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S03: Quantum Ideas

Supplementary notes

Photon polarization practice

- Download the Wolfram Demonstration on Polarization of an Optical Wave through Polarizers and Wave Plates (to run the demo, if you don't have Mathematica, you will need the Mathematica plugin for your browser or the Wolfram CDF Player. Verify that the transformations of the waves under the action of waveplates is consistent with that described in Section C.3 of the book.
- Watch MIT video demonstrations on half-wave plates and quarter-wave plates.

Problem 1. A classical electromagnetic wave with amplitude $\sqrt{A_H^2 + A_V^2} = 1$ V/m and phases $\phi_H = \phi_V$ is polarized at angle 30° to horizontal. The operation $\phi_V \to \phi_V + \Delta\phi$ is applied to this wave. Describe and plot the polarization patterns for $\phi = 0, \pi/2, \pi, 3\pi/2$. Write the corresponding photon states in the canonical basis in the Dirac and matrix notations.

Problem 2. Two states are decomposed in the circular basis according to

$$|\psi\rangle = \frac{1}{5}(3i\,|R\rangle + 4\,|L\rangle), \quad |\phi\rangle = \frac{1}{5}(4i\,|R\rangle - 3\,|L\rangle), \tag{1}$$

a) Show that these states form an orthonormal basis using the fact that the circular basis is orthonormal.

b) Find the decompositions of these states in the canonical basis by finding the matrices of $|\psi\rangle$, $|\phi\rangle$, $|H\rangle$ and $|V\rangle$ in the circular basis and using the inner product rule (A.6) from the lecture notes. Write your answer both in the Dirac and matrix notations.

c) Verify your answer for part (b) by expressing $|R\rangle$ and $|L\rangle$ in the canonical basis and substituting into Eq. (1);

d) Verify that states $|\psi\rangle$ and $|\phi\rangle$ form an orthonormal set using the inner product in the canonical basis.

e) Repeat parts (b)–(d) for the diagonal basis rather than canonical.

f) Decompose states $|H\rangle, |V\rangle, |R\rangle, |L\rangle, (|H\rangle + 2i\,|V\rangle)/\sqrt{5}$ in basis $\{|\psi\rangle, |\phi\rangle\}$. Write your answer both in the Dirac and matrix notations.

g) States $|H\rangle, |V\rangle, |R\rangle, |L\rangle, (|H\rangle + 2i\,|V\rangle)/\sqrt{5}$ are measured in basis $\{|\psi\rangle, |\phi\rangle\}$. What are the probabilities of the outcomes?

Problem 3. States

a) $|\psi\rangle = \sqrt{\frac{1}{4}}\,|H\rangle + \sqrt{\frac{3}{4}}\,|V\rangle$,

b) $|\psi\rangle = \sqrt{\frac{1}{4}}\,|H\rangle - \sqrt{\frac{3}{4}}\,|V\rangle$,

c) A statistical mixture of either $|H\rangle$ with probability $\frac{1}{4}$ or $|V\rangle$ with probability $\frac{3}{4}$

are measured in the diagonal basis. Find the probabilities of the measurement outcomes.

Problem 4. Consider the modified BB84 protocol in which Alice sends and Bob analyzes the photon in the following polarization bases: $\{|H\rangle, |V\rangle\}$ and $\{|\theta\rangle, |\frac{\pi}{2}+\theta\rangle\}$ (this protocol becomes BB84 for $\theta = 45°$). Angle θ is known to Alice, Bob and Eve. Find the bit error rate that Alice and Bob will see in the event of a straightforward "intercept-resend" attack, in which Eve intercepts the photon, measures it in one of the above two bases (randomly chosen with equal probabilities), and resends whatever she detected. There are no losses, all equipment is perfect.

