Appendix D
Dirac delta function and the Fourier transformation

D.1 Dirac delta function

The delta function can be visualized as a Gaussian function (B.15) of infinitely narrow width b (Fig. B.5):

$$G_b(x) = \frac{1}{b\sqrt{\pi}} \mathrm{e}^{-x^2/b^2} \to \delta(x) \qquad \text{for} \qquad b \to 0. \tag{D.1}$$

The delta function is used in mathematics and physics to describe density distributions of infinitely small (*singular*) objects. For example, the position-dependent density of a one-dimensional particle of mass m located at $x = a$, can be written as $m\delta(x-a)$. Similarly, the probability density of a continuous "random variable" that takes on a certain value $x = a$ is $\delta(x-a)$. In quantum mechanics, we use $\delta(x)$, for example, to write the wave function of a particle that has a well-defined position.

The notion of function in mathematics refers to a map that relates a number, x, to another number, $f(x)$. The delta function is hence not a function in the traditional sense: it maps all $x \neq 0$ to zero, but $x = 0$ to infinity, which is not a number. It belongs to the class of so-called *generalized functions*. A rigorous mathematical theory of generalized functions can be found in most mathematical physics textbooks. Here, we discuss only those properties of the delta function that are useful for physicists.

Exercise D.1. Show that, for any smooth[1], bounded function $f(x)$,

$$\lim_{b\to 0} \int\limits_{-\infty}^{+\infty} G_b(x) f(x) \mathrm{d}x = f(0). \tag{D.2}$$

From Eqs. (D.1) and (D.2) and for any smooth function $f(x)$, we obtain

[1] A *smooth* function is one that has derivatives of all finite orders.

A. I. Lvovsky, *Quantum Physics*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-3-662-56584-1

$$\int_{-\infty}^{+\infty} \delta(x) f(x) \mathrm{d}x = f(0) \tag{D.3}$$

This property is extremely important because it allows one to perform meaningful calculations with the delta function in spite of its singular nature. Although the delta function does not have a numerical value for all values of its argument, the integral of the delta function multiplied by another function does. We may write a delta function outside of an integral, but we always keep in mind that it will eventually become a part of an integral, and only then will it produce a quantitative value — for example, a prediction of an experimental result.

In fact, Eq. (D.3) can be viewed as a rigorous mathematical definition of the delta function. Using this definition, we can obtain its other primary properties.

Exercise D.2. Show that

a)

$$\int_{-\infty}^{+\infty} \delta(x) \mathrm{d}x = 1; \tag{D.4}$$

b) for any function $f(x)$,

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x) \mathrm{d}x = f(a); \tag{D.5}$$

c) for any real number a,

$$\delta(ax) = \delta(x)/|a|. \tag{D.6}$$

Exercise D.3. For the Heaviside step function

$$\theta(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}, \tag{D.7}$$

show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \theta(x) = \delta(x). \tag{D.8}$$

Hint: use Eq. (D.3).

Exercise D.4. Show that, for any $c < 0$ and $d > 0$,

$$\int_{c}^{d} \delta(x) \mathrm{d}x = 1 \tag{D.9}$$

D.2 Fourier transformation

Definition D.1. The *Fourier transform* $\tilde{f}(k) \equiv \mathscr{F}[f](k)$ of a function $f(x)$ is a function of the parameter k defined as follows:[2]

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\mathrm{d}x. \tag{D.10}$$

This is an important integral transformation used in all branches of physics. Suppose, for example, that you have a light wave of the form $f(\omega)\mathrm{e}^{-\mathrm{i}\omega t}$, where ω is the frequency and $f(\omega)$ is the complex amplitude, or the *frequency spectrum* of the signal. Then the time-dependent signal from all sources is $\int_{-\infty}^{+\infty} f(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega$ — that is, the Fourier transform of the spectrum. The power density of the spectrum, i.e., the function $|f(\omega)|^2$, can be measured experimentally by means of a dispersive optical element, such as a prism.

Exercise D.5. Show that, if $\tilde{f}(k) = \mathscr{F}[f(x)]$ exists, then

a)
$$\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\mathrm{d}x; \tag{D.11}$$

b) for a real $f(x)$, $\tilde{f}(-k) = \tilde{f}^*(k)$;

c) for $a \neq 0$,
$$\mathscr{F}[f(ax)] = \frac{1}{|a|}\tilde{f}(k/a); \tag{D.12}$$

d)
$$\mathscr{F}[f(x-a)] = \mathrm{e}^{-\mathrm{i}ka}\tilde{f}(k); \tag{D.13}$$

e)
$$\mathscr{F}[\mathrm{e}^{\mathrm{i}\xi x} f(x)] = \tilde{f}(k-\xi); \tag{D.14}$$

f) assuming that $f(x)$ is a smooth function approaching zero at $\pm\infty$,
$$\mathscr{F}[\mathrm{d}f(x)/\mathrm{d}x] = \mathrm{i}k\tilde{f}(k). \tag{D.15}$$

Exercise D.6. Show that the Fourier transform of a Gaussian function is also a Gaussian function:
$$\mathscr{F}[\mathrm{e}^{-x^2/b^2}] = \frac{b}{\sqrt{2}}\mathrm{e}^{-k^2b^2/4}. \tag{D.16}$$

We see from Eq. (D.12) that scaling the argument x of a function results in inverse scaling of the argument k of its Fourier transform. In particular (Ex. D.6), a

[2] There is no common convention as to whether to place the negative sign in the complex exponent of Eqs. (D.10) or (D.21), nor how to distribute the factor of $1/2\pi$ between them. Here I have chosen the convention arbitrarily.

signal with a Gaussian spectrum of width b is a Gaussian pulse of width $2/b$, so the product of the two widths is a constant. This is a manifestation of the *time-frequency uncertainty* that applies to a wide range of wave phenomena in classical physics. In fact, as we see in Sec. 3.3.2, in its application to the position and momentum observables, the Heisenberg uncertainty principle can also be interpreted in this fashion.

