PHYSICS 673
Nonlinear and Quantum Optics

Lecture notes

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Contents

Chapter 1

Some facts from linear optics

1.1 Wave equation

The *Maxwell equations*, assuming that there are no free charges or currents ($\rho = 0,\ \vec{J} = 0$), in SI units, are as follows:

$$\vec{\nabla} \cdot \vec{D} = 0; \tag{1.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0; \tag{1.2}$$

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}; \tag{1.3}$$

$$\vec{\nabla} \times \vec{H} = \dot{\vec{D}}. \tag{1.4}$$

In addition, the following relations are valid:

$$\vec{B} = \mu_0 \vec{H} \tag{1.5}$$

(where we assumed that the material is non-magnetic) and

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \tag{1.6}$$

where $\vec{P}$ is the *polarization* of the material: dipole moment per unit volume.

In polarization, we separate the linear and nonlinear terms:

$$\vec{P} = \vec{P}_{\mathrm{L}} + \vec{P}_{\mathrm{NL}}, \tag{1.7}$$

where

$$\vec{P}_L = \epsilon_0 \chi \vec{E}, \tag{1.8}$$

(χ being the linear electric *susceptibility* of the material) and $\vec{P}_{\mathrm{NL}}$ is the dipole moment due to various effects not linearly dependent on the applied field. In this chapter, we assume the susceptibility to be a scalar, although it is generally a rank 2 tensor (in which case the field and the polarization may have different direction).

Note 1.1 All quantities entering Maxwell equations are functions of position and time. They all represent physical observables, and are thus represented by real numbers.

Problem 1.1 Derive the *electromagnetic wave equation*

$$(\vec{\nabla} \cdot \vec{\nabla})\vec{E} - \frac{1}{c^2}\ddot{\vec{E}} = \frac{1}{c^2 \epsilon_0}\ddot{\vec{P}} \tag{1.9}$$

from Eqs. (1.1)–(1.6).
Hint.

- Eliminate $\vec{H}$ and $\vec{D}$ using Eqs. (1.5) and (1.6).
- Use $\epsilon_0\mu_0 = 1/c^2$.
- Take the curl of both sides of Eq. (1.3) and substitute $\vec{\nabla}\times\vec{B}$ from Eq. (1.4).
- Use the identity
$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\cdot(\vec{\nabla}\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E}. \tag{1.10}$$
- Apply Eq. (1.1) to get rid of the first term in the right-hand side of Eq. (1.10)[1].

Problem 1.2 Show that in the absence of nonlinearity, the wave equation permits the following solution:
$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\vec{k}\vec{r}-i\omega t} + c.c. \tag{1.11}$$
(where "c.c." means "complex conjugate") with
$$k = (\omega/c)\sqrt{1+\chi}. \tag{1.12}$$

Note 1.2 The phase velocity of the electromagnetic wave (1.11) in the absence of absorption is given by $v_{ph} = \omega/k = c/\sqrt{1+\chi}$. The quantity $n = \sqrt{1+\chi}$ is called the *index of refraction.* For $\chi \ll 1$, we have $n = 1+\chi/2$.

Generally speaking, the linear polarization $\vec{P} = \vec{P}(t)$ depends not only on the momentary value of $\vec{E}(t)$ at a particular moment in time, but also on the electric field in the past. To visualize this, think of polarization as displacement of the electrons with respect of the nuclei. The electrons have inertia, so it takes time for the field to displace them. Accordingly, the response of the polarization to the field will be delayed. To account for this delay, we write
$$\vec{P}_{\mathrm{L}}(t) = \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)\vec{E}(t-\tau)\mathrm{d}\tau \tag{1.13}$$
with $\tilde{\chi}(\tau<0)=0$ due to causality: the medium at time t cannot be affected by the field values at times greater than t. The time-dependent susceptibility has the meaning of the medium's response to a very short electric field pulse. If $\vec{E}(t) = \vec{A}\delta(t)$, then we have, according to Eq. (1.13),
$$\vec{P}_{\mathrm{L}}(t) = \vec{A}\frac{\epsilon_0}{2\pi}\tilde{\chi}(t). \tag{1.14}$$
Think of it as follows: the application of a pulse will set the electrons in the medium in motion which will continue for some time. This motion will generate a time-dependent dipole moment that is proportional to $\tilde{\chi}(t)$. For an example, see Ex. 1.13

Problem 1.3 a) Show that the result of Problem 1.2 remains valid if Eq. (1.13) is used rather than Eq. (1.8), but the value of χ in Eq. (1.12) is substituted by the Fourier transform of the time-dependent susceptibiity
$$\chi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{i\omega\tau}\mathrm{d}\tau. \tag{1.15}$$

b) Show that Eq. (1.8) obtains as a particular case of the more general definition (1.13) for $\tilde{\chi}(\tau) = 2\pi\chi\delta(\tau)$ and $\chi(\omega)=\chi=\text{const}$.

Note 1.3 The dependence of the susceptibility (and, accordingly, the refractive index) on the frequency is called *dispersion.* If $\mathrm{d}n/\mathrm{d}\omega > 0$, the dispersion is called *normal*, otherwise *anomalous.*

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[1]This is a bit subtle. Equation (1.1) automatically implies that $\vec{\nabla}\cdot\vec{E}=0$ only if $\vec{D}$ and $\vec{E}$ are proportional to each other, i.e. in the absence of optical nonlinearities. However, by decomposing nonlinear polarization into a Taylor series in electric field (we'll discuss it in the next chapter) one can usually show that $\vec{\nabla}P_{\mathrm{NL}}=0$

Note 1.4 While time-dependent susceptibility $\chi(\tau)$ must be real (why?), its Fourier image (1.15) is a complex number: $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$. The imaginary part of $\chi(\omega)$ leads to complex k in Eq. (1.12) and, subsequently, to a real component in the exponent of Eq. (1.11). This component is interpreted as absorption or amplification of the electromagnetic wave.

Problem 1.4 Show that in the presence of the imaginary part of the susceptibility, the intensity I of the wave propagating along the z axis behaves as

$$I = I_0 e^{-\alpha z}, \tag{1.16}$$

where

$$\alpha(\omega) = \frac{\omega}{c}\chi''(\omega). \tag{1.17}$$

is the absorption coefficient.
Hint: do not forget that the intensity is proportional to the *square* of the amplitude.

Note 1.5 Equation (1.16) is called *Beer's law.* The quantity αL, where L is the length of the optical path inside a medium, is called the *optical depth* of the medium.

Equation (1.17) can be rewritten as $\alpha\lambda/2\pi = \chi''(\omega)$, where λ is the wavelength. The quantity $\chi''(\omega)$ can thus be interpreted as the amount of absorption that light experiences over a distance of $\lambda/2\pi$.

Note 1.6 Absorption is a consequence of delayed response of the medium to an external perturbation. If the response were instantaneous [i.e. Eq. (1.8) were valid], there would be no absorption.

1.2 Slowly-varying envelope approximation

So far, we have been assuming that the field propagating in the medium is a plane wave. In practice, plane waves do not exist; realistic electromagnetic fields have finite spatial and temporal extent. In this case, we can view the field as a *carrier* plane wave multiplied by an *"envelope"* that is varying slowly in space and time (Fig. 1.1):

$$\vec{E}(z,t) = \vec{\mathcal{E}}(z,t)e^{ikz-i\omega t} + c.c. \tag{1.18a}$$

where $\vec{\mathcal{E}}(z,t)$ is the envelope and $e^{ikz-i\omega t}$ is the carrier wave. The polarization is expressed accordingly as

$$\vec{P}(z,t) = \vec{\mathcal{P}}(z,t)e^{ikz-i\omega t} + c.c. \tag{1.18b}$$

Problem 1.5 By "slowly varying" we mean that the envelopes' significant change in time and space occurs at a rate much slower than the period and the wavelength, respectively. Show that this condition can be written mathematically as

$$\begin{pmatrix} \partial_z \\ \partial_t \end{pmatrix}(\vec{\mathcal{E}},\vec{\mathcal{P}}) \ll \begin{pmatrix} k \\ \omega \end{pmatrix}(\vec{\mathcal{E}},\vec{\mathcal{P}}). \tag{1.19}$$

When we study fields described by the slowly varying envelope approximation, we are often only interested in learning about the evolution of the envelope rather than the carrier. If this is the case, we can significantly simplify our math.

Before we proceed to calculate this evolution, let us note that we have some freedom in defining the carrier wave. A small variation in the wave vector or the frequency of that wave may affect the shape of the envelope, but not necessarily invalidate condition (1.19). In the exercise below we study two cases: the carrier wave propagating at the speed of light in vacuum and at the phase velocity determined by the medium's index of refraction.

Problem 1.6 Substitute Eqs. (1.18) into Eq. (1.9) and show that, under Eqs. (1.8) and (1.12), in the first nonvanishing order the equation for the evolution of the envelope takes the following form.

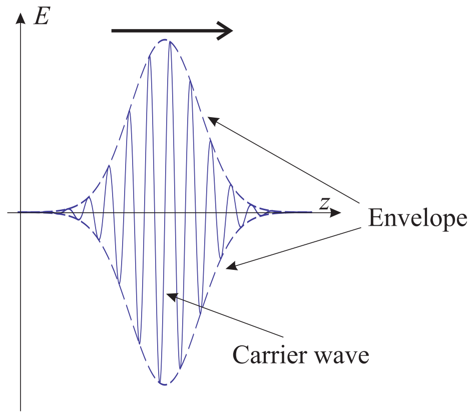


Figure 1.1: A pulse with a slowly varying envelope.

a) If the carrier wave has wave vector $k = \omega/c$

$$\left[\partial_z + \frac{1}{c}\partial_t\right]\vec{\mathcal{E}} = i\frac{\omega}{2\epsilon_0 c}\vec{\mathcal{P}}. \tag{1.20}$$

b) If the carrier wave has wave vector (1.12)

$$\left[\partial_z + \frac{k}{\omega}\partial_t\right]\vec{\mathcal{E}} = i\frac{\omega^2}{2\epsilon_0 k c^2}\vec{\mathcal{P}}_{\rm NL}. \tag{1.21}$$

Neglect dispersion.

Note 1.7 According to the above result, in the absence of nonlinearity, the envelope propagates with the speed equal to the phase velocity $v_{ph} = \omega/k = c/\sqrt{1+\chi}$. This is true only under the simplifying assumption Eq. (1.8), i.e. that no dispersion is present (see Problem 1.3(a)). In the presence of dispersion, the envelope will propagate with the group velocity, as shown in the following problem.