Quantum Computing 1

- In any web browser, open the website `http://quantum-computing.ibm.com`.
- Create an IBMid account and log in.
- When logged in, click on the button 'Launch Composer' on the left-hand side. If this does not work, you can also go to the webpage `http://quantum-computing.ibm.com/composer` instead.
- Multiple windows should appear. Please close the window labeled 'OpenQASM 2.0' or 'Qiskit'.
- Change the window labeled 'Q-sphere' to a 'Statevector' window by clicking on 'Q-sphere' and selecting 'Statevector'.

If all is correct your screen should look like Fig. 1.

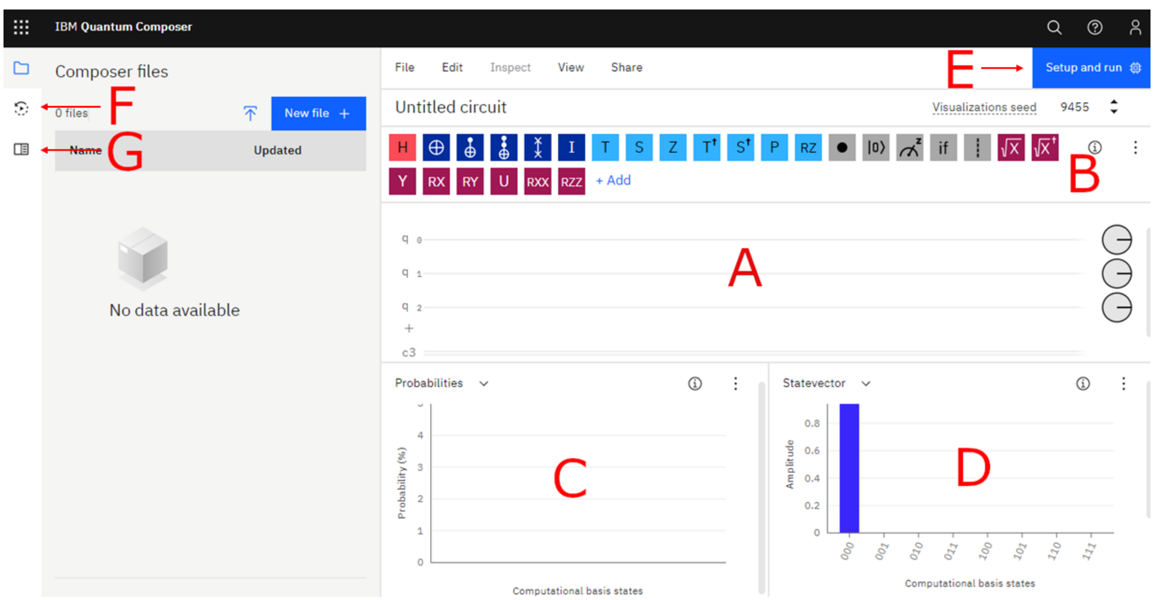


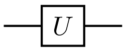
Figure 1: The Quantum Composer user interface. A: The quantum circuit in which gates will be placed; B: Menu with gates available in the quantum composer; C: The simulated results from running the quantum circuit; D: The simulated statevector of all the qubits after running the quantum circuit; E: Button to run the circuit on actual IBM quantum hardware; F: Job status and results; G: Documentation and tutorials by IBM.

The representation of qubits and gates in Figure 1(A) is usually referred to as the *quantum circuit*. The qubits are represented by the horizontal lines labeled $q[0]$, $q[1]$ and so on. The line labeled $c4$ is a classical bit. In this assignment, we will only perform single-qubit operations on $q[0]$, so please delete all other qubuts by clicking on them, and then clicking the trash can icon.

The Quantum Composer will automatically simulate the quantum circuit every time the circuit is changed. The results of this simulation are visible in windows C and D. Window D shows the final state vector of the qubits after all the gates have been applied. Window C shows the probability of getting specific outcomes in the canonical basis $\{|0\rangle, |1\rangle\}$ (called *computational basis* in the quantum computing language) when the qubits are measured. Initially, the qubit is in the $|0\rangle$ state.