Let us now consider two extreme cases of the Fourier transform of Gaussian functions.

Exercise D.7. Show that:

a) in the limit $b \to 0$, Eq. (D.16) takes the form

$$\mathscr{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}; \tag{D.17}$$

b) in the opposite limit, $b \to \infty$, one obtains

$$\mathscr{F}[1] = \sqrt{2\pi}\,\delta(k). \tag{D.18}$$

If the spectrum contains only the zero frequency, the signal, not surprisingly, is time-independent. If, on the other hand, the spectrum is constant, the signal is an instant "flash" occurring at $t = 0$. Here is an interesting consequence of this observation.

Exercise D.8. Show that, for $a \neq 0$,

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}akx}\mathrm{d}x = 2\pi\delta(k)/|a|. \tag{D.19}$$

This result is of paramount importance for many calculations involving the Fourier transform. We will see its utility shortly.

Exercise D.9. Assuming a and b to be real and positive, find the Fourier transforms of the following:

a) $\delta(x+a)+\delta(x-a)$
b) $\cos(ax+b)$;
c) $\mathrm{e}^{-ax^2}\cos bx$;
d) $\mathrm{e}^{-a(x+b)^2}+\mathrm{e}^{-a(x-b)^2}$;
e) $\theta(x)\mathrm{e}^{-ax}$, where $\theta(x)$ is the Heaviside function;
f) a "top-hat function" $\begin{cases} 0 \text{ if } x < -a \text{ or } x > a; \\ A \text{ if } -a \leq x \leq a; \end{cases}$.

The Fourier transform can be inverted: for any given time-dependent pulse one can calculate its frequency spectrum such that the pulse is the Fourier transform of that spectrum. Remarkably, the Fourier transform is very similar to its inverse. This similarity can be observed, for example, by comparing Eqs. (D.13) and (D.14). Displacing the argument of $f(x)$ leads to the multiplication of $\tilde{f}(k)$ by a complex

Box D.1 Interpreting Eq. (D.8)

The result (D.8) seems to tell us that the integral $\int_{-\infty}^{+\infty} e^{ikx}dx$ equals zero for $k \neq 0$. This does not reconcile with traditional calculus, according to which the integral of a finite oscillating function e^{ikx} must diverge for any k. To address this apparent inconsistency, we need to remember that Eq. (D.19) is valid only as a generalized function — that is, as a part of the integral (D.3). Indeed, if we substitute Eq. (D.19) into Eq. (D.3), we obtain a convergent integral

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} e^{ikx}dx\right] f(k)dk = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} e^{ikx} f(k)dk\right] dx = \sqrt{2\pi}\int_{-\infty}^{+\infty} \mathscr{F}[f](-k)dk \stackrel{(D.11)}{=} 2\pi f(0). \tag{D.20}$$

Therefore, while the numerical value of the integral (D.19) for any specific k does not exist, it is meaningfully defined as a generalized function of k.

phase. On the other hand, if we multiply $f(x)$ by a complex phase, the argument of $\tilde{f}(k)$ gets shifted.

Definition D.2. The *inverse Fourier transform* $\mathscr{F}^{-1}[g](x)$ of the function $g(k)$ is a function of the parameter x such that

$$\mathscr{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx} g(k)dk. \tag{D.21}$$

Exercise D.10. Show that

$$\mathscr{F}^{-1}[\mathscr{F}[f]](x) = f(x). \tag{D.22}$$

Exercise D.11. Show that

$$\mathscr{F}^{-1}[f(x)](k) = \mathscr{F}[f(x)](-k) = \mathscr{F}[f(-x)](k). \tag{D.23}$$

Exercise D.12.§ Derive the analogues of the rules found in Ex. D.5 for the inverse Fourier transform.
Answer: Denoting $\breve{g}(x) = \mathscr{F}^{-1}[g(k)]$,

a)

$$\breve{g}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(k)dk; \tag{D.24}$$

b) for a real $g(k)$, $\breve{g}(x) = \breve{g}^*(-x)$;
c)

$$\mathscr{F}^{-1}[g(ak)](x) = \frac{1}{|a|}\breve{g}(k/a); \tag{D.25}$$

d)
$$\mathscr{F}^{-1}[g(k-a)](x) = \mathrm{e}^{\mathrm{i}xa}\breve{g}(k); \tag{D.26}$$

e)
$$\mathscr{F}^{-1}[\mathrm{e}^{\mathrm{i}\xi k}g(k)](x) = \breve{g}(x+\xi); \tag{D.27}$$

f)
$$\mathscr{F}^{-1}[dg(k)/dk] = -ix\breve{g}(x). \tag{D.28}$$