Appendix C
Optical polarization tutorial

C.1 Polarization of light

Consider a classical electromagnetic plane wave propagating along the (horizontal) z-axis with angular frequency ω and wavenumber $k = \omega/c$, where c is the speed of light. The electromagnetic wave is transverse, so its electric field vector lies in the x-y plane:

$$\vec{E}(z,t) = A_H \hat{\imath} \cos(kz - \omega t + \varphi_H) + A_V \hat{\jmath} \cos(kz - \omega t + \varphi_V), \tag{C.1}$$

or in the complex form

$$\vec{E}(z,t) = \mathrm{Re}[(A_H e^{i\varphi_H} \hat{\imath} + A_V e^{i\varphi_V} \hat{\jmath}) e^{ikz - i\omega t}]. \tag{C.2}$$

Here $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along the x and y axes, respectively; A_H and A_V are the real amplitudes of the x and y components (which we will refer to as *horizontal* and *vertical*), and φ_H and φ_V are their phases.

Exercise C.1.§ Show that Eqs. (C.1) and (C.2) are equivalent.

The intensity of light in each polarization is proportional to:

$$I_H \propto A_H^2; \tag{C.3a}$$
$$I_V \propto A_V^2. \tag{C.3b}$$

The total intensity of the wave is the sum of its two components: $I_{\text{total}} \propto A_H^2 + A_V^2$.

Let us study the behavior of the electric field vector at some specific point in space, say $z = 0$. If the two components of the field have different phases, $\vec{E}(z,t)$ will change its direction as a function of time, as illustrated in Fig. C.1. To understand this interesting phenomenon better, try the following exercise.

Exercise C.2. Plot, as a function of time, the horizontal and vertical components of $\vec{E}(0,t)$ for $0 \le \omega t \le 2\pi$, in the following cases:

a) $A_H = 1$ V/m, $A_V = 0$, $\varphi_H = \varphi_V = 0$;

A. I. Lvovsky, *Quantum Physics*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-3-662-56584-1

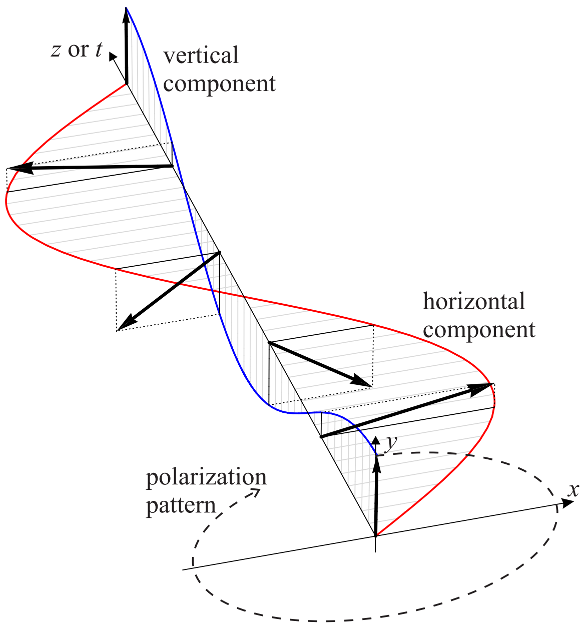

Fig. C.1 Polarization pattern of a plane wave. When the vertical and horizontal components of the electric field vector oscillate with different phases, the direction of that vector (shown with thick arrows) does not remain constant in phase and time. The trajectory of the tip of that vector determines the polarization pattern.

b) $A_H = 5$ V/m, $A_V = -3$ V/m, $\varphi_H = \varphi_V = 0$;
c) $A_H = 5$ V/m, $A_V = -3$ V/m, $\varphi_H = \pi/2$, $\varphi_V = 0$;
d) $A_H = 5$ V/m, $A_V = -3$ V/m, $\varphi_H = \pi/4$, $\varphi_V = -\pi/4$;
e) $A_H = 5$ V/m, $A_V = -3$ V/m, $\varphi_H = 0$, $\varphi_V = \pi/6$.

For each of the above cases, plot the trajectory of the point (E_x, E_y) for a constant z as a function of time.

The field vector trajectory defines the so-called *polarization state (pattern)* of light. The polarization state is one of the primary parameters of an electromagnetic wave; it determines how this field interferes with other waves or interacts with matter. Importantly, the polarization pattern is conserved as the wave is propagating through space and time, with the exception of certain materials which we will study a bit later.

Exercise C.3. Show that the polarization pattern of a plane wave is the same for all values of z.