We will be using only the following two quantum gates in this assignment.

- Arbitrary single-qubit gate.

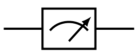
  

  This gate has three parameters — θ, ϕ and λ — and implements the following evolution operator:

  $$\hat{U}(\theta, \phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}. \tag{2}$$

  We will set the third parameter, λ, to zero throughout this problem set.

- Measurement.

  Measures the qubit computational basis.

- Place the single-qubit gate $\hat{U}$ onto the rail $q[0]$. Set the parameters of the latter by double-clicking on it: $\theta = \pi/2, \phi = 0, \lambda = 0$. Convince yourself that this circuit will transform the qubit into the state $1/\sqrt{2}(|0\rangle + |1\rangle)$. Check out the state vector in section C of your interface.

- Add the measurement gate to $q[0]$. After this, the state vector data (section C) will no longer be meaningful because a measurement destroys the quantum state.

- Build a quantum circuit with a NOT gate on q0 and a Hadamard gate on q1 and calculate the probabilities of each outcome. Note that adding the 'measurement' gates collapses the quantum state in the Statevector window as well, so the information in that window is no longer useful.

- Click on the 'Setup and run' button (E) in the top right corner of your screen.

- Under 'Step 1', select the quantum computer you would like to use. Preferably choose the one with the fewest number of pending jobs. Some of the options are simulators, and typically have smaller queues. Feel free to use them — but try to run at least a few jobs on a *bona fide* quantum computer.

- Under 'Step 2', keep the number of shots to 1024. This is the number of times your quantum circuit will be executed.

- Under 'Optional', give your job a recognizable name.

- Click the 'Run on [chosen quantum computer]' to execute your quantum circuit.

- You can find the status and results from your job under button F in Figure 1 on the left side of your screen. Your job may take some time to process depending on how busy the quantum computer is.

Problem 5. What should the parameters θ and ϕ of the U gate be in order to implement, up to an arbitrary overall phase factor,

a) a half-wave plate at angle α to horizontal (where $|0\rangle = |H\rangle, |1\rangle = |V\rangle$);

b) a quarter-wave plate at angle 0 to horizontal;

c) each of the three Pauli operators?

How would you implement these gates optically if you had a half-wave plate and an arbitrary wave plate with controllable phase shift $\delta\varphi$ between the ordinary and extraordinary axes?

Quantum Computing Exercise 1. Construct the circuits to implement single-qubit measurements in

a) canonical;

b) diagonal;

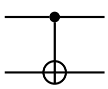
c) circular basis.

Test your circuits by preparing the state $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}(1+i)|1\rangle$ and measuring it in the three bases. Are the resulting measurement statistics consistent with the theoretically expected probabilities?

Quantum Computing 2

- Launch the Composer on the IBM quantum computing web site. As this assignment deals with 2-qubit circuits in the assignment, you will need to remove all extra qubits in the Composer quantum circuit window except $q[0]$ and $q[1]$. The qubit $q[0]$ is the least significant qubit, meaning that it corresponds to the rightmost digit in the basis states.

- An additional gate required in this assignment is the C-not gate.

  The control qubit (top in the diagram above) is marked with a thick dot and the target qubit is marked by the cross. You can use your mouse to drag the dot with respect to the cross to choose the desired control qubit.

Quantum Computing Exercise 2. Design and test the quantum circuits to prepare each of the four Bell states.

Quantum Computing Exercise 3. Design and test a quantum circuit that would enable a measurement in the Bell basis — that is, implement the following transformation on a two-qubit Hilbert space:

$$\begin{aligned} \left|\Psi^-\right\rangle &\to |00\rangle\,; \\ \left|\Psi^+\right\rangle &\to |10\rangle\,; \\ \left|\Phi^-\right\rangle &\to |01\rangle\,; \\ \left|\Phi^+\right\rangle &\to |11\rangle\,. \end{aligned} \tag{3}$$

Quantum Computing Exercise 4. Test the Bell inequality using the IBM quantum computer. Prepare the state $|\Psi^-\rangle$ and measure it in the four bases corresponding to the two different settings of each of the parties' apparatus. You will need to construct a separate circuit for each measurement. Run the quantum computer, collect the measurement result statistics, and check if it violates the Bell inequality.

