Psi_i\right\rangle$, which results in observation of element $\left|v_m\right\rangle$ of her measurement basis, transforms $\left|\Psi_i\right\rangle$ into the unnormalized state $\hat{\Pi}_{A,m}\left|\Psi_i\right\rangle$. Accordingly, the full density matrix becomes

$$
\begin{aligned}
\hat{\rho}_{AB,m} &\to \sum_i p_i\hat{\Pi}_{A,m}\left|\Psi_i\right\rangle\left\langle\Psi_i\right|\hat{\Pi}_{A,m}\\
&= \hat{\Pi}_{A,m}\left(\sum_i p_i\left|\Psi_i\right\rangle\left\langle\Psi_i\right|\right)\hat{\Pi}_{A,m}\\
&= \hat{\Pi}_{A,m}\hat{\rho}_{AB}\hat{\Pi}_{A,m}\\
&= \left|v_m\right\rangle\left\langle v_m\right|\otimes\left\langle v_m\right|\hat{\rho}_{AB}\left|v_m\right\rangle. \qquad \text{(S5.9)}
\end{aligned}
$$

Bob's portion of this bipartite state is $\hat{\rho}_{B,m}=\left\langle v_m\right|\hat{\rho}_{AB}\left|v_m\right\rangle$.

b) If Alice's measurement result is unknown, Bob has a probabilistic ensemble of $\hat{\rho}_{B,m}$ for different m's, so its density operator is the sum of unnormalized individual density matrices as per Ex. 5.4:

$$\hat{\rho}_B=\sum_m\left\langle v_m\right|\hat{\rho}_{AB}\left|v_m\right\rangle=\mathrm{Tr}_A(\hat{\rho}_{AB}).$$

Solution to Exercise 5.40.

For the state of Ex. 2.45(a).

a) The ensemble description of Bob's photon found in the solution to Ex. 2.45 for Alice's measurement in the canonical basis is "either $\left|H\right\rangle$ with probability $1/5$ or $\left|V\right\rangle$ with probability $4/5$". This corresponds to the density matrix $\hat{\rho}_B\simeq\begin{pmatrix}1/5 & 0\\ 0 & 4/5\end{pmatrix}$.

If Alice measures in the diagonal basis, Bob's ensemble becomes "either $\sqrt{1/5}\left|H\right\rangle+\sqrt{4/5}\left|V\right\rangle$ or $\sqrt{1/5}\left|H\right\rangle-\sqrt{4/5}\left|V\right\rangle$ with probabilities $1/2$". The corresponding density matrix,

$$\frac{1}{2}\begin{pmatrix}1/5 & 2/5\\ 2/5 & 4/5\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1/5 & -2/5\\ -2/5 & 4/5\end{pmatrix}=\begin{pmatrix}1/5 & 0\\ 0 & 4/5\end{pmatrix},$$

is the same.

b) Using partial trace,

$$\begin{aligned}\hat{\rho}_B &= \mathrm{Tr}_A\,|\Psi\rangle\langle\Psi| \\ &= \frac{1}{5}\,{}_A\langle H|(|HH\rangle+2\,|VV\rangle)(\langle HH|+2\,\langle VV|)|H\rangle_A \\ &+\frac{1}{5}\,{}_A\langle V|(|HH\rangle+2\,|VV\rangle)(\langle HH|+2\,\langle VV|)|V\rangle_A \\ &= \frac{1}{5}\,|H\rangle\,\langle H|+\frac{4}{5}\,|V\rangle\,\langle V| \\ &\simeq \begin{pmatrix} 1/5 & 0 \\ 0 & 4/5 \end{pmatrix}\end{aligned}$$

This is consistent with part (a).

For the state of Ex. 2.45(b).

a) The verbal descriptions of Bob's photon found in the solution to Ex. 2.45 are "either $|+\rangle$ with probability $2/3$ or $|V\rangle$ with probability $1/3$" and "either $\sqrt{1/5}\,|H\rangle+\sqrt{4/5}\,|V\rangle$ with probability $5/6$ or $|H\rangle$ with probability $1/6$". These ensembles correspond to the same density matrices

$$\frac{2}{3}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}+\frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

and

$$\frac{5}{6}\begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}+\frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix},$$

b)

$$\begin{aligned}\hat{\rho}_B &= \mathrm{Tr}_A\,|\Psi\rangle\langle\Psi| \\ &= \frac{1}{3}\,\langle H|_A\,(|HH\rangle+|HV\rangle+|VV\rangle)(\langle HH|+\langle HV|+\langle VV|)\,|H\rangle_A \\ &+\frac{1}{3}\,\langle V|_A\,(|HH\rangle+|HV\rangle+|VV\rangle)(\langle HH|+\langle HV|+\langle VV|)\,|V\rangle_A \\ &= \frac{1}{3}(|H\rangle+|V\rangle)(\langle H|+\langle V|)+\frac{1}{3}\,|V\rangle\,\langle V| \\ &\simeq \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix},\end{aligned}$$

which is also the same as in part (a).

Solution to Exercise 5.41. For the Bell state $|\Phi^+\rangle$,

$$\begin{aligned}\hat{\rho}_B &= \mathrm{Tr}_A \left|\Phi^+\right\rangle\left\langle\Phi^+\right| \\ &= \frac{1}{2}\langle H|_A\,(|HH\rangle+|VV\rangle)(\langle HH|+\langle VV|)\,|H\rangle_A \\ &+ \frac{1}{2}\langle V|_A\,(|HH\rangle+|VV\rangle)(\langle HH|+\langle VV|)\,|V\rangle_A \\ &= \frac{1}{2}|H\rangle\langle H|+\frac{1}{2}|V\rangle\langle V|,\end{aligned} \tag{S5.10}$$

i.e. the fully mixed state. For the three other Bell states, the calculation is analogous and the result is the same.

Solution to Exercise 5.42. The proof is analogous to that in Ex. 5.26.

Solution to Exercise 5.43. Let us calculate the trace of $\hat{\rho}_{AB}$ in the basis $\{|v_m\rangle\otimes|w_n\rangle\}$, where $\{|v_n\rangle\}$ $\{|w_n\rangle\}$ are orthonormal bases in Alice's and Bob's spaces, respectively. We find

$$\begin{aligned}\mathrm{Tr}\hat{\rho}_{AB} &= \sum_{mn}((\langle v_m|\otimes\langle w_n|)(\hat{\rho}_{AB})(|v_m\rangle\otimes|w_n\rangle)) \\ &= \sum_n \langle w_n|\left(\sum_m \langle v_m|\,\hat{\rho}_{AB}\,|v_m\rangle\right)|w_n\rangle \\ &= \mathrm{Tr}(\mathrm{Tr}_A\hat{\rho}_{AB}).\end{aligned}$$

If the left-hand side of the above equation is 1, so must be its right-hand side.

Solution to Exercise 5.44.

a) If $\hat{\rho}_{AB}=|\phi\rangle\langle\phi|\otimes|\psi\rangle\langle\psi|$ (where states $|\phi\rangle$ and $|\psi\rangle$ live, respectively, in Alice's and Bob's Hilbert spaces), then for any element $|v_m\rangle$ of Alice's basis we have $\langle v_m|\,\hat{\rho}_{AB}\,|v_m\rangle = |\langle v_m|\,\phi\rangle|^2\,|\psi\rangle\langle\psi|$ and hence

$$\mathrm{Tr}_A(\hat{\rho}_{AB}) = \left(\sum_m |\langle v_m|\,\phi\rangle|^2\right)|\psi\rangle\langle\psi|,$$

which is a pure state. The argument for tracing over Bob's space is analogous.

b) Let us first assume the entangled bipartite state to be pure: $\hat{\rho}_{AB}=|\Psi\rangle\langle\Psi|$. We can decompose this state as we did in Sec. 2.2.2: $|\Psi\rangle=\sum_i \frac{1}{N_i}|v_i\rangle\otimes|b_i\rangle$, where $\{|v_i\rangle\}$ is an orthonormal basis in Alice's space and $\{|b_i\rangle\}$ is a set of normalized vectors in Bob's space. Tracing this over Alice's Hilbert space, we have

$$\hat{\rho}_B = \mathrm{Tr}_A\,|\Psi\rangle\langle\Psi| = \sum_i \frac{1}{N_i^2}\,|b_i\rangle\langle b_i|\,.$$

If $|\Psi\rangle$ is entangled, at least two of the $|b_i\rangle$'s are different, so $\hat{\rho}_B$ is mixed.
For a non-pure $\hat{\rho}_{AB}=\sum_i p_i\,|\Psi_i\rangle\langle\Psi_i|$, we have $\mathrm{Tr}_A(\hat{\rho}_{AB})=\sum_i p_i \mathrm{Tr}_A\,|\Psi_i\rangle\langle\Psi_i|$. This is a statistical mixture of mixed states, which, as we have shown in Ex. 5.22, cannot be pure.

Solution to Exercise 5.45. Suppose the system is in the initial state $\hat{\rho}_S=\sum_{ij}\rho_{ij}\,|v_i\rangle\left\langle v_j\right|$ while the initial density matrix of the apparatus is $|w_1\rangle\langle w_1|$. The von Neumann measurement transforms the system and apparatus according to $|v_i\rangle_{\text{system}}\otimes|w_1\rangle_{\text{apparatus}}\to|v_i\rangle_{\text{system}}\otimes|w_i\rangle_{\text{apparatus}}$, and hence it will produce the state

$$\hat{\rho}_{SA} = \sum_{ij} \rho_{ij} (|v_i\rangle\langle v_j|)_{\text{system}} \otimes (|w_i\rangle\langle w_j|)_{\text{apparatus}}$$

Tracing over the apparatus in the basis $\{|w_k\rangle]\}$, we find

$$\begin{aligned}\hat{\rho}'_S &= \sum_{ijk} \rho_{ij} (|v_i\rangle\langle v_j|)_{\text{system}} \langle w_k|\, w_i\rangle \langle w_j|\, w_k\rangle \\ &= \sum_{ijk} \rho_{ij} (|v_i\rangle\langle v_j|)_{\text{system}} \delta_{ik}\delta_{jk} \\ &= \sum_{k} \rho_{kk} (|v_k\rangle\langle v_k|)_{\text{system}},\end{aligned}$$

which corresponds to a diagonal density matrix in the basis $\{|v_k\rangle]\}$.

