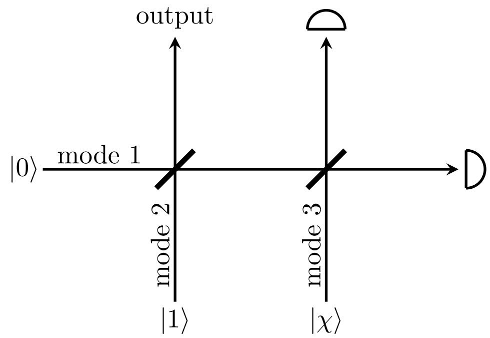
University of Oxford

Department of Physics
Quantum Optics graduate class

Michaelmas term test

November 30, 2021, 9:30–11:30

Problem 1. (50 marks)
The optical scheme shown below, with two symmetric beam splitters, is known as "quantum scissors".



a) (15 marks) The input state is $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ in parts (a–c) and (e). Find the state of the optical modes 1, 2 and 3 prior to the measurement.

$$\begin{aligned}
|\Psi\rangle_{123} &= \hat{U}_{13}\left(\hat{U}_{12}|0\rangle_1 \otimes |1\rangle_2\right) \otimes (\alpha|0\rangle + \beta|1\rangle)_3 \\
&= \frac{1}{\sqrt{2}}\hat{U}_{13}\left(|01\rangle - |10\rangle\right)_{12} \otimes (\alpha|0\rangle + \beta|1\rangle)_3 \\
&= \frac{1}{\sqrt{2}}\hat{U}_{13}\left(\alpha|010\rangle - \alpha|100\rangle + \beta|011\rangle - \beta|101\rangle\right)_{123} \\
&= \frac{1}{2}\left(\sqrt{2}\alpha|010\rangle - \alpha|100\rangle - \alpha|001\rangle + \beta|011\rangle - \beta|110\rangle + \beta|200\rangle - \beta|002\rangle\right)_{123}.
\end{aligned}$$

b) (5 marks) Find the state of mode 2 if the detector in mode 1 detects one photon, and the detector in mode 3 detects no photons. What is the probability p of this event?

$$|\psi_{\rm out}\rangle ={}_{13}\langle 10|\,\Psi\rangle_{123} = -\frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)_2.$$

Since $|\chi\rangle$ is normalized, $|\alpha|^2 + |\beta|^2 = 1$, so the probability of the above state is $p = 1/4$.

c) (5 marks) What quantum communication protocol does this scheme represent? Identify the location of the parties and other main elements of this protocol in the above scheme.

This is a teleportation protocol, with the input into mode 3 coming from Alice and the output of mode 2 going to Bob. The entangled resource $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{12}$ is produced by the first beam splitter in modes 1 and 2 and the Bell measurement is implemented in modes 1 and 3 via the second beam splitter. For the latter, projecting onto $|10\rangle_{13}$ in the beam splitter output is equivalent to projecting onto the Bell state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{13}$ in its input (the beam splitter action is symmetric in the two directions).

d) (5 marks) What is the output if the input state is an arbitrary superposition $|\chi\rangle = \sum_{i=0}^{\infty}\psi_n|n\rangle$? Can you guess why the scheme is called "quantum scissors"?

Any term with $n \geq 2$ in $|\chi\rangle$ will produce at least two photons in the output of the second beam splitter, meaning that the amplitude of projection onto $|1\rangle\, 0$ is zero. Hence the output will be the same as in part (b), with $\alpha = \psi_0$ and $\beta = \psi_1$. "Quantum scirrors" will "cut off" all Fock terms above $n = 1$ from the teleported state.

e) (20 marks) Repeat part (b) if the initial single photon in mode 2 is imperfect and produces a mixture $\hat{\rho}_1 = \eta\,|1\rangle\langle 1| + (1-\eta)\,|0\rangle\langle 0|$. The input state is $|\chi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$.