Quantum Computing Exercise 5. Design and test a quantum circuit that would enable a conditional phase gate as defined in the book.

Quantum Computing Exercise 6. Design and test a quantum circuit that implements quantum teleportation of qubit $q0$ onto $q2$.

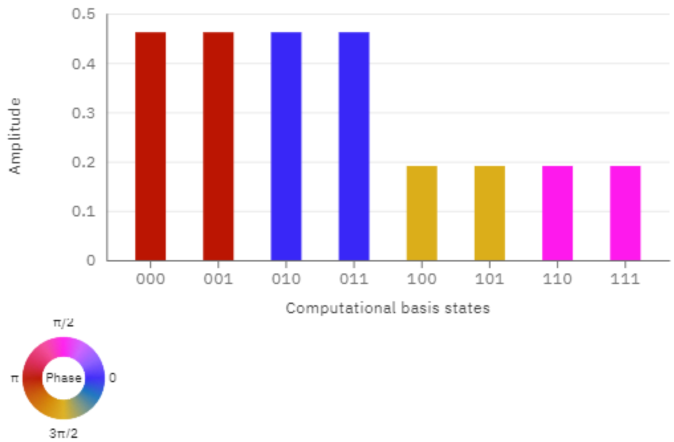


Figure 2: Output *Statevector* appearance after teleportation. The input state is $\hat{U}(\pi/4, \pi/2)\,|0\rangle$.

Hint: You already realized two key elements of this circuit: the preparation of a Bell state and a measurement in the Bell basis. There is, however, a problem with the third element: the transformation of Bob's qubit $q2$ dependent on the result of the Bell measurement (Table 2.3 in the book). To my knowledge, Quantum Composer does not allow such conditional operations. There are two ways to circumvent this problem. One is to use *Qiskit* — IBM's open source SDK for working with their quantum computers, which can be downloaded as a Python package and permits constructing sophisticated and fully customized quantum circuits. You are welcome to explore this on your own.

The other approach is to still use Composer, but implement the required transformations of Bob's qubit as *quantum* conditional operations. Notice that the c-phase and c-not gates can be interpreted, respectively, as the $\hat{\sigma}_z$ and $\hat{\sigma}_x$ operators applied to the target qubit that take place when the control qubit is in the state $|1\rangle$. You can use this property after you transform $q0 \otimes q1$ from the Bell basis into the canonical basis according to

$$\begin{aligned} \left|\Psi^-\right\rangle &\to |00\rangle\,; \\ \left|\Psi^+\right\rangle &\to |10\rangle\,; \\ \left|\Phi^-\right\rangle &\to |01\rangle\,; \\ \left|\Phi^+\right\rangle &\to |11\rangle\,. \end{aligned} \tag{4}$$

If you construct your circuit correctly, the Statevector should appear as shown in Fig. 2. That is, no entanglement is present among the three qubits. The states of the $q0 \otimes q1$ is $-\,|+\rangle \otimes |-\rangle$, respectively, whereas the state of $q2$ is equal to the initially prepared state of $q0$. Please explain why this is the case as a part of your solution to this problem.

- If you are interested to learn more about quantum computing, you can study IBM's tutorial (marked by G in Fig. 1) and try implementing some of the classic quantum computing algorithms, such as Deutsch-Jozsa, Grover, phase estimation and Shor. If you follow this path, you should procure the book *Quantum Computation and Quantum Information* by Nielsen and Chuang, which has been the "bible" of this field for many years.