Solution to Exercise 5.46. For $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, the expectation value of the Pauli observable $\hat{\sigma}_x$ is

$$\langle\hat{\sigma}_x\rangle \overset{(5.16)}{=} \text{Tr}(\hat{\rho}\,\hat{\sigma}_x) = \sum_i p_i \text{Tr}(|\psi_i\rangle\langle\psi_i|\,\hat{\sigma}_x) \overset{(5.11)}{=} \sum_i p_i \langle\psi_i|\,\hat{\sigma}_x|\,\psi_i\rangle \overset{\text{Ex. 4.48(c)}}{=} \sum_i p_i R_{xi}.$$

The arguments for the y and z components of the Bloch vector are analogous.

Solution to Exercise 5.48.

a) An ensemble average $\vec{R}_{\hat{\rho}} = \sum_i p_i \vec{R}_i$ of unequal geometric vectors of length 1 has length less than 1. To prove this rigorously, we find for the length of the Bloch vector

$$\begin{aligned}|R_{\hat{\rho}}|^2 &= \vec{R}_{\hat{\rho}} \cdot \vec{R}_{\hat{\rho}} \\ &= \sum_{ij} p_i p_j \vec{R}_i \cdot \vec{R}_j \\ &< \sum_{ij} p_i p_j |R_i||R_j| \\ &= \left(\sum_i p_i |R_i|\right)^2 \\ &= \left(\sum_i p_i\right)^2 = 1.\end{aligned}$$

We have used the Cauchy-Schwarz inequality. It is strict because at least two of the $\vec{R}_i$'s correspond to unequal states and are hence non-collinear. We also used the fact that $|R_i| = 1$ for a pure state.

b) The fully mixed state is an equal mixture of the states $|\uparrow\rangle$ and $|\downarrow\rangle$. The Bloch vector of state $|\uparrow\rangle$ points along the positive z axis while that of state $|\downarrow\rangle$ along the negative z axis. Both these vectors have length 1, so their sum is zero.

Solution to Exercise 5.49. From the definition (5.20) of the Bloch vector of an ensemble, it follows that

$$\dot{\vec{R}}_{\hat{\rho}} = \sum_i p_i \dot{\vec{R}}_i \stackrel{(4.77)}{=} \sum_i p_i \gamma \vec{R}_i \times \vec{B} = \gamma \vec{R}_{\hat{\rho}} \times \vec{B}.$$

Solution to Exercise 5.50. The density matrix found in Ex. 5.25 is

$$\hat{\rho}(t) \simeq \begin{pmatrix} \frac{1}{2} + \frac{1}{4}\cos\Omega_L t & -\frac{i}{4}\sin\Omega_L t \\ \frac{i}{4}\sin\Omega_L t & \frac{1}{2} - \frac{1}{4}\cos\Omega_L t \end{pmatrix}. \tag{S5.11}$$

The components of the Bloch vector associated with this state are

$$R_x(t) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_x) = 0;$$
$$R_y(t) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_y) = \frac{1}{2}\sin\Omega_L t;$$
$$R_z(t) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_z) = \frac{1}{2}\cos\Omega_L t;$$

The corresponding trajectory of the Bloch vector is a circle of radius $\frac{1}{2}$ in the y-z plane, which corresponds to the precession around the x axis.

Solution to Exercise 5.51. Given the spectral decomposition

$$\hat{\rho} = p\,|v_1\rangle\langle v_1| + (1-p)\,|v_2\rangle\langle v_2|, \tag{S5.12}$$

we find

$$\hat{\rho}^2 = p^2\,|v_1\rangle\langle v_1| + (1-p)^2\,|v_2\rangle\langle v_2|$$

and hence

$$\mathrm{Tr}\hat{\rho}^2 = p^2 + (1-p)^2 = 2p^2 - 2p + 1. \tag{S5.13}$$

To find the length of the Bloch vector corresponding to state (S5.12), we notice that the Bloch vectors of orthogonal pure states $|v_1\rangle$ and $|v_2\rangle$ are opposite (Ex. 4.51) and have length 1. The geometric sum of these vectors with weights p and $1-p$ gives a vector of length

$$|R_{\hat{\rho}}| = |p - (1-p)| = |2p-1|. \tag{S5.14}$$

Combining Eqs. (S5.13) and (S5.14), we obtain Eq. (5.23).

Solution to Exercise 5.52. Let us consider an arbitrary vector $\vec{R}$ of length $0 < |R| \leq 1$. Following the logic of the previous exercise, in order for a normalized state $\hat{\rho}$ to have $\vec{R}$ as its Bloch vector, it must have the spectral decomposition

$$\hat{\rho} = p\,|v_1\rangle\langle v_1| + (1-p)\,|v_2\rangle\langle v_2|. \tag{S5.15}$$

Here $|v_1\rangle$ and $|v_2\rangle$ are orthogonal pure states such that their Bloch vectors, $\vec{R}_1$ and $\vec{R}_2$, satisfy

$$\vec{R} = p\vec{R}_1 + (1-p)\vec{R}_2 \tag{S5.16}$$

with $p \geq 1/2$. The vectors $\vec{R}_1$ and $\vec{R}_2$ have length 1 and are opposite in direction. Therefore, in order to satisfy Eq. (S5.16), these vectors must be collinear with $\vec{R}$, and hence Eq. (S5.16) has a unique solution; $\vec{R}_1 = \vec{R}/|R|$, $\vec{R}_2 = -\vec{R}_1$ and $p = (1+|R|)/2$. These vectors uniquely define the corresponding states $|v_1\rangle$ and $|v_2\rangle$, which, in turn, uniquely define the density operator (S5.15) whose Bloch vector is $\vec{R}$.

Solution to Exercise 5.53. Suppose at a given time t the spin state is given by

$$\hat{\rho}(t) \simeq \begin{pmatrix} \rho_{\uparrow\uparrow}(t) & \rho_{\uparrow\downarrow}(t) \\ \rho_{\downarrow\uparrow}(t) & \rho_{\downarrow\downarrow}(t) \end{pmatrix}.$$

After a short interval Δt, it will decohere, i.e. become

$$\hat{\rho}_{\rm dec}(t) \simeq \begin{pmatrix} \rho_{\uparrow\uparrow}(t) & 0 \\ 0 & \rho_{\downarrow\downarrow}(t) \end{pmatrix}$$

with probability $\Delta t/T_2$, and remain the same with probability $1-\Delta t/T_2$. Accordingly, the density matrix at the moment $t+\Delta t$ is

$$\begin{aligned}\hat{\rho}(t+\Delta t) &= (1-\Delta t/T_2)\hat{\rho}(t) + (\Delta t/T_2)\hat{\rho}_{\rm dec}(t) \\ &\simeq \begin{pmatrix} \rho_{\uparrow\uparrow}(t) & (1-\Delta t/T_2)\rho_{\uparrow\downarrow}(t) \\ (1-\Delta t/T_2)\rho_{\downarrow\uparrow}(t) & \rho_{\downarrow\downarrow}(t) \end{pmatrix}.\end{aligned}$$

The change of the off-diagonal elements during the time Δt can hence be written as

$$\Delta\rho_{ij}(t) = -(\Delta t/T_2)\rho_{ij}(t).$$

Dividing both sides of this equation by Δt, we obtain Eq. (5.24) in the limit $\Delta t \to 0$.

Solution to Exercise 5.54. If the dc field has been turned on for a long enough time for the spins to thermalize, the ratio of their probabilities will be determined by Boltzmann's law:

$$\frac{\mathrm{pr}_\downarrow}{\mathrm{pr}_\uparrow} = \exp\left(-\frac{\gamma\hbar B_0}{kT}\right) \stackrel{(4.71)}{=} \exp\left(-\frac{ge\hbar B_0}{2M_p kT}\right) \approx \exp\left(-1.1\times 10^{-5}\right) \approx 1-1.1\times 10^{-5},$$

where the mass and Landé factor of the proton are found in Table 4.3. Because this ratio is close to unity, both probabilities are close to 0.5 so $\mathrm{pr}_\downarrow - \mathrm{pr}_\uparrow \approx -0.55\times 10^{-5}$.

Solution to Exercise 5.55. Solving the system of equations

$$\begin{cases} \rho_{\downarrow\downarrow,0}/\rho_{\uparrow\uparrow,0} = e^{-\frac{\gamma\hbar B_0}{kT}} \\ \rho_{\downarrow\downarrow,0} + \rho_{\uparrow\uparrow,0} = 1 \end{cases},$$

we find

$$\begin{cases} \rho_{\uparrow\uparrow,0} = \frac{e^{\frac{\gamma\hbar B_0}{2kT}}}{e^{\frac{\gamma\hbar B_0}{2kT}} + e^{-\frac{\gamma\hbar B_0}{2kT}}} \\ \rho_{\downarrow\downarrow,0} = \frac{e^{-\frac{\gamma\hbar B_0}{2kT}}}{e^{\frac{\gamma\hbar B_0}{2kT}} + e^{-\frac{\gamma\hbar B_0}{2kT}}} \end{cases}.$$

According to Eq. (5.20), this corresponds to a Bloch vector of the length

$$\frac{e^{\frac{\gamma\hbar B_0}{2kT}} - e^{-\frac{\gamma\hbar B_0}{2kT}}}{e^{\frac{\gamma\hbar B_0}{2kT}} + e^{-\frac{\gamma\hbar B_0}{2kT}}} = \tanh\frac{\gamma\hbar B_0}{2kT}$$

pointing straight up.

