B3: Quantum, Atomic and Molecular Physics
Problem Set 3

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1. (a) Assume that transitions between two levels in an atom occur only by radiative processes (namely stimulated emission or absorption, and spontaneous emission). Show that the ratio of the steady-state populations is

$$\frac{N_2}{N_1} = \frac{B_{12}\rho(\omega_{21})}{B_{21}\rho(\omega_{21}) + A_{21}}$$

where $\rho(\omega)$ is the energy density per unit (angular) frequency of the radiation field driving the stimulated processes, ω_{21} is the transition frequency, and A and B are the Einstein coefficients.

   (b) What happens to the relative populations in the two levels as the energy density of the radiation is increased to very large values? Would it be possible to create a population inversion this way?

   (c) In thermal equilibrium, the radiation density is given by the Planck black-body distribution. Show that this leads to the following relations between the Einstein coefficients:

$$B_{21} = \frac{g_1}{g_2}B_{12} \qquad\qquad A_{21} = \frac{\hbar\omega_{21}^3}{\pi^2 c^3}B_{21}$$

where g_1 and g_2 are the degeneracies of the lower and upper levels.

   (d) Does the relation between A_{21} and B_{21} still hold if the radiation is *not* black-body? Is it necessary to assume that the atom has *only* two levels?

2. A blob of matter is placed in a cavity and allowed to interact with black-body radiation of temperature T.

   (a) Show that for a transition of angular frequency ω_{21}, the rate of stimulated emission becomes equal to that of spontaneous emission when

$$k_{\mathrm{B}}T = \frac{\hbar\omega_{21}}{\ln 2}$$

   (b) Calculate this temperature for the following transitions:
      i. radio frequencies of 50 MHz
      ii. microwaves at 1 GHz
      iii. visible light of wavelength 500 nm
      iv. X-rays of energy 1 keV

3. (a) Atomic hydrogen is illuminated by light resonant with the $n = 1 \to n = 2$ Lyman-α transition, linearly polarized along the z-axis. Which upper state(s) can be excited?

   (b) Calculate the electric dipole matrix element $\langle 1|ez|2\rangle$ for the transition, expressing your answer in units of ea_0 where a_0 is the Bohr radius. (Look up the relevant hydrogen wavefunctions.)

   (c) Use your result to calculate the Einstein A coefficient for the transition, and hence the lifetime of the upper state.

   (d) A laser capable of producing continuous wave Lyman-α radiation was recently developed, which yielded a power of 1nW in a beam of 1mm diameter. Estimate the Rabi frequency if the laser were tuned to resonance with this transition. Comment on the feasibility of observing Rabi oscillations in this system.

4. (a) A two-level atom has eigenstates $|1\rangle$ and $|2\rangle$ of a time-independent Hamiltonian $\hat{H}$ which are separated by an energy $\hbar\omega_0 = E_2 - E_1$. Monochromatic light of amplitude $\mathbf{E}_0$ and angular frequency $\omega = \omega_0 + \delta$ (where $\delta \ll \omega_0$) is incident on the atom. Writing the wavefunction as

$$|\Psi(t)\rangle = c_1(t)\exp(-\mathrm{i}E_1 t/\hbar)|1\rangle + c_2(t)\exp(-\mathrm{i}E_2 t/\hbar)|2\rangle$$

   show by substitution into the time-dependent Schrödinger equation, with Hamiltonian $\hat{H} + \hat{V}(t)$, that the rate of change of the coefficient c_2 is

$$\dot{c}_2 = -\frac{\mathrm{i}}{\hbar}V_{21}c_1\exp(\mathrm{i}\omega_0 t)$$

   where $V_{21} = V_{12} = \langle 1|\hat{V}|2\rangle = \langle 1|e\mathbf{r}.\mathbf{E}_0|2\rangle\cos\omega t$ and $V_{11} = V_{22} = 0$. What assumptions have you made about the "perturbation" $\hat{V}$?

   (b) Explain what is meant by the *rotating wave approximation* and justify its use here. Make it, and show that this leads to the following coupled differential equations for the coefficients:

$$\dot{c}_2 = -\tfrac{1}{2}\mathrm{i}\Omega c_1\exp(-\mathrm{i}t\delta)$$
$$\dot{c}_1 = -\tfrac{1}{2}\mathrm{i}\Omega c_2\exp(+\mathrm{i}t\delta)$$

   where the Rabi frequency $\Omega = \langle 1|e\mathbf{r}.\mathbf{E}_0|2\rangle/\hbar$.

   (c) Solve for $c_2(t)$ and hence show that, if the atom is in state $|1\rangle$ at $t = 0$, the probability of finding it in state $|2\rangle$ at later time t is given by

$$|c_2(t)|^2 = \frac{\Omega^2}{\Omega^2 + \delta^2}\sin^2\left(\tfrac{1}{2}t\sqrt{\Omega^2 + \delta^2}\right)$$

   Sketch this probability as a function of time for the cases $\delta = 0$ and $\delta = \Omega$.

5. (a) Find the solutions $c_1(t)$ and $c_2(t)$ to the differential equations in the previous question, with the same initial conditions, but for the case of resonant driving ($\delta = 0$).

   (b) What is the state of the system after times given by $\Omega t = \pi/2, \pi, 2\pi$?

   (c) Sketch the positions of the Bloch vector at these times, with the convention that the angular co-ordinates (θ, ϕ) on the Bloch sphere are defined by $c_1 = \sin\left(\frac{\theta}{2}\right)$ and $c_2 = \mathrm{e}^{\mathrm{i}\phi} \cos\left(\frac{\theta}{2}\right)$. Which axis (x, y, z) is the rotation around?

   (d) What happens to a general state at co-ordinates (θ, ϕ) after a π-pulse (that is, after a time $t = \pi/\Omega$)? Is this a rotation about the same axis? Is it possible to perform a rotation about an axis orthogonal to this one?