University of Oxford

Department of Physics
Quantum Optics graduate class

Michaelmas term test — solutions

December 1, 2020

Problem 1. Using the Heisenberg equation, we find

$$\begin{aligned}
\dot{\hat{a}} &= \frac{i}{\hbar}[\hat{H}, a] = -ire^{-i\phi - i\Omega t};\\
\dot{\hat{a}}^\dagger &= ire^{i\phi + i\Omega t};\\
\dot{X} &= \frac{\dot{\hat{a}} + \dot{\hat{a}}^\dagger}{\sqrt{2}} = -\sqrt{2}r\sin(\Omega t + \phi);\\
\dot{P} &= \frac{\dot{\hat{a}} - \dot{\hat{a}}^\dagger}{\sqrt{2}i} = -\sqrt{2}r\cos(\Omega t + \phi).
\end{aligned}$$

Now integrating over time, we obtain

$$\begin{aligned}
\dot{X}(t) &= X(0) + \sqrt{2}\frac{r}{\Omega}[\cos(\Omega t + \phi) - \cos\phi];\\
\dot{P} &= P(0) - \sqrt{2}\frac{r}{\Omega}[\sin(\Omega t + \phi) - \sin\phi].
\end{aligned}$$

The trajectory is circular with the radius $\sqrt{2}\frac{r}{\Omega}$ and period $2\pi/\Omega$.

Problem 2.

a) Since

$$|\psi(0)\rangle = |\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,,$$

we find

$$|\psi(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{e^{-i\nu n^k t}\alpha^n}{\sqrt{n!}}|n\rangle\,.$$

b) For $\nu t = \pi$, $e^{-i\nu n^k t}$ equals 1 for even n and -1 for odd n. Hence one can write

$$|\psi(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{\sqrt{n!}}|n\rangle = |-\alpha\rangle\,.$$

Problem 3.

a) The initial density operator

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{5}\left(4\,|HH\rangle\langle HH| + |VV\rangle\langle VV| + 2\,|HV\rangle\langle HV| + 2\,|VH\rangle\langle VH|\right)$$

becomes after decoherence

$$\hat{\rho}' = |\Psi\rangle\langle\Psi| = \frac{1}{5}\left(4\,|HH\rangle\langle HH| + |VV\rangle\langle VV| + |HV\rangle\langle HV| + |VH\rangle\langle VH|\right)$$

b) The state of Alice's photon after Bob's measurement is

$$
\begin{aligned}
\hat{\rho}_A &= {}_B\langle\theta|\, \hat{\rho}' \,|\,\theta\rangle_B \\
&= \frac{1}{5}\left[4\cos^2\theta\,|H\rangle\langle H| + \sin^2\theta\,|V\rangle\langle V| + \sin\theta\cos\theta(|H\rangle\langle V| + |V\rangle\langle H|)\right] \\
&= \frac{1}{5}\begin{pmatrix} 4\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.
\end{aligned}
$$

The corresponding probability is $\mathrm{Tr}\hat{\rho}_A = \frac{1}{5}(\cos^2\theta + 4\sin^2\theta)$.

c) For unnormalized $\hat{\rho}_A$:

$$
\begin{aligned}
R_x &= \mathrm{Tr}\hat{\rho}_A\hat{\sigma}_x = \frac{2}{5}\sin\theta\cos\theta = \frac{1}{5}\sin 2\theta; \\
R_y &= \mathrm{Tr}\hat{\rho}_A\hat{\sigma}_x = 0; \\
R_z &= \mathrm{Tr}\hat{\rho}_A\hat{\sigma}_x = \frac{1}{5}(4\cos^2\theta - \sin^2\theta) = \frac{3}{10} + \frac{1}{2}\cos 2\theta.
\end{aligned}
$$

Ellipse $[5R_x]^2 + \left[2\left(R_z - \frac{3}{10}\right)\right]^2 = 1$.

For re-normalized $\hat{\rho}'_A = \frac{1}{4\cos^2\theta + \sin^2\theta}\begin{pmatrix} 4\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$:

$$
\begin{aligned}
R_x &= \frac{2\sin 2\theta}{5 + 3\cos 2\theta}; \\
R_y &= 0; \\
R_z &= \frac{3 + 5\cos 2\theta}{5 + 3\cos 2\theta}.
\end{aligned}
$$

Ellipse $(2R_x)^2 + (R_z)^2 = 1$.