Solution to Exercise 5.57. The first term in Eq. (5.32) is due to regular Schrödinger evolution, see Ex. 5.49. The additional term emerging due to the relaxation can be calculated according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{R} = -\gamma\vec{R}\times\vec{B} + \mathrm{Tr}\left[\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\right)_{\mathrm{relax}}\hat{\vec{\sigma}}\right]. \tag{S5.17}$$

We bring together Eqs. (5.24) and (5.30) to write

$$\left(\frac{\mathrm{d}\rho_{ij}}{\mathrm{d}t}\right)_{\mathrm{relax}} = \begin{cases} -[\rho_{ii}(t) - \rho_{ii,0}]/T_1, & i = j \\ -\rho_{ij}(t)/T_2, & i \neq j \end{cases}, \tag{S5.18}$$

or, explicitly,

$$\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\right)_{\mathrm{relax}} \simeq \begin{pmatrix} -[\rho_{\uparrow\uparrow}(t) - \rho_{\uparrow\uparrow,0}]/T_1 & -\rho_{\uparrow\downarrow}(t)/T_2 \\ -\rho_{\downarrow\uparrow}(t)/T_2 & -[\rho_{\downarrow\downarrow}(t) - \rho_{\downarrow\downarrow,0}]/T_1 \end{pmatrix}. \tag{S5.19}$$

From this result, we can calculate the second term of Eq. (S5.17) for each Pauli operator:

$$\mathrm{Tr}\left[\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\right)_{\mathrm{relax}}\hat{\sigma}_x\right] = -[\rho_{\uparrow\downarrow}(t) + \rho_{\downarrow\uparrow}(t)]/T_2;$$
$$\mathrm{Tr}\left[\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\right)_{\mathrm{relax}}\hat{\sigma}_x\right] = -i[\rho_{\uparrow\downarrow}(t) - \rho_{\downarrow\uparrow}(t)]/T_2;$$
$$\mathrm{Tr}\left[\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}\right)_{\mathrm{relax}}\hat{\sigma}_x\right] = -[\rho_{\uparrow\uparrow}(t) - \rho_{\downarrow\downarrow}(t)]/T_1 + [\rho_{\uparrow\uparrow,0}(t) - \rho_{\downarrow\downarrow,0}(t)]/T_1.$$

Relating the components of the Bloch vector to the density matrix elements according to Eq. (5.22), we obtain Eq. (5.33).

Solution to Exercise 5.58. We can start by rewriting Eq. (5.33) explicitly for each component of the Bloch vector:

$$\begin{aligned} \dot{R}_x(t) &= \gamma(R_yB_z - R_zB_y) - R_x(t)/T_2; \\ \dot{R}_y(t) &= \gamma(R_zB_x - R_xB_z) - R_y(t)/T_2; \\ \dot{R}_z(t) &= \gamma(R_xB_y - R_yB_x) - [R_z(t) - R_0]/T_1, \end{aligned} \tag{S5.20}$$

In the absence of the rf field, the fictitious magnetic field (4.87) has only the z component which is determined by the detuning: $B_z = -\Delta/\gamma$. Differential equations (S5.20) therefore simplify to

$$\begin{aligned}\dot{R}_x(t) &= -\Delta R_y - R_x(t)/T_2;\\ \dot{R}_y(t) &= \Delta R_x - R_y(t)/T_2;\\ \dot{R}_z(t) &= -[R_z(t) - R_0]/T_1,\end{aligned} \tag{S5.21}$$

One can verify by direct substitution that Eqs. (5.34) solve these equations.

Solution to Exercise 5.60. Let us work in the rotating frame. Because no rf field is present, we can choose the frequency of the rotating basis equal to the Larmor frequency, so the detuning Δ vanishes. Then the time derivative of the Bloch vector is determined only by the relaxation terms of Eq. (S5.21).

Polar coordinates $(\theta, 0)$ of the initial Bloch vector correspond to Cartesian coordinates $\vec{R}(0) = (\sin\theta, 0, \cos\theta)$. The time derivative of the Bloch vector length is then given by

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}|\vec{R}(t)|^2\right|_{t=0} &= 2\left(R_x\dot{R}_x + R_y\dot{R}_y + R_z\dot{R}_z\right)\big|_{t=0}\\ &\overset{\text{(S5.21)}}{=} -2\frac{\sin^2\theta}{T_2} + 2\cos\theta\frac{1-\cos\theta}{T_1}\\ &= -2\frac{\sin^2\theta}{T_2} + 2\frac{\cos\theta}{T_1} - 2\frac{\cos^2\theta}{T_1},\end{aligned}$$

where we set $\vec{R}_0 = (0,0,1)$ at the absolute zero temperature. Approximating $\sin^2\theta \approx \theta^2$, $\cos\theta \approx 1-\theta^2/2$, $\cos^2\theta \approx 1-\theta^2$ for small θ, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}|\vec{R}(t)|^2\right|_{t=0} \approx -2\frac{\theta^2}{T_2} - \frac{\theta^2}{T_1} + 2\frac{\theta^2}{T_1} = -2\frac{\theta^2}{T_2} + \frac{\theta^2}{T_1}.$$

This derivative cannot be positive because the length of the Bloch vector is already at the maximum possible value of 1 at $t = 0$. This means that $-2/T_2 + 1/T_1 \le 0$ or $T_2 \le 2T_1$.

Solution to Exercise 5.61. Let us first track the evolution of the Bloch vector associated with a specific detuning Δ akin to what we did when solving Ex. 4.74. Applying a $\pi/2$ pulse to the spin-up state will transform it into the state with the spin pointing along the y axis, so $\vec{R}(0) = (0,1,0)$. The subsequent evolution is governed by Eqs. (5.34):

$$\vec{R}(t) = \left(-\sin\Delta t e^{-t/T_2}, \cos\Delta t e^{-t/T_2}, 1 - e^{-t/T_1}\right).$$

The π pulse at $t = t_0$ rotates the spin by 180° around the x axis, resulting in

$$\vec{R}(t_0) = (-\sin\Delta t_0 e^{-t_0/T_2}, -\cos\Delta t_0 e^{-t_0/T_2}, -1 + e^{-t_0/T_1}).$$

Subsequent evolution yields

$$\begin{aligned}\vec{R}(t) &= \big[(-\sin\Delta t_0\cos\Delta(t-t_0)+\cos\Delta t_0\sin\Delta(t-t_0))e^{-t/T_2},\\ &\quad(-\sin\Delta t_0\sin\Delta(t-t_0)-\cos\Delta t_0\cos\Delta(t-t_0))e^{-t_0/T_2},\\ &\quad 1+[-2+e^{-t_0/T_1}]e^{-(t-t_0)/T_1}\big]\\ &=(\sin\Delta(t-2t_0)e^{-t/T_2},-\cos\Delta(t-2t_0)e^{-t/T_2},1-2e^{-(t-t_0)/T_1}+e^{-t/T_1}).\end{aligned}$$

Now integrating the components of that vector over all detunings, we find, by analogy with Ex. 4.76,

$$\begin{aligned}\overline{\langle\mu_x\rangle} &= 0;\\ \overline{\langle\mu_y\rangle} &= -\frac{\hbar\gamma}{2}e^{-\frac{[\Delta_0(t-2t_0)]^2}{4}}e^{-t/T_2};\\ \overline{\langle\mu_z\rangle} &= 1-2e^{-(t-t_0)/T_1}+e^{-t/T_1}.\end{aligned}$$

Solution to Exercise 5.62. The thermal state is characterized by Bloch vector $\vec{R}_0 \stackrel{(5.27)}{=} (0,0,R_{0z})$, where $R_{0z} = e^{-\frac{\gamma\hbar B_0}{kT}}$. The initial π pulse will flip this vector so that $\vec{R}(0) = (0,0,-R_{0z})$. The subsequent evolution according to Eqs. (5.34) is as follows:

$$\vec{R}(t) = (0,0,R_0+[R_z(0)-R_0]e^{-t/T_1}) = (0,0,R_{z0}(1-2e^{-t/T_1})).$$

We see that $\vec{R}(t)=0$ when $e^{-t/T_1}=1/2$ or $t=T_1\ln 2$.

Solution to Exercise 5.63. $\mu_{HH}=3/4,\ \mu_{VH}=1/4, \mu_{HV}=1/3,\ \mu_{VV}=2/3.$

Solution to Exercise 5.64. $\sum_j \mu_{ij}$ is the sum of the probabilities for all possible output states given the ith output of the quantum measurement. Because one and only one of the output states is displayed for each measurement, this sum is equal to one.

Solution to Exercise 5.65. Suppose n photons are incident on the detector. Each of these photons will generate an avalanche with probability η. The "no click" state will occur if none of the photons generate an avalanche, which happens with probability $(1-\eta)^n$. Hence $\mu_{\text{no click},n}=(1-\eta)^n$. Since $\mu_{\text{no click},n}+\mu_{\text{click},n}=1$ (Ex. 5.64), we have $\mu_{\text{click},n}=1-(1-\eta)^n$.

Solution to Exercise 5.66. The Hermitian nature of POVM elements follows from the fact that any projection operator $\hat{\Pi}_i = \langle v_i|\ v_i\rangle$ (where $|v_i\rangle$ is the corresponding basis vector) is Hermitian and all μ_{ji} are real.

To show non-negativity, we write for an arbitrary non-zero vector $|\psi\rangle$:

$$\langle\psi|\ \hat{F}_j|\ \psi\rangle = \left\langle\psi\middle|\sum_i\mu_{ji}\hat{\Pi}_i\middle|\ \psi\right\rangle = \sum_i\mu_{ji}\langle\psi|\ \hat{\Pi}_i|\ \psi\rangle = \sum_i\mu_{ji}|\langle\psi|\ v_i\rangle|^2.$$

The right-hand side of this expression is non-negative because each μ_{ji} is a probability. This implies that $\hat{F}_j$ is non-negative according to Defn. A.22.