Problem 1.7 Show that, if we use Eq. (1.13) instead of (1.8), the equation of motion for the slowly varying envelope will take the form

$$\left[\partial_z + \frac{1}{v_{gr}}\partial_t\right]\vec{\mathcal{E}} = i\frac{\omega^2}{2\epsilon_0 k c^2}\vec{\mathcal{P}}_{\rm NL}. \tag{1.22}$$

with

$$\frac{1}{v_{gr}} = \frac{{\rm d}k}{{\rm d}\omega} = \frac{k}{\omega} + \frac{\omega^2}{2kc^2}\frac{{\rm d}\chi}{{\rm d}\omega}. \tag{1.23}$$

Hint: For a slow envelope, during the short time period when $\tilde{\chi}(\tau)$ is significantly nonzero, we can write $\vec{\mathcal{E}}(t-\tau) \approx \vec{\mathcal{E}}(t) - \tau\dot{\vec{\mathcal{E}}}(t)$. Therefore

$$\begin{aligned} \vec{P} &\overset{(1.13)}{=} \frac{\epsilon_0}{2\pi}\int \tilde{\chi}(\tau)\vec{\mathcal{E}}(t-\tau)e^{ikz-i\omega t+i\omega\tau} + c.c. \\ &\approx \epsilon_0\vec{\mathcal{E}}(t)\underbrace{\frac{1}{2\pi}\int\tilde{\chi}(\tau)e^{i\omega\tau}{\rm d}\tau}_{\chi(\omega)}\,e^{ikz-i\omega t} - \epsilon_0\dot{\vec{\mathcal{E}}}(t)\underbrace{\frac{1}{2\pi}\int\tau\tilde{\chi}(\tau)e^{i\omega\tau}{\rm d}\tau}_{-i{\rm d}\chi(\omega)/{\rm d}\omega}\,e^{ikz-i\omega t} + c.c. \end{aligned}$$

Note 1.8 The group velocity defined in Eq. (1.23) is a complex quantity. It accounts for both the propagation and absorption of the wave. The quantity referred to as the group velocity in the literature consists of only the real part of Eq. (1.23).

1.3 The Kramers-Kronig relations

These relations connect the real and imaginary parts of the frequency-dependent linear susceptibility. They are obtained from the most basic principles: causality and the fact that physical observables are real-valued.

Problem 1.8 Show that

$$\chi(\omega) = \chi^*(-\omega). \tag{1.24}$$

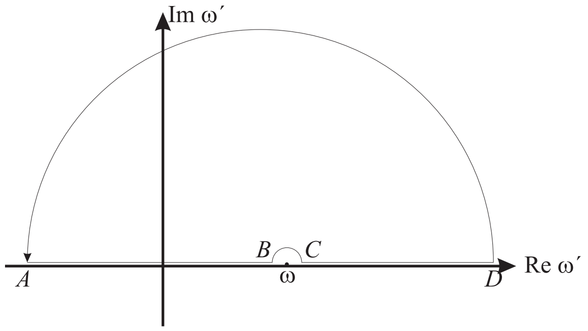


Figure 1.2: Integration contour of Eq. (1.25). Ponts A and D are positioned at $\omega' = \mp\infty$, points B and C at $\omega' = \omega \mp 0$, respectively.

Problem 1.9 The quantity $\chi(\omega)$ is formally defined by Fourier integral (1.15) not only for real, but for arbitrary complex ω. Show that this integral converges as long as $\operatorname{Im}\omega \geq 0$.

Problem 1.10 Analyze the integral

$$\oint\limits_{ABCD} \frac{\chi(\omega')}{\omega' - \omega} \mathrm{d}\omega' \tag{1.25}$$

along the contour shown in Fig. 1.2.

a) Show that the entire integral (1.25) is zero. **Hint:** Are there any poles inside the contour?

b) Show that the integral over part DA of the contour is zero. **Hint:** Check that $\chi(\omega) \to 0$ for $|\omega| \to \infty$.

c) Show that the integral over part BC of the contour is $-i\pi\chi(\omega)$.

d) Conclude that

$$\chi(\omega) = \frac{1}{i\pi}\mathcal{VP}\int\limits_{-\infty}^{+\infty} \frac{\chi(\omega')}{\omega' - \omega}\mathrm{d}\omega', \tag{1.26}$$

where $\mathcal{VP}$ stands for the principal value of the integral, i.e. the sum of parts AB and CD of the contour.

Problem 1.11 Using Eqs. (1.24) and (1.26), derive the Kramers-Kronig relations

$$\chi'(\omega) = \frac{2}{\pi}\int\limits_0^\infty \frac{\omega'\chi''(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.27}$$

$$\chi''(\omega) = -\frac{2\omega}{\pi}\int\limits_0^\infty \frac{\chi'(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega'. \tag{1.28}$$

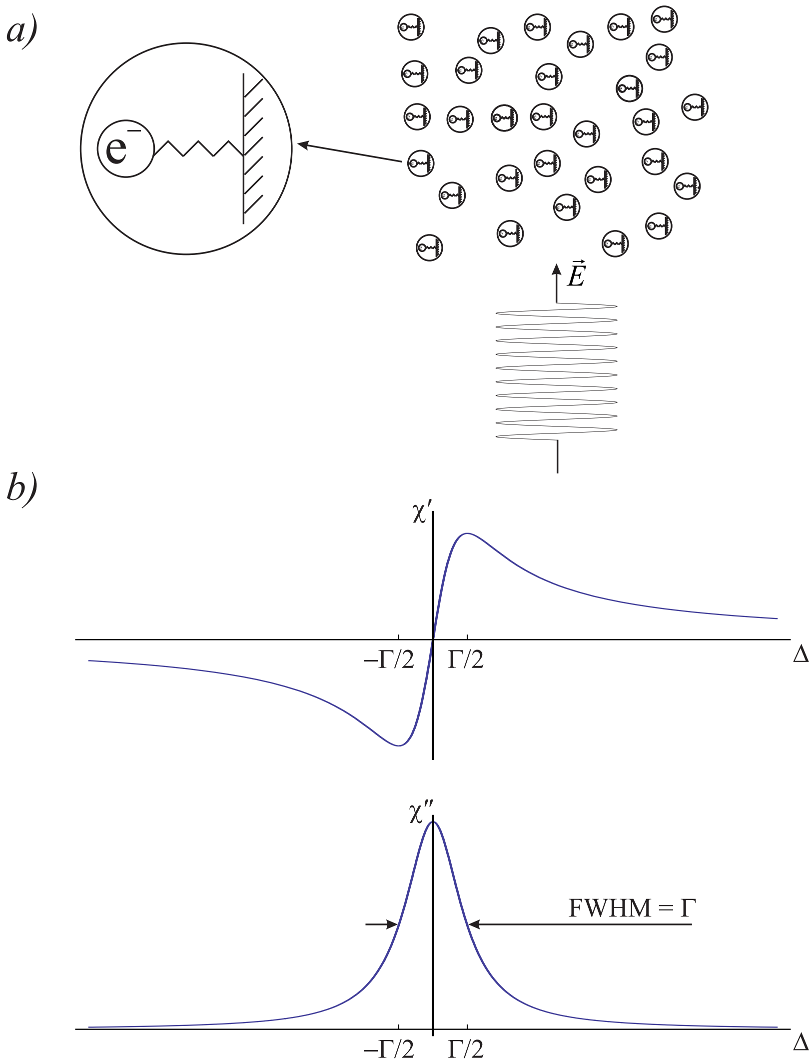


Figure 1.3: The classical theory of dispersion. (a) Model; (b) Real and imaginary parts of the susceptibility.

1.4 Classical theory of dispersion

Problem 1.12 Calculate the susceptibility $\chi(\omega)$ of a gas of electrons shown in Fig. 1.3. Each electron has charge e, mass m, is mounted on a spring with spring constant κ such that $\sqrt{\kappa/m} = \omega_0$ gives the resonant frequency of the oscillators. The damping constant is Γ, meaning that the friction force experienced by each electron is $-\Gamma m\dot{x}$, where x is the position of the electron with respect to the equilibrium position. The number density of electrons is N. Assume $\Gamma \ll \omega_0$.
Hint: Let the electric field be given by

$$E = E_0 e^{-ikz-i\omega t} + c.c.,$$

the coordinate of each electron

$$x = x_0 e^{-ikz-i\omega t} + c.c.,$$

polarization

$$P = P_0 e^{-ikz-i\omega t} + c.c.$$

Calculate x_0 as a function of E_0. Determine the amplitude of polarization as the dipole moment per unit volume. Determine the susceptibility as the proportionality coefficient between the polarization and field amplitudes.

Answer:

$$\chi = \frac{Ne^2}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\Gamma\omega} \approx -\frac{Ne^2}{\epsilon_0 m\omega_0}\frac{1}{2\Delta + i\Gamma}, \tag{1.29}$$

where $\Delta = \omega - \omega_0$, the approximate equality is valid for $\Delta \ll \omega_0$.

Problem 1.13 Let us obtain the same result (1.29) using the notion of time-dependent susceptibility discussed in Sec. 1.1. Suppose the medium is acted upon by a very short ($\ll 1/\omega_0$) electric field pulse $\vec{E}(t) = \vec{A}\delta(t)$.

a) Show that, immediately after the pulse, each electron will start moving with velocity $v_0 = Ae/m$.

b) Show that the motion of the electrons after the pulse is described by (Fig. 1.4)

$$x(t) = \frac{v_0}{\omega_0}\sin(\omega t)e^{-\frac{\Gamma}{2}t}. \tag{1.30}$$

c) Calculate the time-dependent susceptibility $\tilde{\chi}(t)$ and apply the Fourier transform (1.15) to re-derive Eq. (1.29).

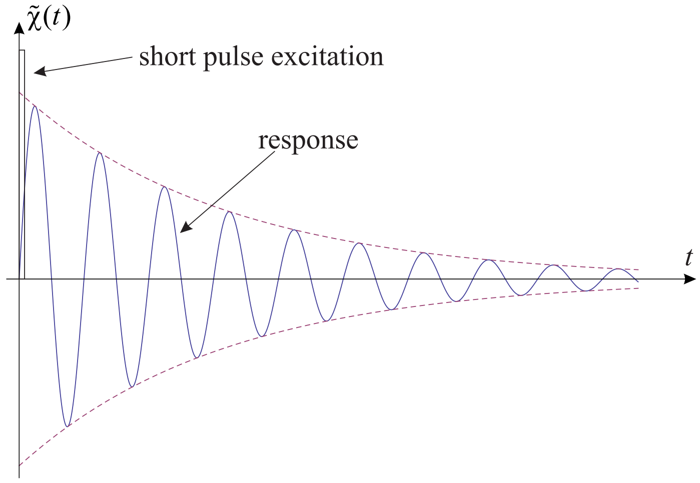


Figure 1.4: The response of a classical charged oscillator to a pulsed excitation.

