University of Oxford

Department of Physics
Subdepartment of Atomic and Laser Physics
Graduate class 2018

Michaelmas term set - solutions

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Problem 1. Suppose K photons are in the state

$$|\Psi_K\rangle = \frac{1}{\sqrt{2}}\left(\Big|\underbrace{H\dots H}_{K\text{ times}}\Big\rangle + e^{i\varphi_K}\Big|\underbrace{V\dots V}_{K\text{ times}}\Big\rangle\right).$$

The first photon is detected to be in the state $|\psi_K\rangle = \frac{1}{\sqrt{2}}\left(\Big|\underbrace{H\dots H}_{K\text{ times}}\Big\rangle + e^{i\varphi_K}\Big|\underbrace{V\dots V}_{K\text{ times}}\Big\rangle\right)$. The resulting $K-1$ photons will be in the state

$$\begin{aligned}\langle\psi_K|\ \Psi_K\rangle &= \frac{1}{2}(\langle H| + e^{-i\theta_K}\langle V|)\left(\Big|\underbrace{H\dots H}_{K\text{ times}}\Big\rangle + e^{i\varphi_K}\Big|\underbrace{V\dots V}_{K\text{ times}}\Big\rangle\right)\\ &= \frac{1}{2}\left(\Big|\underbrace{H\dots H}_{K-1\text{ times}}\Big\rangle + e^{i(\varphi_K-\theta_K)}\Big|\underbrace{V\dots V}_{K-1\text{ times}}\Big\rangle\right)\end{aligned}$$

a) We have $\varphi_n = 0$ and $\sum_{K=2}^{N}\theta_N = (N-M-1)\pi$, so the remaining photon is in the state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|H\rangle + (-1)^{(N-M-1)}|V\rangle)$. The probability for a specific measurement result on each photon is 1/2, so the cumulative probability is $\begin{pmatrix} N-1 \\ M \end{pmatrix}\left(\frac{1}{2}\right)^{N-1}$.

b) We have $\varphi_n = 0$ and $\sum_{K=2}^{N}\theta_N = (2M-N+1)\pi/2$, so the remaining photon is in the state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i^{-(2M-N+1)}|V\rangle)$. The probability is the same.

Problem 2.

a) Heisenberg evolution:

$$\begin{aligned}\dot{\hat{X}} &= \frac{i}{\hbar}[\hat{H},\hat{X}] = -\gamma\hat{X} + d;\\ \dot{\hat{P}} &= \frac{i}{\hbar}[\hat{H},\hat{P}] = \gamma\hat{P}.\end{aligned}$$

Solution:

$$\begin{aligned}\hat{X}(t) &= \frac{d}{\gamma} + e^{-\gamma t}\left(\hat{X}(0) - \frac{d}{\gamma}\right);\\ \hat{P}(t) &= e^{\gamma t}\hat{P}(0),\end{aligned}$$

so $\langle X\rangle = d/\gamma(1-e^{-\gamma t}), \langle P\rangle = 0, \langle\Delta X^2\rangle = \frac{1}{2}e^{-2\gamma t}, \langle\Delta P^2\rangle = \frac{1}{2}e^{2\gamma t}$.

b) Position-squeezed coherent state or displaced position-squeezed vacuum. See the attached file.

c) We rewrite the Hamiltonian as

$$\hat{H} = -\frac{1}{2i}\hbar\gamma(\hat{a}^2 - \hat{a}^{\dagger 2}) + \frac{1}{\sqrt{2}i}\hbar d(\hat{a} - \hat{a}^{\dagger}).$$

Hence in the first order

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(t)\rangle \approx \left(\hat{\mathbf{1}} - \frac{i}{\hbar}\hat{H}t\right)|0\rangle = |0\rangle + \frac{dt}{\sqrt{2}}|1\rangle - \frac{\gamma t}{\sqrt{2}}|2\rangle\,.$$

d)

$$\langle X\rangle = \langle\psi(t)|\frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}|\psi(t)\rangle = dt \text{ (to the first order in } t\text{);}$$
$$\langle P\rangle = \langle\psi(t)|\frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}|\psi(t)\rangle = 0.$$

Problem 3.

a) Rotawing wave approximation Hamiltonian:

$$\hat{H} = -\hbar\begin{pmatrix}\Delta & \Omega\\ \Omega & 0\end{pmatrix}.$$

Stochastic wavefunction:

$$|\psi(t)\rangle = \begin{pmatrix}\psi_a\\ 1\end{pmatrix}.$$

Schrödinger equation:

$$\begin{pmatrix}\dot{\psi}_a\\ 0\end{pmatrix} = -\frac{i}{\hbar}\hat{H}\begin{pmatrix}\psi_a\\ 1\end{pmatrix} + \begin{pmatrix}-\Gamma/2\psi_a\\ 0\end{pmatrix}.$$
$$\dot{\psi}_a = (i\Delta - \Gamma/2)\psi_a + i\Omega.$$

b) Solution:

$$\psi_a(t) = \frac{-i\Omega}{i\Delta - \Gamma/2}\left[1 - e^{(i\Delta-\Gamma/2)t}\right].$$

c) See the attached file.