Solution to Exercise 5.67.

a) Using the result of Ex. 5.63 and summing over all possible outcomes of the quantum mesaurement according to Eq. (5.36), we find

$$F_H = \mu_{HH}|H\rangle\langle H| + \mu_{HV}|V\rangle\langle V| \simeq \begin{pmatrix} 3/4 & 0 \\ 0 & 1/3 \end{pmatrix};$$
$$F_V = \mu_{VH}|H\rangle\langle H| + \mu_{VV}|V\rangle\langle V| \simeq \begin{pmatrix} 1/4 & 0 \\ 0 & 2/3 \end{pmatrix}.$$

b) Similarly, using the result of Ex. 5.65, we obtain

$$\hat{F}_{\text{no click}} = \sum_n \mu_{\text{no click},n}|n\rangle\langle n| = \sum_n (1-\eta)^n |n\rangle\langle n|; \tag{S5.22}$$
$$\hat{F}_{\text{click}} = \sum_n \mu_{\text{click},n}|n\rangle\langle n| = \sum_n [1-(1-\eta)^n]\,|n\rangle\langle n|. \tag{S5.23}$$

Solution to Exercise 5.68.

$$\sum_{j=1}^{M} \hat{F}_j = \sum_{j=1}^{M}\sum_{i=1}^{N} \hat{\Pi}_i \mu_{ij} \stackrel{\text{Ex. 5.64}}{=} \sum_{i=1}^{N} \hat{\Pi}_i = \hat{\mathbf{1}}.$$

In the last equality, we used the resolution of the identity (A.26).

Solution to Exercise 5.69.

a) Using the theorem of total probability (Ex. B.6), we find

$$\text{pr}_j = \sum_i \mu_{ij}\text{pr}_i \stackrel{(5.13)}{=} \sum_i \mu_{ij}\text{Tr}(\hat{\Pi}_i\hat{\rho}) = \text{Tr}(\sum_i \mu_{ij}\hat{\Pi}_i\hat{\rho}) = \text{Tr}(\hat{F}_j\hat{\rho}),$$

b) Similarly,

$$\text{pr}_j = \sum_i \mu_{ij}\hat{\rho}_{B,i} \stackrel{(5.19)}{=} \sum_i \mu_{ij}\text{Tr}_A(\hat{\Pi}_i\hat{\rho}_{AB}) = \text{Tr}_A(\hat{F}_j\hat{\rho}_{AB}),$$

where $\hat{\rho}_{B,i}$ is Bob's state in the event Alice detected $|v_i\rangle$.

Solution to Exercise 5.70.

Method 1: using the pure state and projective measurement formalism

a) We use the model of Fig. 5.2, i.e. assume that Alice's detector consists of a pure quantum polarization measurement device followed by a scrambler. There are four possibilities that may result in Alice's observation of H in the output of her detector.
   - The initial state is $|\Psi_1\rangle$ and Alice's quantum polarization measurement detects $|H\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\,\Psi_1\rangle = (|H\rangle + |V\rangle)/\sqrt{6}$. The probability that Alice's scrambler will map her result onto output state H is $3/4$.
   - The initial state is $|\Psi_1\rangle$ and Alice's quantum polarization measurement detects $|V\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\,\Psi_1\rangle = 2|V\rangle/\sqrt{6}$. The probability that Alice's scrambler will map her result onto output state H is $1/3$.

- The initial state is $|\Psi_2\rangle$ and Alice's quantum polarization measurement detects $|H\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\Psi_2\rangle = |V\rangle$. The probability that Alice's scrambler will map her result onto output state H is $3/4$.
- The initial state is $|\Psi_2\rangle$ and Alice's quantum polarization measurement detects $|V\rangle$. In this case, the state of Bob's photon is $\langle H|\Psi_2\rangle = 0$.

The overall unnormalized Bob's density matrix is therefore

$$\begin{aligned}\rho_{B,H} &= \frac{3}{5}\left[\frac{3}{4}\frac{|H\rangle+|V\rangle}{\sqrt{6}}\frac{\langle H|+\langle V|}{\sqrt{6}} + \frac{1}{3}\frac{2|V\rangle}{\sqrt{6}}\frac{2\langle V|}{\sqrt{6}}\right] + \frac{2}{5}\frac{3}{4}|V\rangle\langle V| \\ &= \frac{3}{40}|H\rangle\langle H| + \frac{3}{40}|H\rangle\langle V| + \frac{3}{40}|V\rangle\langle H| + \frac{61}{120}|V\rangle\langle V|.\end{aligned}$$

b) Proceeding similarly for Alice's observing V, we find the following ensemble:
- The initial state is $|\Psi_1\rangle$ and Alice's quantum polarization measurement detects $|H\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\Psi_1\rangle = (|H\rangle + |V\rangle)/\sqrt{6}$. The probability that Alice's scrambler will map her result onto output state V is $1/4$.
- The initial state is $|\Psi_1\rangle$ and Alice's quantum polarization measurement detects $|V\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\Psi_1\rangle = 2|V\rangle/\sqrt{6}$. The probability that Alice's scrambler will map her result onto output state H is $2/3$.
- The initial state is $|\Psi_2\rangle$ and Alice's quantum polarization measurement detects $|H\rangle$. In this case, the unnormalized state of Bob's photon is $\langle H|\Psi_2\rangle = |V\rangle$. The probability that Alice's scrambler will map her result onto output state H is $1/4$.
- The initial state is $|\Psi_2\rangle$ and Alice's quantum polarization measurement detects $|V\rangle$. In this case, the state of Bob's photon is $\langle H|\Psi_2\rangle = 0$.

The overall unnormalized Bob's density matrix is therefore

$$\begin{aligned}\rho_{B,V} &= \frac{3}{5}\left[\frac{1}{4}\frac{|H\rangle+|V\rangle}{\sqrt{6}}\frac{\langle H|+\langle V|}{\sqrt{6}} + \frac{2}{3}\frac{2|V\rangle}{\sqrt{6}}\frac{2\langle V|}{\sqrt{6}}\right] + \frac{2}{5}\frac{1}{4}|V\rangle\langle V| \\ &= \frac{1}{40}|H\rangle\langle H| + \frac{1}{40}|H\rangle\langle V| + \frac{1}{40}|V\rangle\langle H| + \frac{47}{120}|V\rangle\langle V|.\end{aligned}$$

c) State $|\Psi_1\rangle$, which occurs with probability $3/5$, can be written as

$$|\Psi_1\rangle = |H\rangle \otimes \frac{|H\rangle+|V\rangle}{\sqrt{6}} + |V\rangle \otimes \frac{2|V\rangle}{\sqrt{6}}.$$

If Alice's measurement result is unknown, this is equivalent to the situation when her photon is lost, so Bob's photon is in a mixture of unnormalized states $\frac{|H\rangle+|V\rangle}{\sqrt{6}}$ and $\frac{2|V\rangle}{\sqrt{6}}$. On the other hand, if Alice's photon is lost while the ensemble is in state $|\Psi_2\rangle$ (whose probability is $2/5$), Bob's photon is in the state $|V\rangle$.
This ensemble corresponds to density operator

$$\begin{aligned}\rho_{B,V} &= \frac{3}{5}\left[\frac{|H\rangle+|V\rangle}{\sqrt{6}}\frac{\langle H|+\langle V|}{\sqrt{6}} + \frac{2|V\rangle}{\sqrt{6}}\frac{2\langle V|}{\sqrt{6}}\right] + \frac{2}{5}|V\rangle\langle V| \\ &= \frac{1}{10}|H\rangle\langle H| + \frac{1}{10}|H\rangle\langle V| + \frac{1}{10}|V\rangle\langle H| + \frac{9}{10}|V\rangle\langle V|.\end{aligned}$$

Method 2: using the density matrix and generalized measurement formalism

a) The initial state's density operator is

$$\hat{\rho}_{AB} = \frac{3}{5}\frac{|HH\rangle + |HV\rangle + 2|VV\rangle}{\sqrt{6}}\frac{\langle HH| + \langle HV| + 2\langle VV|}{\sqrt{6}} + \frac{2}{5}|HV\rangle\langle HV|. \tag{S5.24}$$

Alice's detector POVM element corresponding to output state H is, as found in Ex. 5.67, $F_H = \frac{3}{4}|H\rangle_{A\,A}\langle H| + \frac{1}{3}|V\rangle_{A\,A}\langle V|$. Hence

$$\begin{aligned}
F_H\hat{\rho}_{AB} &= \frac{3}{5}\frac{\frac{3}{4}|HH\rangle + \frac{3}{4}|HV\rangle + \frac{1}{3}\times 2|VV\rangle}{\sqrt{6}}\frac{\langle HH| + \langle HV| + 2\langle VV|}{\sqrt{6}} + \frac{2}{5}\frac{3}{4}|HV\rangle\langle HV| \\
&= \frac{3}{40}|HH\rangle\langle HH| + \frac{3}{40}|HH\rangle\langle HV| + \frac{3}{20}|HH\rangle\langle VV| + \frac{3}{40}|HV\rangle\langle HH| + \frac{3}{40}|HV\rangle\langle HV| \\
&+ \frac{3}{20}|HV\rangle\langle VV| + \frac{3}{20}|VV\rangle\langle HH| + \frac{3}{20}|VV\rangle\langle HV| + \frac{2}{15}|VV\rangle\langle VV| + \frac{3}{10}|HV\rangle\langle HV|.
\end{aligned}$$

Now taking the trace over Alice's photon we find the density matrix of Bob's photon.

$$\begin{aligned}
\rho_{B,H} &= \mathrm{Tr}_A[F_H\hat{\rho}_{AB}] = {}_A\langle H|\, F_H\hat{\rho}_{AB}\,|H\rangle_A + {}_A\langle V|\, F_H\hat{\rho}_{AB}\,|V\rangle_A \\
&= \frac{3}{40}|H\rangle\langle H| + \frac{3}{40}|H\rangle\langle V| + \frac{3}{40}|V\rangle\langle H| + \frac{61}{120}|V\rangle\langle V|.
\end{aligned}$$

b) Alice's detector POVM element is in this case $F_V = \frac{1}{4}|H\rangle_{A\,A}\langle H| + \frac{2}{3}|V\rangle_{A\,A}\langle V|$. Hence

$$\begin{aligned}
F_V\hat{\rho}_{AB} &= \frac{3}{5}\frac{\frac{1}{4}|HH\rangle + \frac{1}{4}|HV\rangle + \frac{2}{3}\times 2|VV\rangle}{\sqrt{6}}\frac{\langle HH| + \langle HV| + 2\langle VV|}{\sqrt{6}} + \frac{2}{5}\frac{1}{4}|HV\rangle\langle HV| \\
&= \frac{1}{40}|HH\rangle\langle HH| + \frac{1}{40}|HH\rangle\langle HV| + \frac{1}{20}|HH\rangle\langle VV| + \frac{1}{40}|HV\rangle\langle HH| + \frac{1}{40}|HV\rangle\langle HV| \\
&+ \frac{1}{20}|HV\rangle\langle VV| + \frac{2}{15}|VV\rangle\langle HH| + \frac{2}{15}|VV\rangle\langle HV| + \frac{4}{15}|VV\rangle\langle VV| + \frac{1}{10}|HV\rangle\langle HV|.
\end{aligned}$$