For $|0\rangle$ input in mode 2, we have

$$\begin{aligned}
|\Psi\rangle_{123} &= \hat{U}_{23}\left(\hat{U}_{12}\,|0\rangle_1 \otimes |0\rangle_2\right) \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)_3 \\
&= \hat{U}_{23}\,|00\rangle_{12} \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)_3 \\
&= \hat{U}_{23}\,(\alpha\,|000\rangle + \beta\,|001\rangle)_{123} \\
&= \left(\alpha\,|000\rangle + \frac{1}{\sqrt{2}}\beta\,|001\rangle - \frac{1}{\sqrt{2}}\beta\,|100\rangle\right)_{123}
\end{aligned}$$

and after the photon detection,

$$|\psi'_{\text{out}}\rangle =_{13} \langle 10|\,\Psi\rangle_{123} = \frac{1}{\sqrt{2}}\beta\,|1\rangle_2\,.$$

Since the input in mode 2 is $|1\rangle$ with probability η and $|0\rangle$ with probability $1-\eta$, the (unnormalized) output ensemble is

$$\begin{aligned}
\hat{\rho} &= \eta\,|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}| + (1-\eta)\,|\psi'_{\text{out}}\rangle\langle\psi'_{\text{out}}| \\
&= \left(\frac{\eta}{4}|\alpha|^2 + \frac{1-\eta}{2}|\beta|^2\right)|0\rangle\langle 0| + \frac{\eta}{4}\alpha\beta^*\,|0\rangle\langle 1| + \frac{\eta}{4}\alpha^*\beta\,|1\rangle\langle 0| + \frac{\eta}{4}|\beta|^2\,|1\rangle\langle 1|\,.
\end{aligned}$$

The probability is the sum of the diagonal elements,

$$p = \frac{\eta}{4} + \frac{1-\eta}{2}|\beta|^2.$$

You can use the beam splitter evolution operator $|U\rangle$ in the Fock basis as derived in class:

$$\begin{aligned}
\hat{U}\,|00\rangle &= |00\rangle\,; \\
\hat{U}\,|10\rangle &= \frac{1}{\sqrt{2}}\,|10\rangle + \frac{1}{\sqrt{2}}\,|01\rangle\,; \\
\hat{U}\,|01\rangle &= \frac{1}{\sqrt{2}}\,|01\rangle - \frac{1}{\sqrt{2}}\,|10\rangle\,; \\
\hat{U}\,|11\rangle &= -\frac{1}{\sqrt{2}}\,|20\rangle + \frac{1}{\sqrt{2}}\,|02\rangle\,,
\end{aligned}$$

where the operator of the first beam splitter acts on modes (1,2) and the second one on modes (1,3).

Problem 2. (50 marks) A general waveplate with its optical axis oriented at angle θ transforms the polarization state of a photon as follows:

$$\hat{U}\,|\theta\rangle = e^{i\Delta\varphi}\,|\theta\rangle\,; \tag{1}$$

$$\hat{U}\left|\frac{\pi}{2} + \theta\right\rangle = \left|\frac{\pi}{2} + \theta\right\rangle, \tag{2}$$

where $\hat{U}$ is the evolution operator and $\Delta\varphi$ is the difference of the phase shifts of the extraordinary ($|\theta\rangle$) and ordinary ($\left|\frac{\pi}{2} + \theta\right\rangle$) photons.

a) (5 marks) Write down $\hat{U}$ explicitly in its eigenbasis.

$$\hat{U} = e^{i\Delta\varphi}\,|\theta\rangle\langle\theta| + \left|\frac{\pi}{2}+\theta\right\rangle\left\langle\frac{\pi}{2}+\theta\right|.$$

b) (5 marks) Suppose this evolution occurs under the action of a fictitious Hamiltonian $\hat{H}$ for the time period t_0. Write down $\hat{H}$ in its eigenbasis.