Problem 1.14 Use the classical theory of dispersion to calculate the index of absorption and the index of refraction of a gas at wavelength $\lambda = 800$ nm with the resonance width $\gamma/2\pi = 6$ MHz and number density 10^{10} cm^{-3}.

a) Assume that the oscillators do not move.

b) Take into account Doppler broadening. Assume room temperature. The mass of each oscillator is equal to that of a rubidium atom. **Hint:** because the natural oscillator's linewidth can be assumed much narrower than the Doppler width.

Problem 1.15 Show that, in the framework of the classical theory of dispersion, the maximum value of the index of refraction obtains at $\Delta = -\Gamma/2$ and that it is related to the maximum (on-resonance) absorption index as $n_{\max} - 1 = \frac{\lambda}{8\pi}\alpha_{\max}$, where λ is the wavelength (Fig. 1.5). Calculate $n_{\max} - 1$ for the example of Ex. 1.14.

Note 1.9 We see that in spite of significant absorption in the medium on resonance, the variation of its refractive index is relatively small. This is because the imaginary part of the susceptibility equals the amount of absorption over a path length on a scale of a wavelength (see Note 1.5). While this quantity may be small, the absorption may become significant over a macroscopic path length. On the other hand, both χ' and χ'' are of the same order of magnitude near resonance, and so is $n_{\max} - 1$.

Off resonance, the situation is drastically different. The real part of susceptibility falls with the detuning as $\propto 1/\Delta$, whereas the imaginary part as $\propto 1/\Delta^2$ (check this!). Therefore, the off-resonant refraction can be significant while absorption negligible. This explains, for example, why glass, whose optical resonances are in the ultraviolet region, is transparent, yet exhibits high refraction.

Problem 1.16 Verify the Kramers-Kronig relations for the linear and imaginary parts of susceptibility (1.29). Near-resonance, narrow-resonance approximations can be used.

1.5 Fast and slow light

The phase velocity of light in the medium, given by $v_{ph} = \omega/k = c/\sqrt{1+\chi'}$, is superluminal (exceeds the speed of light in vacuum) whenever $\chi' < 0$. Under the classical theory of dispersion, this happens for all $\omega > \omega_0$ (Fig. 1.5). This is not surprising given that the phase velocity is not related to the speed with which the energy or information carried by the light wave travel.

The latter are usually associated with the group velocity. However, as we will see in this section, this quantity can also exceed the speed of light or even be negative, albeit without violating special relativity or causality.

Let us make an order-of-magnitude estimate of the real part of the inverse group velocity given by Eq. (1.23) and use the assumption that the susceptibility remains small at all frequencies. Then $\omega/k \approx c$ and we have

$$\mathrm{Re}\frac{1}{v_{gr}} \approx \frac{1}{c} + \frac{\omega}{2c}\frac{\mathrm{d}\chi}{\mathrm{d}\omega}. \tag{1.31}$$

The quantity $\mathrm{d}\chi'/\mathrm{d}\omega$ is on a scale of $\chi'_{\max}/\Gamma$ (Fig. 1.5). Accordingly, the scale of the second term is $(1/c)(\omega/\Gamma)\chi'_{\max}$.

Specializing to the example of Exercises 1.14 and 1.15, we have that, although $\chi'_{\max} \sim 10^{-3}$ is small, the second term is enhanced by the ratio of the optical frequency and the resonance width: $\omega/\Gamma \sim 10^8$. Accordingly, the second term can be on a scale of $10^5/c$, greatly exceeding the first one. Dependent on the detuning, we can have the following situations.

a) In the region of normal dispersion, v_{gr} is positive but many orders of magnitude lower than the speed of light in vacuum. This is the so-called phenomenon of *slow light*.

b) Within the region of anomalous dispersion near the extremum points, where the second term of Eq. (1.31) is negative but its absolute value is less than $1/c$, the group velocity is *superluminal*.

c) In most of the anomalous dispersion region, the *group velocity is negative*.

The latter case is most curious. What does it mean to have a negative group velocity? Suppose a pulse arrives at a sample of length L at time t_{in}. According to the definition of the group velocity, it should leave the sample at time $t_{out} = t_{in} + L/v_{gr}$. If $v_{gr} < 0$, the pulse will leave the sample *before* it has arrived.

Doesn't this imply violation of causality?

To answer, we have to remember that the optical pulse is not instantaneous; it has finite duration. The timing of the pulse is determined by its middle point. If the absorption of the pulse is non-unifirm, so its shape changes while traversing, its central point may shift. This shift may affect the apparent "check-in" and "check-out" timings of the pulse.

Think of the pulse as a long train approaching a bridge — the sample. Suppose just after the locomotive has passed the bridge, the bridge crumbles and the remainder of the train falls into the river. How will this accident affect the timings?

The time t_{in} is the moment the middle car of the train enters the train. This moment will occur *after* the bridge has collapsed and the locomotive has cleared the bridge. Since the "transmitted" train consists only of the locomotive, it is that latter moment that we designate as t_{out}. *Ergo*, $t_{in} > t_{out}$, and the causality is not violated.

This allegory surprisingly accurately describes the negative group velocity phenomenon associated with an atomic absorption resonance. Similarly to the locomotive crossing the bridge, the leading trail of the pulse is almost unaffected by the absorption. To understand this, let us recall the ball-on-a-spring model we used as the basis of the classical theory of dispersion.

Absorption occurs because the oscillation of the electrons is damped and the energy is dissipated. The characteristic time of the dissipation is $1/\Gamma$ (Fig. 1.4). During the initial part of the pulse, shorter than $1/\Gamma$, the dissipation will not "catch up" with the oscillation, and hence no significant absorption will occur.

One can think of this in the spectral domain as well. The spectral width of the pulse must be narrower than the width Γ of the absorption resonance, so the pulse duration must exceed $1/\Gamma$. If the pulse is shorter, most of it will miss the resonance and will not be absorbed. But when responding to the leading edge of the pulse, the medium "does not yet know" if the incoming field is a short pulse or a beginning of a long pulse. Hence it will respond to it in the same way it would to a short pulse, i.e. let it pass without significant loss.

Whatever the interpretation, we see that absorption is essential for the medium to exhibit negative group velocity. Let us now verify this mathematically.

Problem 1.17 Consider a pulse of duration $\tau_p \sim 1/\Gamma$. Show that, in order to advance the pulse in time by $t_{out} - t_{in} = -M\tau_p$ (where M is an arbitrary number), the optical depth of the medium must be on a scale of $\alpha L \sim M$.

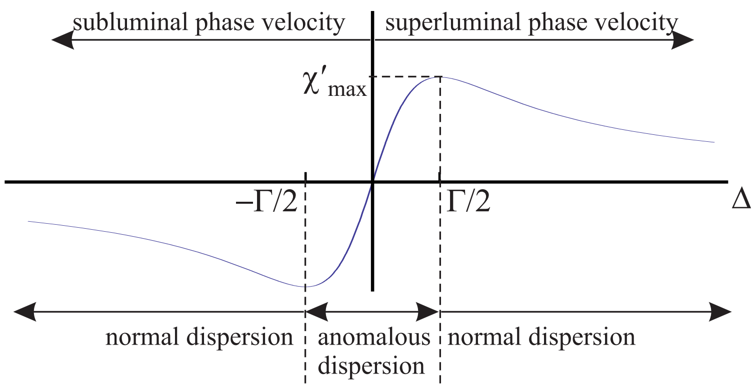


Figure 1.5: The real part of the susceptibility and the dispersion regions.

A closely related phenomenon is that of slow light. While it does occur in the normal dispersion regions of an absorption resonance (Fig. 1.5), it is more instructive and practically relevant to study it on a different system whose spectral properties are largely opposite to those studied so far. Specifically, let us analyze a medium with a narrow transparency "window" in a broadband absorption spectrum (Fig. 1.6):

$$\chi'' = \frac{\alpha_0 c}{\omega_0}\left(1 - \frac{\Gamma^2}{4(\omega-\omega_0)^2 + \Gamma^2}\right); \tag{1.32a}$$

$$\chi' = \frac{\alpha_0 c}{\omega_0}\left(\frac{2(\omega-\omega_0)\Gamma}{4(\omega-\omega_0)^2 + \Gamma^2}\right). \tag{1.32b}$$

(as usual, we assume $\chi' \sim \chi'' \ll 1$). Because of the opposite sign of the absorption feature, this system's dispersion must also be opposite to that of the absorption resonance. We have normal dispersion (slow light) on resonance and anomalous dispersion (superluminal and negative group velocities) off resonance. Thanks to the similarities between the two systems, we can readily find the delays associated with the slow light.

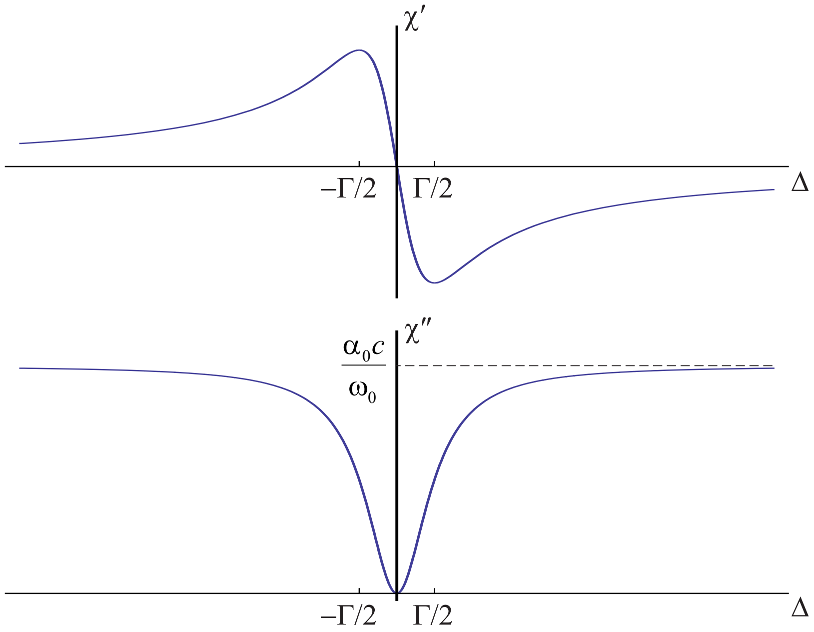


Figure 1.6: Susceptibility of a medium with a transparency window in the spectrum.