Tracing over Alice's photon we find:

$$\begin{aligned}
\rho_{B,H} &= \mathrm{Tr}_A[F_H\hat{\rho}_{AB}] = {}_A\langle H|\, F_H\hat{\rho}_{AB}\,|H\rangle_A + {}_A\langle V|\, F_H\hat{\rho}_{AB}\,|V\rangle_A \\
&= \frac{1}{40}|H\rangle\langle H| + \frac{1}{40}|H\rangle\langle V| + \frac{1}{40}|V\rangle\langle H| + \frac{47}{120}|V\rangle\langle V|.
\end{aligned}$$

c) Tracing the bipartite density matrix (S5.24) over Alice's photon, we find

$$\begin{aligned}
\hat{\rho}_B = \mathrm{Tr}_A\hat{\rho}_{AB} &= \frac{1}{10}|H\rangle\langle H| + \frac{1}{10}|H\rangle\langle V| + \frac{1}{10}|V\rangle\langle H| + \left(\frac{1}{10} + \frac{4}{10} + \frac{2}{5}\right)|V\rangle\langle V| \\
&= \frac{1}{10}|H\rangle\langle H| + \frac{1}{10}|H\rangle\langle V| + \frac{1}{10}|V\rangle\langle H| + \frac{9}{10}|V\rangle\langle V|.
\end{aligned}$$

We see that the results from both methods are consistent with each other and the density matrix of part (c) is the sum of those from parts (a) and (b), as expected. Furthermore, the trace of the density matrix from part (c), which gives the state of Bob's photon independently of Alice's measurement result, equals one as expected.

Solution to Exercise 5.71.

a) In the event of the measurement result $|v_i\rangle$, the unnormalized density operator of the system becomes $\hat{\Pi}_i\hat{\rho}\hat{\Pi}_i$ (Ex. 5.33). For the event that the measurement result $|v_i\rangle$ occurred *and* the scrambler pointed to output state j, the unnormalized density operator is $\mu_{ij}\hat{\Pi}_i\hat{\rho}\hat{\Pi}_i$. Summing over all i's, we find the density matrix corresponding to the event of observing output state j:

$$\hat{\rho}_j = \sum_i \mu_{ji}\hat{\Pi}_i\hat{\rho}\hat{\Pi}_i.$$

b) Using the above equation, for $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, we find for the unnormalized measurement outputs:

$$\hat{\rho}_H = \frac{3}{4}\hat{\Pi}_H\hat{\rho}\hat{\Pi}_H + \frac{1}{3}\hat{\Pi}_V\hat{\rho}\hat{\Pi}_V \simeq \frac{3}{4}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/8 & 0 \\ 0 & 1/6 \end{pmatrix};$$
$$\hat{\rho}_V = \frac{1}{4}\hat{\Pi}_H\hat{\rho}\hat{\Pi}_H + \frac{2}{3}\hat{\Pi}_V\hat{\rho}\hat{\Pi}_V \simeq \frac{1}{4}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/8 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

The traces of these density matrices' give the probabilities of occurrence for each of the detector output states. Their sum equals unity, which is consistent with the fact that one of the outputs must occur with certainty.
On the other hand, using the POVM found in Ex. 5.67, we have

$$\hat{F}_H\hat{\rho}\hat{F}_H \simeq \begin{pmatrix} 9/32 & 1/8 \\ 1/8 & 1/18 \end{pmatrix};$$
$$\hat{F}_V\hat{\rho}\hat{F}_V \simeq \begin{pmatrix} 1/32 & 1/12 \\ 1/12 & 2/9 \end{pmatrix}.$$

Solution to Exercise 5.72. Normalizing the first result of Ex. 5.71(b), we obtain

$$\hat{\rho}_H = \mathcal{N}[\hat{\rho}_H] \simeq \frac{24}{13}\begin{pmatrix} 3/8 & 0 \\ 0 & 1/6 \end{pmatrix} = \begin{pmatrix} 9/13 & 0 \\ 0 & 4/13 \end{pmatrix}.$$

Repeating the measurement produces the following unnormalized matrices:

$$\begin{aligned}\hat{\rho}_{H\to H} =&\frac{3}{4}\hat{\Pi}_H N[\hat{\rho}_H]\hat{\Pi}_H + \frac{1}{3}\hat{\Pi}_V N[\hat{\rho}_H]\hat{\Pi}_V\\ &\simeq \frac{3}{4}\begin{pmatrix} 9/13 & 0 \\ 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & 4/13\end{pmatrix}\\ &= \begin{pmatrix} 27/52 & 0 \\ 0 & 4/39\end{pmatrix};\\ \hat{\rho}_{H\to V} =&\frac{1}{4}\hat{\Pi}_H N[\hat{\rho}_H]\hat{\Pi}_H + \frac{2}{3}\hat{\Pi}_V N[\hat{\rho}_H]\hat{\Pi}_V\\ &\simeq \frac{1}{4}\begin{pmatrix} 9/13 & 0 \\ 0 & 0\end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 4/13\end{pmatrix}\\ &= \begin{pmatrix} 9/52 & 0 \\ 0 & 8/39\end{pmatrix}.\end{aligned}$$

The probabilities of each result are equal to the traces $\text{Tr}\hat{\rho}_{H\to H} = 97/156$, $\text{Tr}\hat{\rho}_{H\to V} = 59/156$.

Solution to Exercise 5.73.

a) The photon incident on the first beam splitter has equal probability to be transmitted and reflected: $\text{pr}_r = \text{pr}_t = \frac{1}{2}$. Let us consider these cases separately.

- If the photon is transmitted, it is measured in the canonical basis. Then the probability to obtain outputs 1 and 2 are, respectively,

$$\text{pr}_{1|t} = \langle H|\,\hat{\rho}\,|H\rangle = \rho_{HH};$$
$$\text{pr}_{2|t} = \langle V|\,\hat{\rho}\,|V\rangle = \rho_{VV}.$$

- If the photon is reflected, it is measured in the diagonal basis. Then the probability to obtain outputs 1 and 2 are, respectively,

$$\text{pr}_{1|r} = \langle +|\,\hat{\rho}\,|+\rangle = \frac{1}{2}(\rho_{HH} + \rho_{HV} + \rho_{VH} + \rho_{VV});$$
$$\text{pr}_{2|r} = \langle -|\,\hat{\rho}\,|-\rangle = \frac{1}{2}(\rho_{HH} - \rho_{HV} - \rho_{VH} + \rho_{VV}).$$

Now using the theorem of total probabilities (B.6), we find

$$\text{pr}_1 = \text{pr}_t\text{pr}_{1|t} + pr_r\text{pr}_{1|r} = \frac{3}{4}\rho_{HH} + \frac{1}{4}\rho_{HV} + \frac{1}{4}\rho_{VH} + \frac{1}{4}\rho_{VV};$$
$$\text{pr}_2 = \text{pr}_t\text{pr}_{2|t} + pr_r\text{pr}_{2|r} = \frac{1}{4}\rho_{HH} - \frac{1}{4}\rho_{HV} - \frac{1}{4}\rho_{VH} + \frac{3}{4}\rho_{VV}.$$

b) Let the POVM element for the jth detector output be $\hat{F}_j \simeq \begin{pmatrix} F_{jHH} & F_{jHV} \\ F_{jVH} & F_{jVV}\end{pmatrix}$. Then we have

$$\text{pr}_j \overset{(5.39)}{=} F_{jHH}\rho_{HH} + F_{jHV}\rho_{VH} + F_{jVH}\rho_{HV} + F_{jVV}\rho_{VV}.$$

Comparing this with the result of part (a), we find

$$\hat{F}_1 \simeq \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}; \quad \hat{F}_2 \simeq \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}.$$

As expected, $\hat{F}_1 + \hat{F}_2 = \hat{\mathbf{1}}$.