This can be restated more generally: adding an arbitrary shift to both phases φ_H and φ_V will not change the polarization pattern. One can say that the pattern depends not on the individual phases of its two components, but on their difference $\varphi_H - \varphi_V$

[see Ex. C.2(c,d) for an example]. This property of classical polarization patterns has a direct counterpart in the quantum world: applying an overall phase shift to a quantum state does not change its physical properties (see Sec. 1.3 for a more detailed discussion).

In general, the polarization pattern is elliptical; however, as we have seen above, there exist special cases when the ellipse collapses into a straight line or blows out into a circle. Let us look at these cases more carefully.

Exercise C.4. Show the following:

a) The polarization pattern is linear if and only if $\varphi_H = \varphi_V + m\pi$, where m is an integer, or $A_H = 0$ or $A_V = 0$. The angle θ of the field vector with respect to the x axis is given by $\tan\theta = A_V/A_H$.
b) The polarization pattern is circular if and only if $\varphi_H = \varphi_V \frac{\pi}{2} + m\pi$, where m is an integer, and $A_H = \pm A_V$.

Important specific cases of linear polarization are horizontal ($A_V = 0$), vertical ($A_H = 0$), and $\pm 45°$ ($A_V = \pm A_H$). For circular polarization, one can distinguish two cases according to the helicity of the wave: right and left circular.

- For right circular polarization, $A_V = A_H$ and $\varphi_V = \varphi_H + \frac{\pi}{2} + 2\pi m$ or $A_V = -A_H$ and $\varphi_V = \varphi_H - \frac{\pi}{2} + 2\pi m$, where m is an integer.
- For left circular polarization, $A_V = A_H$ and $\varphi_V = \varphi_H - \frac{\pi}{2} + 2\pi m$ or $A_V = -A_H$ and $\varphi_V = \varphi_H + \frac{\pi}{2} + 2\pi m$, where m is an integer[1].

Exercise C.5.* Show that, when none of the conditions of Ex. C.4 are satisfied, the tip of the electric field vector follows an elliptical pattern.

C.2 Polarizing beam splitter

The *polarizing beam splitter (PBS)* (Fig. C.2) is an important optical device for analyzing polarization. It is a transparent cube consisting of two triangular prisms glued to each other, constructed to transmit horizontally polarized light, but reflect vertically polarized. If a classical wave (C.2) is incident on such a beam splitter, the intensities of the transmitted and reflected light will be proportional to A_H^2 and A_V^2, respectively.

[1] Defining what circular polarization pattern should be called "left" or "right" is a matter of convention. Here we follow the convention that is common in the quantum optics community. In the right-circular pattern, the trajectory of the electric field vector is clockwise when viewed from the "back" of the wave (from the source). However, it is counterclockwise when viewed from the "front", or in the x-y plane with the traditional axis orientation. In space, this trajectory has the shape of a *left-handed* screw.

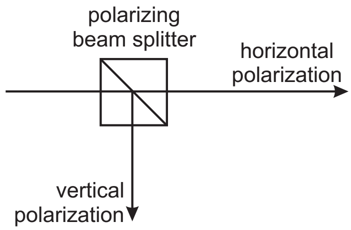

Fig. C.2 Polarizing beam splitter.

C.3 Waveplates

It is sometimes necessary to change the polarization state of light without splitting the vertical and horizontal components spatially. This is normally achieved using an optical instrument called a *waveplate*. The waveplate relies on *birefringence*, or *double refraction* — an optical property displayed by certain materials, primarily crystals, for example quartz or calcite. Birefringent crystals have an anisotropic structure, such that a light wave propagating through them will not conserve its polarization pattern unless it is linearly polarized along one of the two directions: either along or perpendicular to the crystal's *optic axis*. Traditionally, these directions are referred to as *extraordinary* and *ordinary*, respectively.

A birefringent material exhibits different indices of refraction for these two polarizations. Therefore, after propagation through the crystal, the ordinary and extraordinary waves will acquire different phases: $\Delta\varphi_e$ and $\Delta\varphi_o$, respectively. Because an overall phase shift has no effect on the polarization pattern, the quantity of interest is the difference $\delta\varphi = \Delta\varphi_e - \Delta\varphi_o$.

Exercise C.6. The indices of refraction for light polarized along and perpendicular to the optic axis are n_e and n_o, respectively, the length of the crystal is L, and the wavelength in vacuum is λ. Find $\delta\varphi$.

A waveplate is a birefringent crystal of a certain length, so $\Delta\varphi$ is precisely known. Two kinds of waveplates are manufactured commercially: $\lambda/2$-*plate (half-wave plate)* with $\delta\varphi = \pi$ and $\lambda/4$-*plate (quarter-wave plate)* with $\delta\varphi = \pi/2$.