Problem 1.18 Consider a slab of length L consisting of the substance described by Eq. (1.32).

a)* Verify the Kramers-Kronig relations for the real and imaginary parts of the susceptibility.

b) Calculate the group velocity v_{gr} on resonance. Show it to be subluminal.
Answer: $1/v_{gr} = 1/c + \alpha_0/\Gamma$

c) A laser pulse of frequency ω_0 and temporal width $\tau_p \approx 1/\Gamma$ (so its spectrum just fits into the transparency window) propagates through the medium of length L. Find the delay compared to propagation through vacuum. Express it in the units of τ_p.
Answer: delay $= \alpha_0 L/\Gamma \approx (\alpha_0 L)\tau_p$

As previously, significant absorption is a requirement for the abnormal group velocity. In the present case, however, the pulse is within the transparency window, i.e. it need not experience absorption and its shape need not change in order for the slowdown to occur.

This result is relevant to quantum memory based on electromagnetically-induced transparency (EIT). In this memory protocol, the signal pulse is slowed down inside a transparency window, and its geometric length is reduced accordingly. The result of Problem 1.18(c) shows that the slowdown is sufficient for the cell to fully accommodate the signal pulse only if its optical depth outside the EIT window exceeds one.

Chapter 2

Basics of nonlinear optics

2.1 Nonlinear susceptibility

According to the classical theory of dispersion, the polarization response of the medium is entirely linear. This is because in the harmonic oscillator potential, the displacement of a particle is directly proportional to the electric field force acting on it: $\vec{x} \propto \vec{E}$.

For a physically realistic system, the harmonic oscillator picture is an approximation valid only in the limit of weak fields (Fig. unharmonicFig). In general, the displacement is a complex function of the field. This function can be decomposed into a Taylor series: $x = \alpha_1 E + \alpha_2 E^2 + \alpha_3 E^3 + \ldots$. This gives class to a large class of linear optical phenomena, some of which are studied in this chapter.

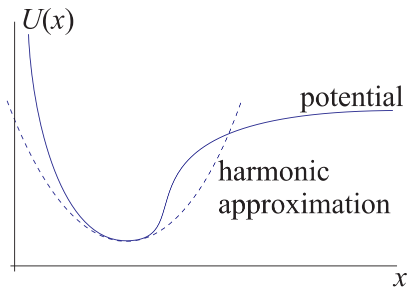


Figure 2.1: Harmonic potential is a valid approximation only for weak displacements.

We begin by calculating the nonlinear polarization. The polarization of the medium is proportional to the average displacement of its microscopic oscillators, so we write (assuming for now that the medium's response is instant):

$$\vec{P} = \vec{\vec{\chi}}^{(1)} : \vec{E} + \vec{\vec{\chi}}^{(2)} : \vec{E}\vec{E} + \vec{\vec{\chi}}^{(3)} : \vec{E}\vec{E}\vec{E} + \ldots, \tag{2.1}$$

where $\vec{\vec{\vec{\chi}}}^{(i)}(\tau_1, \ldots, \tau_i)$ is the *nonlinear susceptibility tensor of order* i. The tensor notation should be interpreted as follows (using the second-order term as an example):

$$P_i^{(2)} = (\vec{\vec{\chi}}^{(2)} : \vec{E}\vec{E})_i = \sum_{j,k=1}^{3} \chi_{ijk}^{(2)} E_j E_k. \tag{2.2}$$

We see that the susceptibility tensor of of order i has rank $i+1$. Below, we will sometimes omit the tensor notation and treat all quantities as scalars if the tensor properties are not of interest.

Following the route established in Chapter 1, we generalize Eq. (2.1) to a realistic situation of a medium with non-instantaneous response to perturbations:

$$\begin{aligned}\vec{P}(t) &= \epsilon_0 \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \tilde{\chi}^{(1)}(\tau)\vec{E}(t-\tau)\mathrm{d}\tau \\ &+\epsilon_0 \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{+\infty} \tilde{\chi}^{(2)}(\tau_1,\tau_2)\vec{E}(t-\tau_1)\vec{E}(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ &+\epsilon_0 \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{+\infty} \tilde{\chi}^{(3)}(\tau_1,\tau_2,\tau_3)\vec{E}(t-\tau_1)\vec{E}(t-\tau_2)\vec{E}(t-\tau_3)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3 + \ldots .\end{aligned} \tag{2.3}$$

This expression appears complicated, but, similarly to the linear case, simplifies significantly in the frequency domain.

Problem 2.1 Show that in the Fourier domain Eq. (2.3) takes the form[1]

$$\begin{aligned}\vec{P}_F(\omega) &= \epsilon_0 \chi^{(1)}(\omega)E_F(\omega) \\ &+\epsilon_0 \iint\limits_{-\infty}^{+\infty} \chi^{(2)}(\omega_1,\omega_2)\delta(\omega-\omega_1-\omega_2)\vec{E}_F(\omega_1)\vec{E}_F(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2 \\ &+\epsilon_0 \iiint\limits_{-\infty}^{+\infty} \chi^{(3)}(\omega_1,\omega_2,\omega_3)\delta(\omega-\omega_1-\omega_2-\omega_3)\vec{E}_F(\omega_1)\vec{E}_F(\omega_2)\vec{E}_F(\omega_3)\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3 \\ &+\ldots,\end{aligned} \tag{2.4}$$

where subscript F indicates the Fourier transform defined similarly to Eq. (1.15).

The Dirac delta functions in the nonlinear terms of Eq. (2.4) permits simple interpretation. The frequency of the nonlinear response must always be the *sum of frequencies* of the input fields. Caveat: if the input field is a sinusoidal wave oscillating at frequency ω, its Fourier image also contains a component at $-\omega$, and this component must be taken into account when evaluating the polarization spectrum[2].

Before we analyze this matter in detail, let us establish an important symmetry, one of the many we will encounter in this chapter.

Problem 2.2 Show that, without loss of generality, one can assume that

$$\chi^{(2)}_{ijk}(\omega_1,\omega_2) = \chi^{(2)}_{ikj}(\omega_2,\omega_1); \tag{2.5a}$$

$$\tilde{\chi}^{(2)}_{ijk}(\tau_1,\tau_2) = \tilde{\chi}^{(2)}_{ikj}(\tau_2,\tau_1). \tag{2.5b}$$

That is, any nonlinear medium can be described by a susceptibility compliant with Eqs. (2.5)

Note 2.1 The above relations can be generalized to higher-order nonlinearities.

Let us now familiarize ourselves with a few physical phenomena that arise due to nonlinear optical polarization.

Problem 2.3 Suppose the field has only two components:

$$E(t) = E_1 e^{-i\omega_1 t + ik_1 z} + E_2 e^{-i\omega_2 t + ik_2 z} + c.c. \tag{2.6}$$

[1] For educational purposes, it is sufficient to verify Eq. (2.4) for the second-order term. Higher-order terms are derived similarly.

[2] the delta functions in the right-hand side of Eq. (2.4) can be easily eliminated by additional integration. However, we prefer to keep this form of the equation because it explicitly shows the symmetry with respect to the input field frequencies.

Calculate $P(t)$ due to the second-order nonlinearity. Discuss the physics of each term.
Answer:

$$
\begin{aligned}
P^{(2)}(t) \quad = \quad & \epsilon_0 \Big[\underbrace{\chi^{(2)}(\omega_1, \omega_1) E_1^2 e^{-2i\omega_1 t + 2ik_1 z} + \chi^{(2)}(\omega_2, \omega_2) E_2^2 e^{-2i\omega_2 t + 2ik_2 z}}_{\text{second-harmonic generation (SHG)}} \qquad (2.7) \\
& + \underbrace{2\chi^{(2)}(\omega_1, \omega_2) E_1 E_2 e^{-i(\omega_1+\omega_2)t + i(k_1+k_2)z}}_{\text{sum-frequency generation (SFG)}} + \underbrace{2\chi^{(2)}(\omega_1, -\omega_2) E_1 E_2^* e^{-i(\omega_1-\omega_2)t + i(k_1-k_2)}}_{\text{difference-frequency generation (DFG)}} \\
& + \underbrace{\chi^{(2)}(\omega_1, -\omega_1)|E_1|^2 + \chi^{(2)}(\omega_2, -\omega_2)|E_2|^2}_{\text{optical rectification (OR)}} + c.c. \Big]
\end{aligned}
$$

In other words, if we have two input fields of frequencies 300 THz and 500 THz, the polarization will have components of twice each frequency (600 and 1000 THz), sum and difference frequencies (800 and 200 THz) and zero frequency. The latter phenomenon, known as optical rectification, implies that the medium will obtain DC polarization when the optical fields are applied.

Note 2.2 Equation (2.7) has a factor of 2 in front of the SFG and DFG terms that is absent in SHG. It may appear that the nonlinear effect of two slightly non-degenerate fields is different, by a factor of two, from the nonlinear effect of two identical fields. This is, of course, not so. If we have $E_1 = E_2$ and $\omega_1 = \omega_2$, then the second harmonic and sum frequency components of the polarization (the first three terms of Eq. (2.7)) become frequency degenerate, giving in combination a term that is four times as large as the second harmonic of each of the fields alone. It is this term that must be compared with the total effect of the second harmonic and sum frequency terms that are present when the input fields are non-identical.

Note 2.3 Because of the delta functions in Eq. (2.4), the frequency-domain second-order susceptibilities are sometimes written in the following notation,

$$\chi^{(2)}(\omega_1, \omega_2) \equiv \chi^{(2)}(\omega, \omega_1, \omega_2) \equiv \chi^{(2)}(\omega = \omega_1 + \omega_2). \qquad (2.8)$$

Problem 2.4 Verify that the dimension of $\chi^{(1)}(\omega)$ is unity; the dimension of $\chi^{(2)}(\omega)$ is m/V; the dimension of $\chi^{(3)}(\omega)$ is $\mathrm{m}^2/\mathrm{V}^2$.

Problem 2.5 Repeat Problem 2.3 for the third-order nonlinearity with three fields.

2.2 Evaluating nonlinear susceptibilities

In the previous section we discussed the general framework for analyzing nonlinear optical effect. Our next goal is to develop tools for evaluating the nonlinear susceptibilities quantitatively. To that end, we utilize our old friend, the classical theory of dispersion, with a modification. We assume that the springs on which the charges are hanging are unharmonic, so the potential energy is given by

$$U(x) = \frac{1}{2}\kappa x^2 + \frac{1}{3}max^3 + \frac{1}{4}mbx^4 + \dots, \qquad (2.9)$$

where a and b are the constants quantifying the unharmonicity, which we will estimate them later in this section.