Solution to Exercise 5.74.

a) We rely on the fact that the probability (5.39) is a real number for any physical state $\hat{\rho}$. Let us first set $\hat{\rho} = |v_k\rangle\langle v_k|$, where $\{|v_k\rangle\}$ is an arbitrary orthonormal basis of the Hilbert space. We then have

$$\mathrm{Tr}(\hat{F}_j\hat{\rho}) = \left\langle v_k\right| \hat{F}_j \left| v_k\right\rangle \in \mathbb{R},$$

which shows that all diagonal elements $(F_j)_{kk}$ of the matrix of $\hat{F}_j$ are real. Next, let us prove that any off-diagonal elements $(F_j)_{kl}$ and $(F_j)_{lk}$ are complex conjugate of each other. Consider the states $\hat{\rho}_{\mathrm{re,im}} = \left|\psi_{\mathrm{re,im}}\right\rangle\left\langle\psi_{\mathrm{re,im}}\right|$ with $\psi_{\mathrm{re}} = (|v_k\rangle + |v_l\rangle)/\sqrt{2}$ and $\psi_{\mathrm{im}} = (|v_k\rangle + i|v_l\rangle)/\sqrt{2}$. For these states, $(\rho_{\mathrm{re,im}})_{kk} = (\rho_{\mathrm{re,im}})_{ll} = (\rho_{\mathrm{re}})_{kl} = (\rho_{\mathrm{re}})_{lk} = \frac{1}{2}$, $(\rho_{\mathrm{im}})_{kl} = -\frac{i}{2}$, and $(\rho_{\mathrm{im}})_{lk} = \frac{i}{2}$. Hence

$$\mathrm{Tr}[\hat{F}_j\,|\psi_{\mathrm{re}}\rangle\langle\psi_{\mathrm{re}}|] = \frac{1}{2}[(F_j)_{kk} + (F_j)_{ll} + (F_j)_{kl} + (F_j)_{lk}] \in \mathbb{R};$$
$$\mathrm{Tr}[\hat{F}_j\,|\psi_{\mathrm{im}}\rangle\langle\psi_{\mathrm{re}}|] = \frac{1}{2}[(F_j)_{kk} + (F_j)_{ll}] + \frac{i}{2}[(F_j)_{kl} - (F_j)_{lk}] \in \mathbb{R}.$$

Because, as we found out, both $(F_j)_{kk}$ and $(F_j)_{ll}$ are real, it follows from the above that

$$\mathrm{Im}\,[(F_j)_{kl} + (F_j)_{lk}] = 0;$$
$$\mathrm{Re}\,[(F_j)_{kl} - (F_j)_{lk}] = 0.$$

This means that $\mathrm{Im}\,(F_j)_{kl} = -\mathrm{Im}\,(F_j)_{lk}$ and $\mathrm{Re}\,(F_j)_{kl} = \mathrm{Re}\,(F_j)_{lk}$, i.e. $(F_j)_{kl} = (F_j)^*_{lk}$.

b) Suppose a POVM element $\hat{F}_j$ is not non-negative, i.e. there exists a state $|\psi\rangle$ such that $\left\langle\psi\right| \hat{F}_j \left|\psi\right\rangle < 0$. But this value, according to Eq. (5.39), equals the probability $\mathrm{pr}_j = \mathrm{Tr}(\hat{F}_j\,|\psi\rangle\langle\psi|)$ of observing the jth detector output when measuring the state $|\psi\rangle$. Because negative probabilities are impossible, we have arrived at a contradiction.

c) Suppose we perform a measurement on a physical state with the density matrix $\hat{\rho}$. Because this measurement will with certainty result in exactly one output state of the detector, we can write using Eq. (5.39):

$$\sum_j \mathrm{pr}_j = \sum_j \mathrm{Tr}(\hat{F}_j\hat{\rho}) = \mathrm{Tr}(\hat{X}\hat{\rho}) = 1, \tag{S5.25}$$

where we denoted $\sum_j \hat{F}_j = \hat{X}$.

Because all $\hat{F}_j$ are Hermitian operators, so is $\hat{X}$. Hence there exists an orthonormal basis $\{|v_k\rangle\}$ in which $\hat{X}$ diagonalizes (see Ex. A.60). Setting $\hat{\rho} = |v_k\rangle\langle v_k|$ and substituting this state into Eq. (S5.25), we then have for any k

$$\mathrm{Tr}(\hat{X}\hat{\rho}) = \left\langle v_k\right| \hat{X} \left| v_k\right\rangle = 1.$$

Because the matrix of $\hat{X}$ is diagonal in basis $\{|v_k\rangle\}$, it follows from the above relation that this matrix corresponds to the identity operator.

Solution to Exercise 5.75. For any POVM element $\hat{F}_j$, there exists an orthonormal basis $\{|v_i\rangle\}$ in which $\hat{F}_j$ diagonalizes. Substituting elements of this basis into Eq. (5.39), we find for the probability of the jth measurement outcome

$$\mathrm{pr}_j = \mathrm{Tr}(\hat{F}_j\,|v_i\rangle\langle v_i|) = \left\langle v_i\right|\hat{F}_j\left|v_i\right\rangle.$$

The detector's not being able to provide information about the input quantum system's state means that p_j is the same for all input states, so $\left\langle v_i\right|\hat{F}_j\left|v_i\right\rangle$ is the same for all values of i. In other words, the matrix of $\hat{F}_j$ in the basis $\{|v_i\rangle\}$ is diagonal with all diagonal elements equal to p_j.

Solution to Exercise 5.76.

a) The density matrix contains N^2 elements, which implies that N^2 *complex* parameters suffice to fully describe it. However, since the density operator is Hermitian, $\rho_{ij} = \rho_{ji}^*$, so one pair of real numbers provides information about both these matrix elements (and only one real number is required for each diagonal element). Hence N^2 *real* parameters are in fact sufficient. Furthemore, physical density matrices have the unity trace restriction, which means that if we know any $N-1$ diagonal elements, we can calculate the Nth one. This further reduces the number of needed real parameters to N^2-1.
Note that physical density matrices are also restricted by condition (5.3). However, this condition is an inequality and hence does not further reduce the number of required parameters.
b) Projective measurements in a given basis $\{\left|v_j\right\rangle\}$ yield N real probabilities $\mathrm{pr}_j = \left\langle v_j\right|\hat{\rho}\left|v_j\right\rangle$ associated with the N basis elements. However, because these probabilities add up to one, the information about them can be contained in $N-1$ real numbers.

Solution to Exercise 5.77. Using the results of Exercise 5.3, we find

$$\begin{aligned}
\rho_{VV} &= \mathrm{pr}_H;\\
\rho_{VV} &= \mathrm{pr}_V;\\
\rho_{HV}+\rho_{VH} &= 2\mathrm{pr}_+ - \mathrm{pr}_H - \mathrm{pr}_V = 2\mathrm{pr}_+ - 1;\\
\rho_{HV}-\rho_{VH} &= -i(2\mathrm{pr}_R - \mathrm{pr}_H - \mathrm{pr}_V) = -i(2\mathrm{pr}_R - 1),
\end{aligned}$$

where we used $\mathrm{pr}_H + \mathrm{pr}_V = 1$. The latter two equations give

$$\begin{aligned}
\rho_{HV} &= \mathrm{pr}_+ - i\mathrm{pr}_R + \frac{-1+i}{2};\\
\rho_{VH} &= \mathrm{pr}_+ + i\mathrm{pr}_R + \frac{-1-i}{2}.
\end{aligned}$$

Solution to Exercise 5.78. We look for the two photons' density matrix in the canonical basis

$$\hat{\rho} \simeq \begin{pmatrix} \rho_{HHHH} & \rho_{HHHV} & \rho_{HHVH} & \rho_{HHVV} \\ \rho_{HVHH} & \rho_{HVHV} & \rho_{HVVH} & \rho_{HVVV} \\ \rho_{VHHH} & \rho_{VHHV} & \rho_{VHVH} & \rho_{VHVV} \\ \rho_{VVHH} & \rho_{VVHV} & \rho_{VVVH} & \rho_{VVVV} \end{pmatrix},$$

where e.g. $\rho_{HHHV} = \langle HH|\,\hat{\rho}\,|HV\rangle$.

- Measurements in Alice's and Bob's canonical bases yield the diagonal elements

$$\begin{aligned}
\mathrm{pr}_{HH} &= \langle HH|\,\hat{\rho}\,|\,HH\rangle = \rho_{HHHH};\\
\mathrm{pr}_{HV} &= \langle HV|\,\hat{\rho}\,|\,HV\rangle = \rho_{HVHV};\\
\mathrm{pr}_{VH} &= \langle VH|\,\hat{\rho}\,|\,VH\rangle = \rho_{VHVH};\\
\mathrm{pr}_{VV} &= \langle VV|\,\hat{\rho}\,|\,VV\rangle = \rho_{VVVV}.
\end{aligned}$$

- Measurements in which Alice's basis is canonical and Bob's diagonal and circular yield

$$\begin{aligned}
\mathrm{pr}_{H+} &= \frac{1}{2}\left[\langle H|\otimes(\langle H|+\langle V|)\right]\hat{\rho}\left[|H\rangle\otimes(|H\rangle+|V\rangle)\right]\\
&= \frac{1}{2}(\rho_{HHHH}+\rho_{HHHV}+\rho_{HVHH}+\rho_{HVHV});\\
\mathrm{pr}_{HR} &= \frac{1}{2}\left[\langle H|\otimes(\langle H|-i\langle V|)\right]\hat{\rho}\left[|H\rangle\otimes(|H\rangle+i|V\rangle)\right]\\
&= \frac{1}{2}(\rho_{HHHH}+i\rho_{HHHV}-i\rho_{HVHH}+\rho_{HVHV}),
\end{aligned}$$

  from which, using the existing knowledge of ρ_{HHHH} and ρ_{HVHV}, we find $\rho_{HHHV}\pm\rho_{HVHH}$ and, subsequently, both ρ_{HHHV} and ρ_{HVHH}. Similarly, from pr_{V+} and pr_{VR} we find ρ_{VVVH} and ρ_{VHVV}.
- Measurements in which Bob's basis is canonical and Alice's diagonal and circular yield, by the same token, ρ_{HHVH}, ρ_{VHHH}, ρ_{VVHV} and ρ_{HVVV}.
- The matrix elements that remain to be found are ρ_{HHVV}, ρ_{VVHH}, ρ_{HVVH} and ρ_{VHHV}. We can calculate them by from measurements in which Alice's and Bob's bases are diagonal and circular. In particular,

$$\begin{aligned}
\mathrm{pr}_{++} &= \frac{1}{2}\left[(\langle H|+\langle V|)\otimes(\langle H|+\langle V|)\right]\hat{\rho}\left[(|H\rangle+|V\rangle)\otimes(|H\rangle+|V\rangle)\right]\\
&= \frac{1}{4}(\ldots+\rho_{HHVV}+\rho_{VVHH}+\rho_{HVVH}+\rho_{VHHV}+\ldots);\\
\mathrm{pr}_{+R} &= \frac{1}{2}\left[(\langle H|+\langle V|)\otimes(\langle H|-i\langle V|)\right]\hat{\rho}\left[(|H\rangle+|V\rangle)\otimes(|H\rangle+i|V\rangle)\right]\\
&= \frac{1}{4}(\ldots+i\rho_{HHVV}-i\rho_{VVHH}-i\rho_{HVVH}+i\rho_{VHHV}+\ldots);\\
\mathrm{pr}_{R+} &= \frac{1}{2}\left[(\langle H|-i\langle V|)\otimes(\langle H|+\langle V|)\right]\hat{\rho}\left[(|H\rangle+i|V\rangle)\otimes(|H\rangle+|V\rangle)\right]\\
&= \frac{1}{4}(\ldots+i\rho_{HHVV}-i\rho_{VVHH}+i\rho_{HVVH}-i\rho_{VHHV}+\ldots);\\
\mathrm{pr}_{RR} &= \frac{1}{2}\left[(\langle H|-i\langle V|)\otimes(\langle H|-i\langle V|)\right]\hat{\rho}\left[(|H\rangle+i|V\rangle)\otimes(|H\rangle+i|V\rangle)\right]\\
&= \frac{1}{4}(\ldots-\rho_{HHVV}-\rho_{VVHH}+\rho_{HVVH}+\rho_{VHHV}+\ldots),
\end{aligned}$$

  where the ellipses stand for those density matrix elements that are known from previous steps. The above four equations can be readily solved to find the four remaining unknown matrix elements.

