**University of Oxford**

**Keble College**
**Michaelmas term**
**Tutor: Alex Lvovsky**

**B3: Quantum, Atomic and Molecular Physics**

# Additional homework problem set

**Problem A.1.** Perform full calculation of the energy eigenvalues of an electron in an atom as a function of the applied magnetic field. In addition to the Zeeman interaction, the spin-orbit interaction is present, so the total Hamiltonian is

$$\hat{H} = A\hat{\vec{L}} \cdot \hat{\vec{S}} - \hat{\vec{\mu}} \cdot \vec{B}.$$

The quantization axis $z$ is chosen along the magnetic field; $A > 0$ is constant. The orbital angular momentum $l = 1$. The gyromagnetic ratios for the orbital and spin angular momenta are, respectively,

$$\gamma_l = -\frac{e}{2M} = -\frac{\mu_B}{\hbar}; \quad \gamma_s = -\frac{e}{M} = -\frac{2\mu_B}{\hbar},$$

where $\mu_B$ is the Bohr magneton, and the negative sign accounts for the negative charge of the electron.

a) Find the matrix of the Hamiltonian in the $\{|j, m_j\rangle]\}$ basis, with $(j, m_j)$ taking on all possible values for $l = 1, s = \frac{1}{2}$:

$$|v_1\rangle = \left|j = \frac{3}{2}, m_j = \frac{3}{2}\right\rangle;$$
$$|v_2\rangle = \left|j = \frac{3}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$|v_3\rangle = \left|j = \frac{1}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$|v_4\rangle = \left|j = \frac{3}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$|v_5\rangle = \left|j = \frac{1}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$|v_6\rangle = \left|j = \frac{3}{2}, m_j = -\frac{3}{2}\right\rangle.$$

Use the notation

$$\alpha = \frac{A\hbar^2}{2}; \ \beta = \mu_B B.$$

**Hint:** First write the first term of the Hamiltonian in the $\{|j, m_j\rangle]\}$ basis, and the second term in the $\{|m_l, m_s\rangle\}$ basis. Then, knowing that the two bases are related via the Clebsch-Gordan coefficients (`https://www.physicsforums.com/attachments/cg-table-jpg.32635/`)

$$\left|m_l = 1, m_s = \frac{1}{2}\right\rangle = \left|j = \frac{3}{2}, m_j = \frac{3}{2}\right\rangle;$$
$$\left|m_l = 1, m_s = -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|j = \frac{3}{2}, m_j = \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|j = \frac{1}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$\left|m_l = 0, m_s = \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|j = \frac{3}{2}, m_j = \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|j = \frac{1}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$\left|m_l = 0, m_s = -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|j = \frac{3}{2}, m_j = -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|j = \frac{1}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$\left|m_l = -1, m_s = \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|j = \frac{3}{2}, m_j = -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|j = \frac{1}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$\left|m_l = -1, m_s - = \frac{1}{2}\right\rangle = \left|j = \frac{3}{2}, m_j = -\frac{3}{2}\right\rangle,$$

rewrite the second term in the $\{|j, m_j\rangle]\}$ basis using the resolution of the identity.

b) Find the eigenvalues $E_i(\alpha, \beta)$ of the Hamiltonian.
**Hint:** Notice that the matrix is block-diagonal.

c) Decompose the expressions for the energy eigenvalues into a power series in $\beta$ for $\beta \ll \alpha$ (weak Zeeman effect) and in $\alpha$ for $\beta \gg \alpha$ (strong Zeeman effect) up to the two leading terms. Check for consistency with the approximations derived in class.

**Problem A.2.** What temperature will a thermometer show in space? Assume an arbitrary shape/construction of the thermometer. The thermometer is one astronomical unit away from the Sun, and is not shaded from the Sun by any other object.