Problem 2.6 Assuming that the nonlinearities are small, show that the second- and third-order nonlinear susceptibility in a system of oscillators described by Eq. (2.9) is given by

$$
\begin{aligned}
\chi^{(2)}(\omega_1, \omega_2) \quad &= \quad -\frac{Ne^3 a}{\epsilon_0 m^2} \frac{1}{D(\omega_1+\omega_2) D(\omega_1) D(\omega_2)}; \qquad (2.10) \\
\chi^{(3)}(\omega_1, \omega_2, \omega_3) \quad &= \quad -\frac{Ne^4 b}{\epsilon_0 m^3} \frac{1}{D(\omega_1+\omega_2+\omega_3) D(\omega_1) D(\omega_2) D(\omega_3)}, \qquad (2.11)
\end{aligned}
$$

where

$$D(\omega) = \omega_0^2 - \omega^2 - i\omega\gamma. \tag{2.12}$$

Hint. In order to calculate the second-order susceptibility, assume that two fields are applied akin to Problem 2.3. Write the equation of motion and calculate $x(t)$ initially neglecting the nonlinear term. Then substitute this solution into the equation of motion and solve this equation again, for example, for the sum frequency component of the oscillation spectrum.

Note 2.4 Based on the previous calculation, we obtain *Miller's rule*: the quantity

$$\frac{\chi^{(2)}(\omega_1,\omega_2)}{\chi^{(1)}(\omega_1+\omega_2)\chi^{(1)}(\omega_1)\chi^{(1)}(\omega_2)}$$

is frequency-independent.

Problem 2.7 Estimate the order of magnitude of the first-, second- and third-order nonlinear susceptibilities in a crystal assuming that the anharmonicity of the potential becomes significant when the position x of the electron is on a scale of the lattice constant d:

$$\kappa d \sim mad^2 \sim mbd^3, \tag{2.13}$$

where $\kappa = m\omega_0^2$ is the spring constant associated with the harmonic oscillator. Assume that all frequencies are in the optical range, but the optical fields are far away from the resonance at ω_0.
Answer:

$$\begin{aligned} \chi^{(1)} &\sim \frac{e^2}{\epsilon_0 d^3 m\omega^2} \sim 1 \\ \chi^{(2)} &\sim \frac{e^3}{\epsilon_0 d^4 m^2\omega^4} \sim 10^{-10}\mathrm{m/V} \\ \chi^{(3)} &\sim \frac{e^4}{\epsilon_0 d^5 m^3\omega^6} \sim 10^{-20}\mathrm{m^2/V^2} \end{aligned} \tag{2.14}$$

Problem 2.8 Using the model of Problem 2.6, obtain the following relation:

$$\chi^{(2)}(\omega_3=\omega_1+\omega_2) = \chi^{(2)}(\omega_1=\omega_3-\omega_2) = \chi^{(2)}(\omega_2=\omega_3-\omega_1). \tag{2.15}$$

Note 2.5 A more careful calculation taking into account tensor nature of the susceptibilities shows that

$$\chi^{(2)}_{kij}(\omega_3=\omega_1+\omega_2) = \chi^{(2)}_{ikj}(\omega_1=\omega_3-\omega_2) = \chi^{(2)}_{jki}(\omega_2=\omega_3-\omega_1) \tag{2.16}$$

(where i,j,k are spatial coordinates), i.e. the tensor indices must be permuted together with the frequencies. A similar result is valid for the third-order nonlinearity.

An even stronger relation is obtained under the assumption that the susceptibilities are frequency independent. In this case, any permutation of indices is allowed, for example:

$$\chi^{(2)}_{xyz} = \chi^{(2)}_{xzy} = \chi^{(2)}_{yxz} = \chi^{(2)}_{yzx} = \chi^{(2)}_{zxy} = \chi^{(2)}_{zyx},$$

or

$$\chi^{(2)}_{xxz} = \chi^{(2)}_{xzx} = \chi^{(2)}_{zxx}.$$

This result is called *Kleinman's symmetry*; it is also valid for the third-order nonlinearity. Note that Kleinman's symmetry does not mean that, for example, $\chi^{(2)}_{xxz} = \chi^{(2)}_{xxy}$.

Problem 2.9 Show that there are only 10 independent tensor elements of $\chi^{(2)}$ under Kleinman's symmetry.

2.3 Symmetries

In the previous section we obtained several relations among elements of the nonlinear susceptibility tensor. Although these relations turn out to be quite general, they rely on a specific physical model. And any model, however sophisticated, has its limitations. Important additional insight about the nonlinear properties of a material can be recovered from intrinsic symmetries that its molecular structure may possess. This is the subject of the present section. In this basic course, we discuss two classes of such symmetry: rotational symmetry (isotropicity) and mirror symmetry.

Isotropicity implies that the optical susceptibility tensor is invariant under rotations. Isotropic materials include gasses and liquids (with the exception of liquid crystals). On the other hand, most crystals have an unisotropic lattice and therefore have unisotropic optical properties. Such crystals, however, may be symmetric with respect to mirror reflection about one or several planes.

Problem 2.10 a) How does the susceptibility tensor of an arbitrary material transform under a 180° rotation around each of the Cartesian axes?

b) Show that the only tensor elements of the second-order susceptibility that may not vanish in an isotropic medium are those with all three indices different (i.e. $\chi^{(2)}_{xyz}, \chi^{(2)}_{xzy}, \chi^{(2)}_{yxz}, \chi^{(2)}_{yzx}, \chi^{(2)}_{zxy}, \chi^{(2)}_{zyx}$).

Problem 2.11 a) How does the susceptibility tensor of an arbitrary material transform under a mirror reflection about Cartesian planes xy, yz, xz?

b) Show that the second-order nonlinear susceptibility in any medium with mirror symmetry with respect to these three planes completely vanishes.

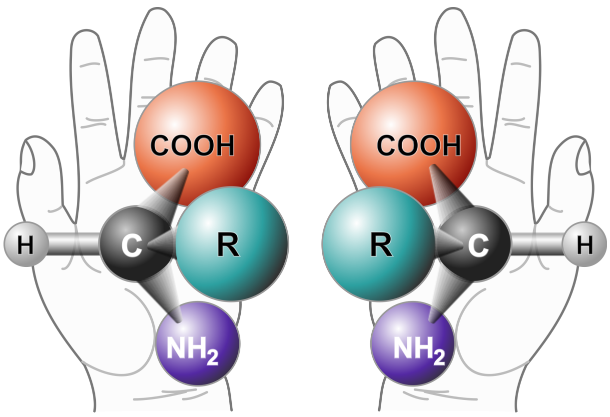


Figure 2.2: A chiral molecule of an amino acid (adapted from *Wikipedia*).

Materials without mirror symmetry are called *chiral*. Chirality may arise both due to the structure of a crystal lattice or that of individual molecules composing the material (Fig. 2.2). In the latter case, even an isotropic medium (e.g. a liquid or a gas) can exhibit second-order nonlinearity.

In non-chiral materials second-order nonlinear effects are impossible, unless the symmetry is broken, for example, by an external field. This explains why ferroelectric crystals comprise the bulk of materials used for second-order nonlinear optical applications. For the third-order nonlinearity, however, this restrictive rule does not hold, as demonstrated in the following exercise.

Problem 2.12 Show that in an isotropic non-chiral medium, the only nonvanishing elements of the third order nonlinear susceptibility are $\chi^{(3)}_{iijj}$, $\chi^{(3)}_{ijij}$, and $\chi^{(3)}_{ijji}$. Show that each of these elements is the same for all i and j as long as $i \neq j$ (i.e., for example, $\chi^{(3)}_{xxyy} = \chi^{(3)}_{yyzz}$). Show that

$$\chi^{(3)}_{xxxx} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}.$$

Note 2.6 An alternative notation[3] for the second-order susceptibility with Kleinman's symmetry is tensor d_{il} with

$$d_{il} = \frac{1}{2}\chi^{(2)}_{ijk}$$

and the pair of indices jk map onto a single index l as follows:

xx	yy	zz	$yz = zy$	$xz = zx$	$xy = yx$
1	2	3	4	5	6

For example, $d_{12} = \frac{1}{2}\chi^{(2)}_{xyy}$, $d_{36} = \frac{1}{2}\chi^{(2)}_{zyx}$, etc.

Problem 2.13 Which of the d's are equal to each other under Kleinman's symmetry?

Note 2.7 The values of d's for most important nonlinear crystals, defined with respect to crystallographic axes, are tabulated and can be found, for example, in the *Handbook of nonlinear optical crystals* together with the recipe to determine the effective nonlinearity (see below). Due to external symmetries, many crystals have fewer independent d's than 10.

2.4 Frequency conversion

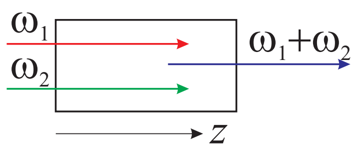


Figure 2.3: Sum frequency generation.

Now that we are familiar with the basic properties of nonlinear susceptibilities, let us discuss their practical applications. Suppose two waves of given frequencies are propagating through a second-order nonlinear medium with known properties. As we know, they will generate polarization, i.e. a bulk dipole moment, oscillating at the second-harmonic, sum and difference frequencies. This dipole moment, in turn, emits an electromagnetic wave. In this section, we will study methods of calculating the amplitude of these waves, their dependence on the propagation distance, and their back-action upon the initial (*fundamental*) waves.

Before we proceed with the calculation, let us introduce a notational agreement to simplify the bookkeeping. Consider, for example, the sum frequency term in Eq. (2.7) (Fig. 2.3:

$$P^{(2)}_i(z,t) = \sum_{j,k} 2\chi^{(2)}_{ijk}(\omega_1,\omega_2)\vec{E}_{1j}\vec{E}_{2k}e^{-i(\omega_1+\omega_2)t+i(k_1+k_2)z} + c.c. \tag{2.17}$$

If we are interested in the *magnitude* P of the polarization vector as well as those E_1, E_2 of the field vectors, we prefer not to drag the tensor notation through the calculation. Therefore we rewrite Eq. (2.17) in the scalar form

$$P^{(2)}(z,t) = 2\chi^{(2)}_{\text{eff}}(\omega_1,\omega_2)E_1E_2e^{-i(\omega_1+\omega_2)t+i(k_1+k_2)z} + c.c., \tag{2.18}$$

where $\chi^{(2)}_{\text{eff}}$ is the *effective susceptibility.*

For example, suppose we are generating sum frequency in an isotropic, chiral medium, with $\vec{E}_1$ oriented along the x axis, and $\vec{E}_2$ along the y axis. In this case, the only nonzero element of the susceptibility tensor that can contribute to the sum-frequency effect is $\chi^{(2)}_{zxy}$ (see Problem 2.10).