Solution to Exercise 5.79. As found in Ex. 5.23(b), $E(\hat{\rho}) = \hat{U}\hat{\rho}\hat{U}^\dagger$. The matrix of $E(\hat{\rho})$ in basis $\{|v_i\rangle\}$ is therefore simply the product of matrices $\hat{U}$, $\hat{\rho}$ and $\hat{U}^\dagger$. The matrix of $\hat{U}$ is known because we know the state $\hat{U}\left|v_j\right\rangle$, i.e. the matrix element $\left\langle v_i\right|\hat{U}\left|v_j\right\rangle$ for all i and j.

Solution to Exercise 5.80. As we found in Ex. 5.22(a), state $\alpha\hat{\rho}_1 + \beta\hat{\rho}_2$ is equivalent, in all its physical properties, to the ensemble in which state $\hat{\rho}_1$ occurs with probability α, and state $\hat{\rho}_2$ with probability β. So we can, without loss of generality, assume that it is the ensemble entering the "black box". After passing through the "black box", states $\hat{\rho}_{1,2}$ produce states $\mathbf{E}(\hat{\rho}_{1,2})$, respectively. Therefore, the output has an ensemble in which the state $\mathbf{E}(\hat{\rho}_1)$ occurs with probability α, and state $\mathbf{E}(\hat{\rho}_2)$ with probability β. The density operator of this ensemble is $\alpha\mathbf{E}(\hat{\rho}_1) + \beta\mathbf{E}(\hat{\rho}_2)$.

Solution to Exercise 5.81. By construction, each element in Q (the set defined in the Hint to this Exercise) corresponds to a physical state. The number of elements in Q is N^2. According to Ex. A.7, all that needs to be proven to demonstrate that Q is a basis is that it forms a spanning set in the space of linear operators.

To this end, we express the operator $|v_k\rangle\langle v_l|$, for any k and l, through the elements of Q. For $k = l$, this expression is trivial: $|v_k\rangle\langle v_l| = \rho_{kk}$. For $k \neq l$ we write

$$\hat{\rho}_{\text{re},kl} = \frac{1}{2}(\hat{\rho}_{kk} + \hat{\rho}_{ll} + |v_k\rangle\langle v_l| + |v_l\rangle\langle v_k|);$$
$$\hat{\rho}_{\text{im},kl} = \frac{1}{2}(\hat{\rho}_{kk} + \hat{\rho}_{ll} - i\,|v_k\rangle\langle v_l| + i\,|v_l\rangle\langle v_k|),$$

from which it follows that

$$|v_k\rangle\langle v_l| = \hat{\rho}_{\text{re},kl} + i\hat{\rho}_{\text{im},kl} - \frac{1+i}{2}(\hat{\rho}_{kk} + \hat{\rho}_{ll});$$
$$|v_l\rangle\langle v_k| = \hat{\rho}_{\text{re},kl} - i\hat{\rho}_{\text{im},kl} - \frac{1-i}{2}(\hat{\rho}_{kk} + \hat{\rho}_{ll}).$$

Because the set $\{|v_k\rangle\langle v_l|\}$ forms a basis in the space of linear operators (see Ex. A.42), so does Q.

Solution to Exercise 5.82. The statement of this exercise is a direct generalization of Ex. 5.80.

Solution to Exercise 5.83. Because the linear space of 2×2 matrices is 4-dimensional, and

$$Q = \left\{\hat{\rho}_\uparrow \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \hat{\rho}_\downarrow \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \hat{\rho}_+ \simeq \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \hat{\rho}_R \simeq \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\right\} \tag{S5.26}$$

has four elements, it suffices to check that Q is a spanning set (Ex. A.7). Decomposing an arbitrary matrix $\hat{\rho} \simeq \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}$ into the basis Q means finding the coefficients of the decomposition

$$\hat{\rho} = \lambda_\uparrow\hat{\rho}_\uparrow + \lambda_\downarrow\hat{\rho}_\downarrow + \lambda_+\hat{\rho}_+ + \lambda_R\hat{\rho}_R,$$

which we can rewrite in the matrix form as

$$\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \lambda_\uparrow + \frac{1}{2}(\lambda_+ + \lambda_R) & \frac{1}{2}(\lambda_+ - i\lambda_R) \\ \frac{1}{2}(\lambda_+ + i\lambda_R) & \lambda_\downarrow + \frac{1}{2}(\lambda_+ + \lambda_R) \end{pmatrix}.$$

Solving the above equation for the λ's, we find

$$\begin{aligned}
\lambda_+ &= \rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow};\\
\lambda_R &= i(\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow});\\
\lambda_\uparrow &= \rho_{\uparrow\uparrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow};\\
\lambda_\downarrow &= \rho_{\downarrow\downarrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow}.
\end{aligned} \tag{S5.27}$$

or

$$\begin{aligned}
\hat\rho \simeq &\left(\rho_{\uparrow\uparrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow}\right)\hat\rho_\uparrow \qquad \text{(S5.28)}\\
&+\left(\rho_{\downarrow\downarrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow}\right)\hat\rho_\downarrow\\
&+(\rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow})\hat\rho_+ + i(\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow})\hat\rho_R.
\end{aligned}$$

We see that a decomposition into the elements of Q exists for every $\hat\rho$, so Q is indeed a spanning set.

Solution to Exercise 5.84.

a)

$$\mathbf{E}(\hat\rho_\uparrow) \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \tag{S5.29a}$$

$$\mathbf{E}(\hat\rho_\downarrow) \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \tag{S5.29b}$$

$$\mathbf{E}(\hat\rho_+) \simeq \frac{1}{2}\begin{pmatrix} 1 & e^{-t/T_2} \\ e^{-t/T_2} & 1 \end{pmatrix}; \tag{S5.29c}$$

$$\mathbf{E}(\hat\rho_R) \simeq \frac{1}{2}\begin{pmatrix} 1 & -ie^{-t/T_2} \\ ie^{-t/T_2} & 1 \end{pmatrix}. \tag{S5.29d}$$

b) Using decomposition (S5.28), we find

$$\begin{aligned}
\mathbf{E}(\hat\rho) \simeq &\left(\rho_{\uparrow\uparrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow}\right)\mathbf{E}(\hat\rho_\uparrow) + \left(\rho_{\downarrow\downarrow} - \frac{1+i}{2}\rho_{\uparrow\downarrow} - \frac{1-i}{2}\rho_{\downarrow\uparrow}\right)\mathbf{E}(\hat\rho_\downarrow)\\
&+(\rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow})\mathbf{E}(\hat\rho_+) + i(\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow})\mathbf{E}(\hat\rho_R).
\end{aligned}$$

Substituting the equations from part (a), we obtain Eq. (5.45).

Solution to Exercise 5.85. Any density operator is written in the basis $\{|v_n\rangle\}$ as

$$\hat\rho = \sum_{n,m} \langle v_n|\,\hat\rho\,|v_m\rangle\,|v_n\rangle\langle v_m|\,.$$

Substituting Eq. (5.47) into this decomposition, we have

$$\hat{\rho} = \sum_{n,m,i} \lambda_{nmi} \langle v_n | \hat{\rho} | v_m \rangle \hat{\rho}_i \tag{S5.30}$$

so that

$$\mathbf{E}(\hat{\rho}) = \sum_{n,m,i} \lambda_{mni} \langle v_n | \hat{\rho} | v_m \rangle \mathbf{E}(\hat{\rho}_i)$$

and hence

$$\begin{aligned}\langle v_l | \mathbf{E}(\hat{\rho}) | v_k \rangle &= \sum_{n,m,i} \lambda_{nmi} \langle v_l | \mathbf{E}(\hat{\rho}_i) | v_k \rangle \langle v_n | \hat{\rho} | v_m \rangle \\ &= \sum_{n,m} \left[\sum_i \lambda_{nmi} \langle v_l | \mathbf{E}(\hat{\rho}_i) | v_k \rangle \right] \langle v_n | \hat{\rho} | v_m \rangle\end{aligned}$$

(the summation over m and n runs from 1 to N while the summation over i from 1 to N^2). Comparing the above equation with Eq. (5.46), we see that the expression in the brackets is $\mathbf{E}_{lk}^{nm}$.