Since $\hat{U} = \exp(-\frac{i}{\hbar}\hat{H}t_0)$, we have $\hat{H} = \frac{\hbar\Delta\varphi}{t_0}\,|\theta\rangle\langle\theta|$.

c) (20 marks) Find the differential equations for the evolution of the three Pauli operators under this Hamiltonian in the Heisenberg picture.

Since $|\theta\rangle = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \equiv \begin{pmatrix}c\\ s\end{pmatrix}$, we find

$$\hat{H} = \begin{pmatrix}c^2 & cs\\ cs & s^2\end{pmatrix} = \frac{1}{2}\hat{\mathbf{1}} + \frac{c^2-s^2}{2}\hat{\sigma}_z + cs\hat{\sigma}_x.$$

Using $[\hat{\sigma}_i, \hat{\sigma}_j] = 2\varepsilon_{ijk}\hat{\sigma}_k$, we have

$$\begin{aligned}
\dot{\hat{\sigma}}_x &= \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_x] = -(c^2-s^2)\frac{\Delta\varphi}{t_0}\hat{\sigma}_y;\\
\dot{\hat{\sigma}}_y &= \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_y] = (c^2-s^2)\frac{\hbar\Delta\varphi}{t_0}\hat{\sigma}_x - 2cs\frac{\Delta\varphi}{t_0}\hat{\sigma}_z;\\
\dot{\hat{\sigma}}_z &= \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_z] = 2cs\frac{\Delta\varphi}{t_0}\hat{\sigma}_y.
\end{aligned}$$

d) (15 marks) Specializing to a quarter-wave plate at $\theta = \frac{\pi}{4}$, solve these equations and find the corresponding evolution of the Pauli operators.

For $\theta = \frac{\pi}{4}$, we hace $c = s = \frac{1}{\sqrt{2}}$, hence

$$\begin{aligned}
\dot{\hat{\sigma}}_x &= 0;\\
\dot{\hat{\sigma}}_y &= -\frac{\Delta\varphi}{t_0}\hat{\sigma}_z;\\
\dot{\hat{\sigma}}_z &= \frac{\Delta\varphi}{t_0}\hat{\sigma}_y.
\end{aligned}$$

The solution is

$$\begin{aligned}
\hat{\sigma}_x(t) &= \hat{\sigma}_x(0);\\
\hat{\sigma}_y(t) &= \hat{\sigma}_y(0)\cos\frac{\Delta\varphi}{t_0}t - \hat{\sigma}_z(0)\sin\frac{\Delta\varphi}{t_0}t;\\
\hat{\sigma}_z(t) &= \hat{\sigma}_z(0)\cos\frac{\Delta\varphi}{t_0}t + \hat{\sigma}_y(0)\sin\frac{\Delta\varphi}{t_0}t.
\end{aligned}$$

For a quarter-wave plate, $\Delta\varphi = \frac{\pi}{2}$, hence $\hat{\sigma}_x(t) = \hat{\sigma}_x(0); \hat{\sigma}_y(t) = -\hat{\sigma}_z(0); \hat{\sigma}_z(t) = \hat{\sigma}_y(0)$.

e) (5 marks) Interpret your results: is the evolution consistent with the known properties of waveplates?

The states linearly polarized at $\theta = \frac{\pi}{4}$ and $\frac{\pi}{2} + \theta = \frac{3\pi}{4}$ are unaffected by the waveplate. At the same time, they are the eigenstates of $\hat{\sigma}_x$ (and hence of the Hamiltonian), hence it is not surprising that this operator does not evolve.

The operators $\hat{\sigma}_z$ and $\hat{\sigma}_y$ evolve into each other. Again, this is consistent with the property of the quarter-wave plate at angle $\frac{\pi}{4}$ to interconvert between horizontal and vertical polarizations (eigenstates of $\hat{\sigma}_z$) and circular polarizations (eigenstates of $\hat{\sigma}_x$).