[3]This convention does not appear very logical. First, the factor of 1/2 does not reflect any physics. Second, there are $3 \times 6 = 18$ d's while the number of independent χ's is only 10.

Therefore the polarization vector will be oriented along the z axis and the effective susceptibility is $\chi^{(2)}_{\text{eff}} = \chi^{(2)}_{zxy}$.

The definition of effective nonlinearity can be straightforwardly extended to other nonlinear effects. We can also define $d_{\text{eff}} = \chi^{(2)}_{\text{eff}}/2$ in accordance with Note 2.6.

Problem 2.14 Consider a sum-frequency experiment in which both fundamental waves are propagating along z. Wave 1 is polarized linearly along the x axis and wave 2 at a 30° angle with respect to the x axis. Without making any assumptions about the symmetry of the medium, determine the direction of the polarization vector and the effective susceptibility. The susceptibility tensor of the medium $\chi^{(2)}_{ijk}$ is known.

Now that we know how to calculate the polarization, how can we use this information to determine the generated field? The most useful approach turns out to be the slowly varying envelope approximation discussed in Sec. 1.2. Applying Eq. (1.22) to each of the waves involved permits us to write differential equations that determine the wave amplitudes' evolution in space and time.

In what follows, we continue to specialize to the sum-frequency generation process. We will assume the continuous-wave (CW) regime, i.e. that the fundamental waves' amplitudes are constant in time. The nonlinear polarization and the generated wave amplitude will then not depend on time, either, so we can concentrate on their behavior as a function of the propagation distance.

Problem 2.15 Derive *the coupled-wave equations for sum-frequency generation* $(1 + 2 \to 3)$ in the continuous-wave regime, for all waves propagating along the z axes:

$$\partial_z \mathcal{E}_1 = \frac{2id_{\text{eff}}\omega_1}{n_1 c}\mathcal{E}_3\mathcal{E}_2^* e^{-i\Delta k z}, \tag{2.19a}$$

$$\partial_z \mathcal{E}_2 = \frac{2id_{\text{eff}}\omega_2}{n_2 c}\mathcal{E}_3\mathcal{E}_1^* e^{-i\Delta k z}, \tag{2.19b}$$

$$\partial_z \mathcal{E}_3 = \frac{2id_{\text{eff}}\omega_3}{n_3 c}\mathcal{E}_1\mathcal{E}_2 e^{i\Delta k z}, \tag{2.19c}$$

where $\delta_k = k_1 + k_2 - k_3$ is the *phase mismatch*, $\mathcal{E}$'s are the time-independent slowly-varying envelope amplitudes.

It is generally not enough to write the propagation equation for the generated wave alone. As you know, any electromagnetic field carries energy. The energy of the sum frequency field must come from the fundamental fields, so the amplitudes of these fields must accordingly diminish with the propagation distance z. This, in turn, will affect the sum-frequency wave.

Another way of looking at it is as follows. As the sum-frequency field is generated, it becomes involved in nonlinear interaction with fields 1 and 2, generating difference-frequency waves at frequencies ω_2 and ω_1, respectively. These waves destructively interfere with the fundamental fields, resulting in a loss of amplitude.

Let us study this argument in more detail. As we know, the energy carried by an electromagnetic wave is described by its *intensity* I, which is the energy flux per unit area and per unit time. If the amplitudes of the waves change with z, the energy conservation demands that

$$\partial_z I_1(z) + \partial_z I_2(z) + \partial_z I_3(z) = 0. \tag{2.20}$$

A stronger relation comes from the quantum picture of SFG. According to that picture, the SFG process consists of a pair of photons from each fundamental wave fusing together to produce a single sum-frequency photon: $\hbar\omega_1 + \hbar\omega_2 \to \hbar(\omega_1\omega_2)$. The flux of photons in each wave is proportional to its intensity divided by the energy of one photon, i.e. $I_{1,2,3}(z)/\omega_{1,2,3}$. Therefore we have the so-called *Manley-Rowe relations*:

$$\frac{1}{\omega_1}\partial_z I_1 = \frac{1}{\omega_2}\partial_z I_2 = -\frac{1}{\omega_3}\partial_z I_3. \tag{2.21}$$

In other words, the rates at which photons are removed from waves 1 and 2 are equal to each other and equal to the rate at which photons are added to wave 3.

Let us now bring these findings together.

Problem 2.16 a) Derive the energy conservation condition (2.20) from the Manley-Rowe relations.

b) Derive the Manley-Rowe relations from the classical coupled-wave equations for sum-frequency generation (2.19). **Hint:** $I_i = 2n_i\epsilon_0 c|\mathcal{E}_i|^2$.

Let us now proceed with solving the coupled-wave equations. A general solution is rather complicated, so we have to make a few simplifying assumptions.

Problem 2.17 Waves 1 and 2 with intensities I_1 and I_2, respectively, are incident on the front face ($z = 0$) of a nonlinear crystal with effective nonlinearity d_{eff} and generate a sum-frequency wave. Solve the coupled-wave equations to find the dependencies of I_1 and the sum-frequency intensity I_3 on the propagation distance z. Assume that $I_2 \gg I_1$ so wave 2 does not get substantially depleted in the process.

a) Assume that wave 1 does not get depleted, so its intensity remains constant.
Answer:

$$I_3(z) = I_1(0)\frac{\omega_3}{\omega_1}\frac{1}{L_{\text{NL}}^2}\frac{\sin^2\frac{\Delta kz}{2}}{\frac{\Delta k}{2}}, \tag{2.22}$$

where

$$L_{\text{NL}} = \sqrt{\frac{\epsilon_0 c^3 n_1 n_2 n_3}{2\omega_1\omega_3 I_2 d_{\text{eff}}^2}} \tag{2.23}$$

is the so-called *nonlinear length.*

b) Assume perfect phase matching ($\Delta k = 0$) instead.
Answer:

$$I_1(z) = I_1(0)\cos^2\frac{z}{L_{\text{NL}}}; \tag{2.24}$$

$$I_3(z) = I_1(0)\frac{\omega_3}{\omega_1}\sin^2\frac{z}{L_{\text{NL}}}. \tag{2.25}$$

c)* Do not assume either (a) or (b).
Answer:

$$I_1(z) = I_1(0)\left[\frac{\Delta k^2}{4g^2} + \frac{1}{L_{\text{NL}}^2}\frac{\cos^2 gz}{g^2}\right]; \tag{2.26}$$

$$I_3(z) = I_1(0)\frac{\omega_3}{\omega_1}\frac{1}{L_{\text{NL}}^2}\frac{\sin^2 gz}{g^2}, \tag{2.27}$$

where

$$g = \sqrt{\frac{1}{L_{\text{NL}}^2} + \frac{\Delta k^2}{4}}. \tag{2.28}$$

Let us stop and discuss this important set of results. In the beginning, before the effects of depletion or mode mismatch kick in, the amplitude of the sum frequency field initially grows linearly with z in accordance with Eq. (2.19c), so its intensity grows quadratically. In the absence of mode mismatch, this quadratic growth is limited by the depletion of wave 1. This complete depletion occurs at $z/L_{\text{NL}} = \pi/2$, after which the sum frequency process fully yields its place to the difference frequency process. Wave 1 is completely regenerated at $z/L_{\text{NL}} = \pi$, and then the story repeats itself [Fig. 2.17(a)].

If the phase mismatch is significant ($\delta k \gg 1/L_{\text{NL}}$), it becomes the dominant limiting factor in SFG. The growth of the sum frequency wave stops at $\Delta kz \approx \pi$, after which it reverses, so, again, we obtain a squared sinusoidal behavior of I_3 as a function of z [Fig. 2.17(b)]. In this case, wave 1 almost does not get depleted, so the approximation used in part (b) of Problem 2.17 is valid.

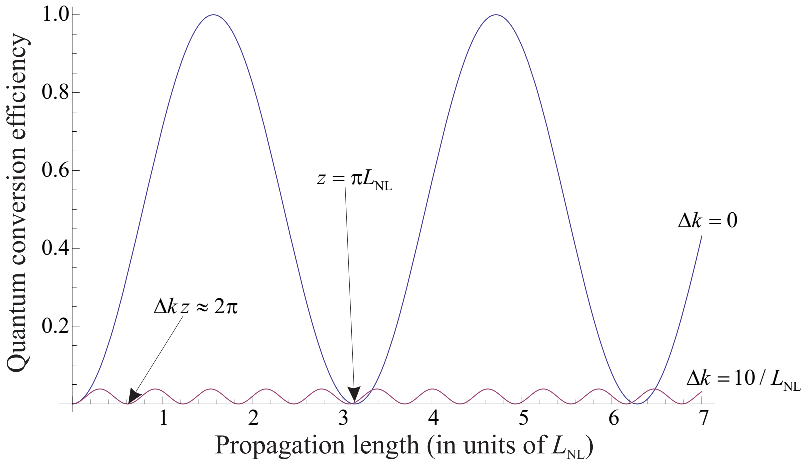


Figure 2.4: SFG conversion efficiency as a function of the propagation distance (Problem 2.17).

The nature of the phase mismatch is illustrated in Fig. 2.5(a). The phase of the sum-frequency wave *generated* at each particular point z is defined by the phase of the nonlinear polarization $P_3(z)$ at that point, which, in turn, is the sum of phases of waves 1 and 2:

$$P_3(z) \propto e^{i(k_1+k_2)z} \tag{2.29}$$

(we neglect the time dependence in this argument).

But the SFG field amplitude at z is determined not only by the nonlinear effect at that specific point. The sum frequency field is generated everywhere in the crystal, and the amplitude at point z is the sum (interference) of the "wavelets" that have been generated at all points $0 \le z' \le z$ and then *propagated* to point z.

But the wave vector k_3 with which the sum frequency wave propagates may be different from $k_1 + k_2$. Accordingly, all wavelets have slightly different phases and their interference is not entirely constructive. In fact, if the phases associated with wavelets from all points $0 \le z' \le z$ are distributed over the angle of 2π (this happens when $\Delta k z \approx 2N\pi$, N being an integer), the interference wipes out the SFG wave completely.

Therefore it is important to have the phase mismatch low: less than or on the order of the inverse length of the crystal. This is illustrated in Fig. 2.5(b), which shows the intensity of the output SFG field as a function of the phase mismatch for a constant crystal length z, under the assumption that $z \le L_{\mathrm{NL}}$ so that the depletion of wave 1 never plays a role.