If the polarization pattern is not strictly ordinary or extraordinary, propagation through a birefringent crystal will transform it. In order to determine this transformation, we decompose the wave into the extraordinary and ordinary components. The phase shift of each component is known. Knowing the new phases of both components, we can combine them to find the new polarization pattern.

Exercise C.7. For each of the polarization states of Ex. C.2, plot the polarization patterns that the waves will acquire when they propagate through (a) a half-wave plate, (b) a quarter-wave plate with the optical axes oriented vertically.

Solving the above exercise, you may have noticed that the half-wave plate "flips" the polarization pattern around the vertical (or horizontal) axis akin to a mirror.

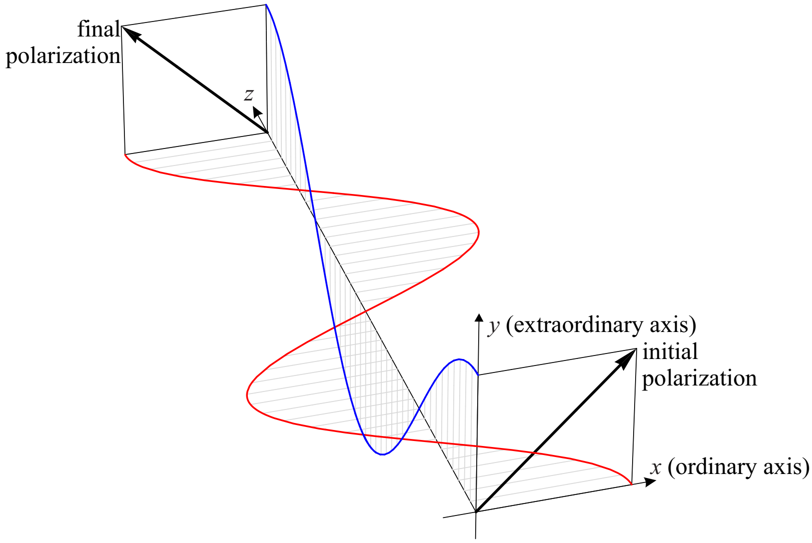

Fig. C.3 Action of a $\lambda/2$-plate with optic axis oriented vertically. Different refractive indices for the ordinary and extraordinary polarizations result in different optical path lengths, thereby rotating the polarization axis.

This is not surprising: the phase shift of π in the vertical component is equivalent to multiplication of A_V by -1. Of course, this mirroring property applies, not only when the optic axis is oriented vertically, but for *any* orientation, making the half-wave plate a universal tool for rotating the polarization of an electromagnetic field. Specifically, a light wave that is linearly polarized at angle θ to the horizontal, after propagating through a half-wave plate with its optic axis oriented at angle α to the horizontal, will transform into a linearly polarized wave at angle $2\alpha - \theta$ (Fig. C.4).

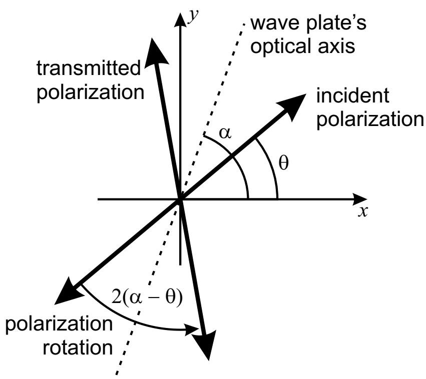

Fig. C.4 Polarization rotation by a $\lambda/2$ plate.

Exercise C.8.[§] Show that a $\lambda/2$-plate with the optic axis oriented at 22.5° to the horizontal interconverts between the horizontal and 45° polarizations, as well as between the vertical and -45° polarizations.

However, rotations alone do not provide a full set of possible transformations. For example, half-wave plates cannot transform between linear and circular/elliptical patterns. To accomplish this, we would need a quarter-wave plate.

Exercise C.9. Show that a $\lambda/4$-plate with the optic axis oriented horizontally or vertically interconverts between the circular and $\pm 45^\circ$ polarizations.

Exercise C.10. Linearly polarized light at angle θ to the horizontal propagates through a $\lambda/4$-plate with the optic axis oriented vertically. For the resulting elliptical pattern, find the angle between the major axis and the horizontal and the ratio of the minor to major axes.

Exercise C.11.[*] Suppose you have a source of horizontally polarized light. Show that, by using one half-wave plate and one quarter-wave plate, you can obtain light with an arbitrary polarization pattern.
Hint: It is easier to tackle this problem using geometric arguments, particularly the result of Ex. C.5, rather than formal algebra.

Exercise C.12.[*] Linearly polarized light propagates through a half-wave plate, then through a quarter-wave plate at angle 45° to the horizontal, then through a polarizing beam splitter. Show that the transmitted intensity does not depend on the angle of the half-wave plate.