Solution to Exercise 5.86. Using decomposition (S5.28), we have

$$\begin{aligned}|\uparrow\rangle\langle\uparrow| &\simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \simeq \hat{\rho}_\uparrow; \\ |\downarrow\rangle\langle\downarrow| &\simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \simeq \hat{\rho}_\downarrow; \\ |\uparrow\rangle\langle\downarrow| &\simeq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \simeq \hat{\rho}_+ + i\hat{\rho}_R + \frac{-1-i}{2}\hat{\rho}_\uparrow + \frac{-1-i}{2}\hat{\rho}_\downarrow; \\ |\downarrow\rangle\langle\uparrow| &\simeq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \simeq \hat{\rho}_+ - i\hat{\rho}_R + \frac{-1+i}{2}\hat{\rho}_\uparrow + \frac{-1+i}{2}\hat{\rho}_\downarrow.\end{aligned}$$

Therefore

$$\begin{array}{cccc} \lambda_{\uparrow\uparrow\uparrow} = 1; & \lambda_{\uparrow\uparrow\downarrow} = 0; & \lambda_{\uparrow\uparrow+} = 0; & \lambda_{\uparrow\uparrow R} = 0; \\ \lambda_{\downarrow\downarrow\uparrow} = 0; & \lambda_{\downarrow\downarrow\downarrow} = 1; & \lambda_{\downarrow\downarrow+} = 0; & \lambda_{\downarrow\downarrow R} = 0; \\ \lambda_{\uparrow\downarrow\uparrow} = \frac{-1-i}{2}; & \lambda_{\uparrow\downarrow\downarrow} = \frac{-1-i}{2}; & \lambda_{\uparrow\downarrow+} = 1; & \lambda_{\uparrow\downarrow R} = i; \\ \lambda_{\downarrow\uparrow\uparrow} = \frac{-1+i}{2}; & \lambda_{\downarrow\uparrow\downarrow} = \frac{-1+i}{2}; & \lambda_{\downarrow\uparrow+} = 1; & \lambda_{\downarrow\uparrow R} = -i. \end{array} \tag{S5.31}$$

Solution to Exercise 5.87. We can think of the process tensor (5.48) as a set of matrices $\mathbf{E}^{nm}$ (where $n,m \in \{1,\ldots,N\}$) each of which is given by

$$\mathbf{E}^{nm} = \sum_{i=1}^{N^2} \lambda_{nmi} \mathbf{E}(\hat{\rho}_i). \tag{S5.32}$$

Using Eqs. (S5.29) and (S5.31), we find

$$\mathbf{E}^{\uparrow\uparrow} = \lambda_{\uparrow\uparrow\uparrow}\mathbf{E}(\hat\rho_\uparrow) + \lambda_{\uparrow\uparrow\downarrow}\mathbf{E}(\hat\rho_\downarrow) + \lambda_{\uparrow\uparrow+}\mathbf{E}(\hat\rho_+) + \lambda_{\uparrow\uparrow R}\mathbf{E}(\hat\rho_R) \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$\mathbf{E}^{\downarrow\downarrow} = \lambda_{\downarrow\downarrow\uparrow}\mathbf{E}(\hat\rho_\uparrow) + \lambda_{\downarrow\downarrow\downarrow}\mathbf{E}(\hat\rho_\downarrow) + \lambda_{\downarrow\downarrow+}\mathbf{E}(\hat\rho_+) + \lambda_{\downarrow\downarrow R}\mathbf{E}(\hat\rho_R) \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\begin{aligned}\mathbf{E}^{\uparrow\downarrow} &= \lambda_{\uparrow\downarrow\uparrow}\mathbf{E}(\hat\rho_\uparrow) + \lambda_{\uparrow\downarrow\downarrow}\mathbf{E}(\hat\rho_\downarrow) + \lambda_{\uparrow\downarrow+}\mathbf{E}(\hat\rho_+) + \lambda_{\uparrow\downarrow R}\mathbf{E}(\hat\rho_R) \\ &\simeq \frac{-1-i}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{-1-i}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & e^{-t/T_2} \\ e^{-t/T_2} & 1 \end{pmatrix} + i\frac{1}{2}\begin{pmatrix} 1 & -ie^{-t/T_2} \\ ie^{-t/T_2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{-t/T_2} \\ 0 & 0 \end{pmatrix};\end{aligned}$$

$$\begin{aligned}\mathbf{E}^{\downarrow\uparrow} &= \lambda_{\downarrow\uparrow\uparrow}\mathbf{E}(\hat\rho_\uparrow) + \lambda_{\downarrow\uparrow\downarrow}\mathbf{E}(\hat\rho_\downarrow) + \lambda_{\downarrow\uparrow+}\mathbf{E}(\hat\rho_+) + \lambda_{\downarrow\uparrow R}\mathbf{E}(\hat\rho_R) \\ &\simeq \frac{-1+i}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{-1+i}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & e^{-t/T_2} \\ e^{-t/T_2} & 1 \end{pmatrix} - i\frac{1}{2}\begin{pmatrix} 1 & -ie^{-t/T_2} \\ ie^{-t/T_2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ e^{-t/T_2} & 0 \end{pmatrix}.\end{aligned}$$

Solution to Exercise 5.88. Following in the footsteps of Ex. 5.80, we assume state $\alpha\hat\rho_1 + \beta\hat\rho_2$ to represent the ensemble in which state $\hat\rho_1$ occurs with probability α, and state $\hat\rho_2$ with probability β. Then, using conditional probabilities [Eq. (B.6)], we can write the probability for the detector to display output state j after the measurement as follows:

$$\mathrm{pr}_j = \mathrm{pr}_{\hat\rho_1}\mathrm{pr}_{j|\hat\rho_1} + \mathrm{pr}_{\hat\rho_2}\mathrm{pr}_{j|\hat\rho_2} = \alpha\,\mathrm{pr}_j(\hat\rho_1) + \beta\,\mathrm{pr}_j(\hat\rho_2).$$

Solution to Exercise 5.89. Using the result of the previous exercise, and since $\hat\rho = \sum\limits_i \lambda_i\hat\rho_i$, we find that

$$\mathrm{pr}_j(\hat\rho) = \sum_i \lambda_i \mathrm{pr}_j(\hat\rho_i).$$

Solution to Exercise 5.90. Using decomposition (S5.30), which applies to the present case, and the result of the previous exercise, we find that

$$\mathrm{pr}_j(\hat\rho) = \sum_{n,m}\left[\sum_i \lambda_{nmi}\mathrm{pr}_j(\hat\rho_i)\right]\langle v_n|\,\hat\rho\,|v_m\rangle\,. \tag{S5.33}$$

On the other hand, Eq. (5.39) can be rewritten in the form

$$\mathrm{pr}_j(\hat\rho) = \sum_{n,m}[F_j]_{mn}\langle v_n|\,\hat\rho\,|v_m\rangle\,.$$

Comparing these two equations, we see that the expression in the brackets in Eq. (S5.33) is in fact the matrix of the j'th POVM element, i.e.

$$[F_j]_{mn} = \sum_i \lambda_{nmi} \mathrm{pr}_j(\hat{\rho}_i). \tag{S5.34}$$

Solution to Exercise 5.91.

a) We calculate the probability $\mathrm{pr}_j(\hat{\rho}_i)$ of the jth detector output for all $\hat{\rho}_i \in Q$ and $j \in \{1,2\}$ using the result of Ex. 5.73. We find

$$\mathrm{pr}_1(\hat{\rho}_H) = \frac{3}{4};\ \mathrm{pr}_2(\hat{\rho}_H) = \frac{1}{4};$$
$$\mathrm{pr}_1(\hat{\rho}_V) = \frac{1}{4};\ \mathrm{pr}_2(\hat{\rho}_V) = \frac{3}{4};$$
$$\mathrm{pr}_1(\hat{\rho}_+) = \frac{3}{4};\ \mathrm{pr}_2(\hat{\rho}_+) = \frac{1}{4};$$
$$\mathrm{pr}_1(\hat{\rho}_R) = \frac{1}{2};\ \mathrm{pr}_2(\hat{\rho}_R) = \frac{1}{2}.$$

b) We notice that our set of probe states is the same as (5.44), except that now we work with a photon polarization qubit rather than the spin qubit. Hence we can use the decomposition (5.47) with the coefficients given by Eq. (S5.31) (with states $|\uparrow\rangle$ and $|\downarrow\rangle$ replaced by $|H\rangle$ and $|V\rangle$, respectively). Now, using the result of Ex. refPOVMgenEx, we obtain

$$\begin{aligned}
\hat{F}_1 = {} & |H\rangle\langle H| \left(\mathrm{pr}_1(\hat{\rho}_H)\lambda_{HHH} + \mathrm{pr}_1(\hat{\rho}_V)\lambda_{HHV} + \mathrm{pr}_1(\hat{\rho}_+)\lambda_{HH+} + \mathrm{pr}_1(\hat{\rho}_R)\lambda_{HHR}\right) \\
& + |V\rangle\langle V| \left(\mathrm{pr}_1(\hat{\rho}_H)\lambda_{VVH} + \mathrm{pr}_1(\hat{\rho}_V)\lambda_{VVV} + \mathrm{pr}_1(\hat{\rho}_+)\lambda_{VV+} + \mathrm{pr}_1(\hat{\rho}_R)\lambda_{VVR}\right) \\
& + |H\rangle\langle V| \left(\mathrm{pr}_1(\hat{\rho}_H)\lambda_{VHH} + \mathrm{pr}_1(\hat{\rho}_V)\lambda_{VHV} + \mathrm{pr}_1(\hat{\rho}_+)\lambda_{VH+} + \mathrm{pr}_1(\hat{\rho}_R)\lambda_{VHR}\right) \\
& + |V\rangle\langle H| \left(\mathrm{pr}_1(\hat{\rho}_H)\lambda_{HVH} + \mathrm{pr}_1(\hat{\rho}_V)\lambda_{HVV} + \mathrm{pr}_1(\hat{\rho}_+)\lambda_{HV+} + \mathrm{pr}_1(\hat{\rho}_R)\lambda_{HVR}\right) \\
= {} & |H\rangle\langle H| \frac{3}{4} + |V\rangle\langle V| \frac{1}{4} \\
& + |H\rangle\langle V| \left(\frac{3}{4} \times \frac{-1+i}{2} + \frac{1}{4} \times \frac{-1+i}{2} + \frac{3}{4} - \frac{1}{2} i\right) \\
& + |H\rangle\langle V| \left(\frac{3}{4} \times \frac{-1-i}{2} + \frac{1}{4} \times \frac{-1-i}{2} + \frac{3}{4} + \frac{1}{2} i\right) \\
\simeq {} & \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}.
\end{aligned}$$

Similarly,

$$\hat{F}_2 \simeq \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}.$$