Problem 2.18 A wave with intensity I_1 and frequency ω is incident on the front face ($z = 0$) of a nonlinear crystal with effective nonlinearity d_{eff} and generates a second-harmonic wave. Write the coupled-wave equations and find the behavior of I_1 and the second-harmonic intensity I_2 as functions of z.

a) Assume that the propagation distance is sufficiently short so wave 1 does not get depleted.
Answer [Fig. 2.6(a)]:

$$I_2(z) = I_1 \frac{z^2}{L_{\mathrm{NL}}^2} \operatorname{sinc}^2 \frac{\Delta k z}{2} \tag{2.30}$$

where

$$L_{\mathrm{NL}} = \sqrt{\frac{\epsilon_0 c^3 n_1^2 n_2}{2\omega^2 I_1 d_{\mathrm{eff}}^2}}. \tag{2.31}$$

b)* Assume perfect phase matching ($\Delta k = 0$) instead. Verify energy conservation.
Answer [Fig. 2.6(b)]:

$$I_2(z) = I_1 \tanh^2 \frac{z^2}{L_{\mathrm{NL}}^2} \tag{2.32}$$

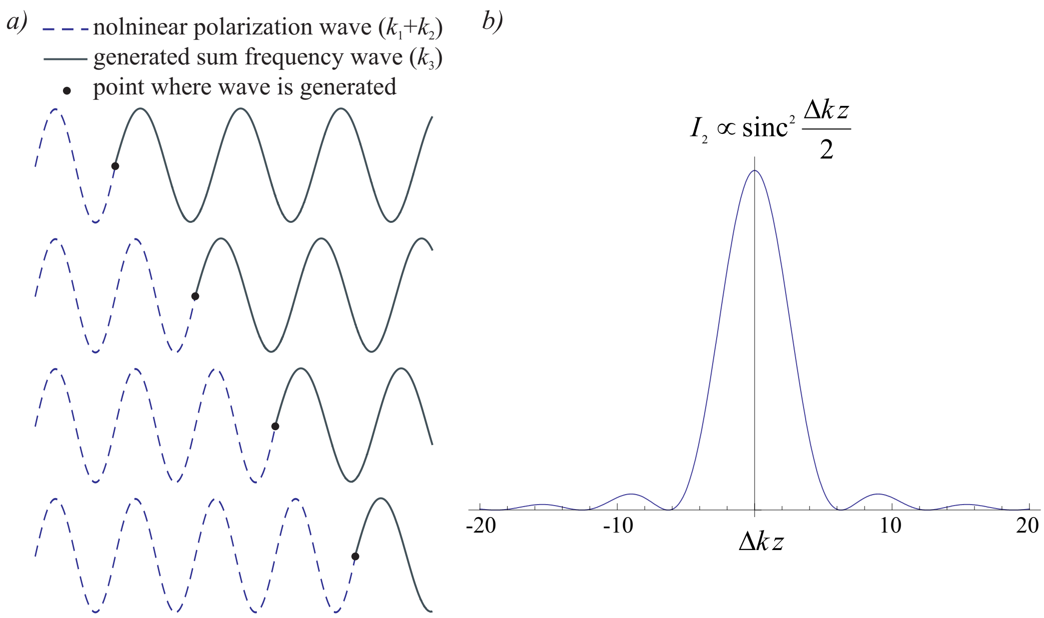


Figure 2.5: : (a) Explanation of phase mismatch. The sum frequency wave at the end of the crystal is the result of interference of waves generated at different points within the crystal. In the presence of nonzero mismatch, these waves have different phases. (b) SFG/SHG intensity as a function of the phase mismatch for a constant crystal length.

c)* Do not make any of the above assumptions.
Answer [Fig. 2.6(b)]:

$$I_2(z) = I_1\kappa\ \mathrm{sn}^2(\kappa^{-1/2}z/L_{\mathrm{NL}},\kappa) \tag{2.33}$$

where sn(·, ·) is Jacobi's elliptic sine and

$$\kappa = \left[\sqrt{1+\frac{\Delta k^2}{16L_{\mathrm{NL}}^2}} - \frac{\Delta k}{4L_{\mathrm{NL}}}\right]^2. \tag{2.34}$$

From part (b) of Problems 2.17 and 2.18 we see that the spatial evolution of the second harmonic field in the case of perfect phase matching is significantly different from that of the sum frequency. This happens not due to different nature of these two processes, but because in our SFG calculations we assumed that fundamental wave 1 is much weaker than wave 2. Then wave 2 does not get depleted, so the generated sum frequency wave interacts with it, leading to periodic interconversion of waves 1 and 3. In Problem 2.18, on the other hand, when the single fundamental wave becomes depleted, there is no process that brings about conversion of the SHG wave back into the fundamental wave[4].

Suppose that we are given a nonlinear medium with given properties and a laser source of given average power. How should we configure an SHG or SFG setup in order to optimize the frequency conversion process — that is, to minimize the nonlinear length? Prior to discussing this question, let us perform a numerical estimate.

Problem 2.19 A laser beam of power $P = 1$ W is focused onto a spot of $d = 0.1$ mm diameter in a crystal with effective second-order susceptibility of $d_{\mathrm{eff}} = 1$ pm/V. Estimate the order of magnitude of the nonlinear length L_{NL} for second-harmonic generation in the cases of (a) CW laser; (b) pulsed laser with a pulse width of $\tau = 1$ ps and the repetition rate $f = 100$ MHz.

[4]This is not strictly correct. Parametric down-conversion, which we will study shortly, converts photons into pairs of lower energy photons. However, this phenomenon is of pure quantum nature; it is impossible classically and very weak. Hence it can be neglected in this discussion.

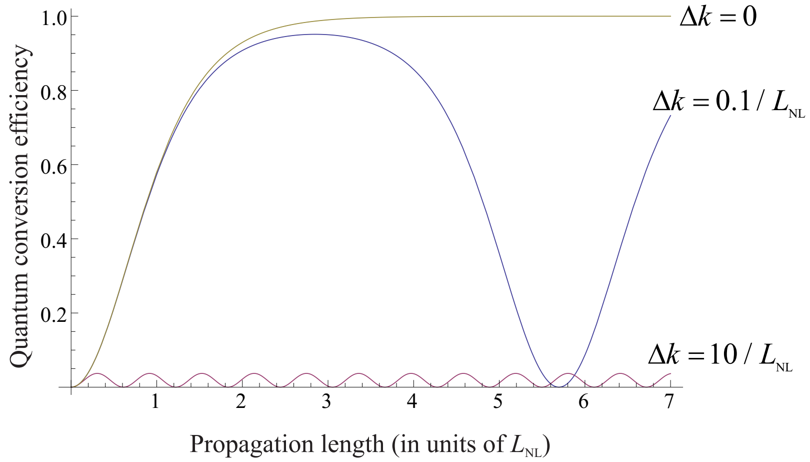


Figure 2.6: SHG conversion efficiency as a function of the propagation distance for various values of the phase mismatch [Problem 2.18].

As evident from Eqs. (2.23) and (2.31), the nonlinear length is inversely proportional to the intensity of the fundamental wave — that is, its power per unit cross sectional area. One way of maximizing the intensity is proposed in Problem 2.19: using a pulsed laser. With sufficiently short pulses, one can achieve sufficient conversion in a single pulse of a laser beam through a crystal. In the CW case, a single pass is typically insufficient; one needs to build a Fabry-Perot cavity around the crystal to enhance the intensity and conversion efficiency.

SHG power can also be enhanced by focusing a laser into the crystal. However, one should be careful not to focus too tightly. If one focuses into an excessively narrow spot, the beam will quickly diverge due to diffraction, so the advantage of high intensity is present only over a short length (Fig. 2.7).

One should therefore focus into a spot that is as narrow as possible as long as the diffraction divergence does not play a major role. This translates into a requirement that the Rayleigh range[5] of the focused beam be on the order of the length L of the crystal:

$$\frac{\pi w_0^2}{\lambda} \sim \frac{L}{2}, \tag{2.35}$$

where w_0 is the $1/e^2$ radius of the beam waist. A more precise calculation, performed in 1968 by G. D. Boyd and D. A. Kleinman, shows that the focusing is optimized when $\pi w_0^2/\lambda \approx L/5.68$.

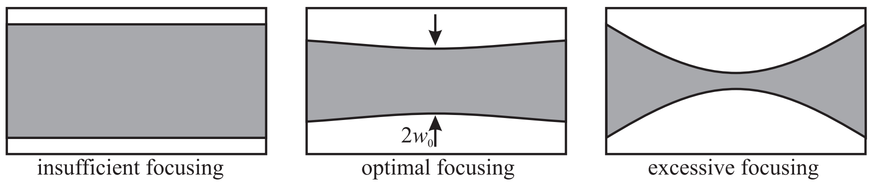


Figure 2.7: Focusing a laser beam into the crystal for optimal conversion.

[5]The Rayleigh range is the distance along the propagation direction of a beam from the waist to the place where the area of the cross section is doubled.

2.5 Linear optics in crystals

Because second-order processes vanish in symmetric media, in practice these processes are often implemented in crystals with asymmetric lattices. Such crystals possess peculiar optical properties, and design of any nonlinear conversion experiment requires deep understanding of these properties. This is the subject of the present section.

In fact, complex optical properties of crystals is not a bug, but a feature. As we discussed, precise phase matching is paramount for the efficiency of a nonlinear process. This phase matching is not automatic. For example, in the case of SHG, $\Delta k = 0$ implies that $n(\omega) = n(2\omega)$. But most media have dispersion, and refractive indices at different wavelengths are different.

This is where peculiar properties of crystals become useful. Most crystals are *birefringent*: their index of refraction depends on the polarization. Furthermore, for some polarizations the refractive index can change gradually as a function of the propagation direction with respect to the crystallographic axes. It is sometimes possible to pick a propagation angle such that $n(\omega)$ for one of the polarizations equals $n(2\omega)$ for the other, thereby achieving phase matching.

We proceed by recalling that the first-order susceptibility $\chi_{ij}^{(1)}$ is a rank-2 tensor, so it has nine components. However, it follows from linear algebra that the first-order susceptibility tensor can be diagonalized by re-orientation of the reference frame[6].

$$\begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \to \begin{pmatrix} \chi_{xx} & 0 & 0 \\ 0 & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix} \tag{2.36}$$

We will be working in the reference frame in which χ is diagonalized. The coordinate axes are then called *dielectric* axes of the crystal[7]. The planes xy, xz and yz are called *principal planes.* With each χ_{ii}, we associate a *principal value of the refractive index* $n_i = \sqrt{1 + \chi_{ii}}$.

For example, let us consider a wave propagating along the z axis. If it is polarized along $\hat{x}$, its index of refraction equals n_x, and for $\hat{y}$ polarization it is n_y. This is an example of birefringence: dependence of the refractive index on polarization. The physics becomes even more complicated if we consider a wave that is propagating diagonally with respect to the crystal's axes.

Problem 2.20 Consider a plane wave of frequency ω propagating in the principal plane xz with the k vector oriented at angle θ with respect to the z axis. The wave is polarized so that its electric field vector is also in the xz plane.

a) Show that such a wave, with the index of refraction given by

$$\frac{1}{n_e(\theta)^2} = \frac{\sin^2\theta}{n_z^2} + \frac{\cos^2\theta}{n_x^2}, \tag{2.37}$$

is a valid solution of Maxwell's equations.

b) Show that the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is not parallel to $\vec{k}$, but oriented at angle

$$\tan\theta' = \frac{n_x^2}{n_z^2}\tan\theta \tag{2.38}$$

with respect to the z axis.

Hint: Because the susceptibility is non-scalar, we have $\vec{P} \nparallel \vec{E}$ and thus $\vec{D} \nparallel \vec{E}$. Therefore you have to start from Maxwell's equations and determine the magnitude of k and the direction of $\vec{E}_0$

[6] In this section, we talk only about $\chi^{(1)}$. The superscstipt (1) will be omitted. The crystal is assumed non-magnetic.

[7] The dielectric (crystallophysical) axes coincide with the crystallographic axes, but are denoted differently. Crystallographic axes are denoted a, b, c, and dielectric x, y, z. The conventional assignment of axes may not be straightforward, e. g. for LBO it is $abc \to xzy$. In the literature, the table in Note 2.6 may refer to either crystallographic or dielectric axes. In this course, however, we will always define objects with respect to the dielectric axes, unless otherwise indicated.

under which the wave $\vec{E}_0 e^{i\vec{k}\vec{r}-i\omega t}$ satisfies these equations. It is convenient to solve the problem in the vector form. Notice that because of Eq. (1.1) we must have $\vec{D} \perp \vec{k}$. Therefore, if $\vec{k} = \begin{pmatrix} k\sin\theta \\ 0 \\ k\cos\theta \end{pmatrix}$, then $\vec{D}_0 = \begin{pmatrix} D_0\cos\theta \\ 0 \\ -D_0\sin\theta \end{pmatrix}$ and thus $\vec{E}_0 = \frac{1}{\epsilon_0}\begin{pmatrix} 0 \\ D_0\cos\theta/n_y^2 \\ -D_0\sin\theta/n_z^2 \end{pmatrix}$.

The behavior of the refractive index (2.37) is not surprising. If $\theta = 0$, the wavevector is parallel to the z axis, so the polarization is along $\hat{x}$ and the refractive index $n_e(0) = n_x$. On the other hand, for $\theta = \pi/2$, the wavevector is along $\hat{x}$, polarization is along $\hat{z}$ and $n_e(\pi/2) = n_z$. For the propagation angle between 0 and $\pi/2$, the index of refraction also takes on values between n_x and n_z.

The second result of Problem 2.20 is more astonishing. It implies that the direction in which a laser beam propagates through a crystal is not perpendicular to the wave front. This phenomenon is called *walk-off*, and the angle between $\vec{k}$ and $\vec{S}$ the *walk-off angle*. The walk-off effect is the principle of operation of many optical devices. In particular, it permits clean filtering of polarizations, because, as we shall see below, walk-off is present for only one polarization direction.

Problem 2.21 Show that, if the wave in Problem 2.20 is polarized orthogonally to the yz plane, it behaves as a regular electromagnetic wave in an isotropic medium, i.e. $\vec{E} \perp \vec{k}$ and there is no walk-off. Verify that the index of refraction for this wave is n_x.

We see that the two polarizations of a wave propagating diagonally in one of the crystal's principal planes have only different refractive indices, but significantly different physical properties. The wave polarized in the direction perpendicular to the plane is called *ordinary* while that polarized in the plane is *extraordinary.*

A natural question one might ask is what behavior one can expect from a wave that is neither ordinary nor extraordinary, i.e. a wave with the amplitude vector $\vec{E}_0$ oriented at an arbitrary angle with respect to the propagation plane. The answer is that no expression of the form $\vec{E}_0 e^{i\vec{k}\vec{r}-i\omega t}$ will satisfy Maxwell's equations, so such a wave cannot propagate in the crystal.

So what will happen if such a wave enters the crystal? This wave will decompose into an ordinary and extraordinary components, which will experience different indices of refraction. Accordingly, the wave's polarization vector will not be preserved. It will evolve (rotate) with the propagation distance, akin to the evolution that optical polarization experiences in a wave plate.

Dependent on the lattice symmetry, we can identify three types of crystals.

- Crystals with all three refractive indices equal (e.g. with a cubic lattice) are called *isotropic.*
- Crystals with only two refractive indices being equal, and the third one unequal to the first two, are called *uniaxial.* Examples: trigonal, tetragonal, hexagonal lattices.
- Crystals with all three refractive indices being unequal are *biaxial.* Examples: triclinic, monoclinic, orthorhombic lattices.

Let us study each of these types separately.

An isotropic crystal will exhibit the same index of refraction no matter what the propagation and polarization directions of the wave are. This is because the susceptibility tensor (2.36) in this case is proportional to a unity matrix, so it will not be affected by any transformations of the reference frame.

In a uniaxial crystal, the dielectric axes are conventionally defined so that $n_x = n_y$. The properties of the crystal are then axially symmetric with respect to the z axis. Any wave propagating at angle θ with respect to the z axis is then decomposed into ordinary and extraordinary wave, with the indices of refraction given by $n_O \equiv n_x = n_y$ and Eq. (2.37), respectively.

A propagation direction for which all polarizations are ordinary so there is no birefringence is called the *optical axis* of the crystal. In a uniaxial crystal, the optical axis is the z axis. Note that optical axes need not necessarily coincide with one of the crystallographic axes, as we will see shortly.

In the context of uniaxial crystals, the refractive index n_z is sometimes called *extraordinary* ($n_E \equiv n_z$). Note a possible confusion: the index of refraction $n_e(\theta)$ for the extraordinary wave varies with θ and equals n_E only if $\theta = \pi/2$ [see Eq. (2.37)]. Thus the terms "extraordinary index of refraction" and "index of refraction for the extraordinary wave" mean different things. In this course, we use a capital E subscript to denote the former and a small e for the latter. If $n_E > n_O$, the crystal is called *positive*, otherwise *negative*.

Figure 2.8 illustrates the behavior of the refractive index as a function of the propagation direction. This diagram should be interpreted as follows: for a given propagation angle, the length of a radius from the origin to the curve associated with either ordinary or extraordinary wave signifies the magnitude of the associated index of refraction.

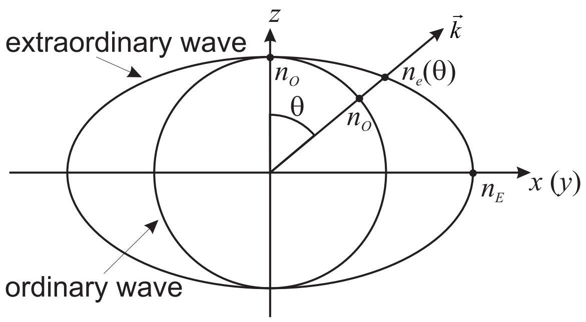


Figure 2.8: Dependence of the refractive index on the propagation direction in a positive uniaxial crystal.

Let us now consider a biaxial crystal with $n_z < n_y < n_x$. For each principal plane, we perform an analysis similar to that in Problems 2.20 and 2.21 and arrive at the diagrams shown in Fig. 2.9. In this case, we have two optical axes, both in the xz plane.

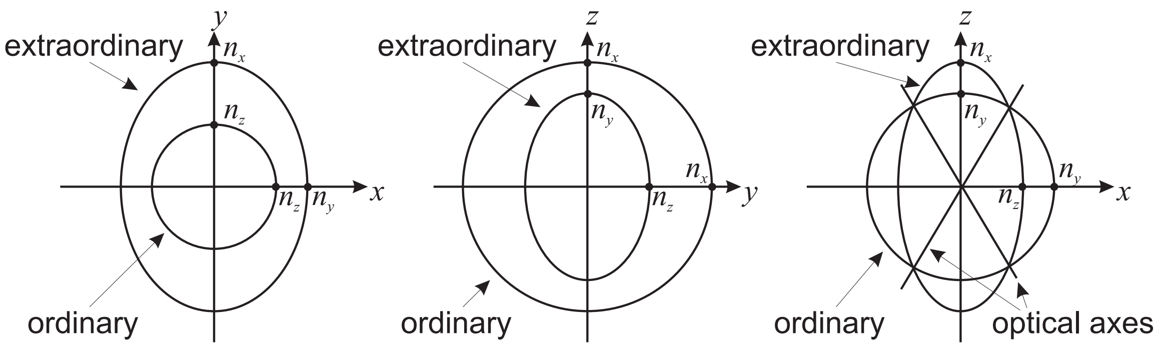


Figure 2.9: Dependence of the refractive index on the propagation direction in a biaxial crystal for the three principal planes.

Problem 2.22 Show that the angle between the optical axes and the z axis is given by

$$\sin^2\theta = \frac{1/n_x^2 - 1/n_y^2}{1/n_x^2 - 1/n_z^2}. \tag{2.39}$$

Note 2.8 The dielectric axes of biaxial crystals are conventionally defined in such way that either $n_x < n_y < n_z$ or $n_z < n_y < n_x$ so the optical axes are always in the xz plane (their azimuthal angle $\varphi = 0$).

2.6 Phase matching

As discussed in the previous section, complex refractive properties of optical crystals are helpful in achieving phase matching for nonlinear optical frequency conversion. The technique of *critical phase matching* consists of finding a propagation direction such that the refractive indices match for one of the waves being ordinary, and the other extraordinary. The crystal must be manufactured with its optical surfaces roughly perpendicular to the intended propagation direction consistent with the chosen phase matching configuration.

Suppose we need to generate the second harmonic of laser light at wavelength $\lambda = 1064$ nm in a beta barium borate (BBO) crystal. This is a uniaxial crystal with $n_O(\omega) = 1.6551$, $n_E(\omega) = 1.5425$ at 1064 nm and $n_O(2\omega) = 1.6749$, $n_E(2\omega) = 1.5555$ at 532 nm.

Notice that $n_E(2\omega) < n_O(\omega) < n_O(2\omega)$ According to Eq. (2.37), the index of refraction $n_e(2\omega, \theta)$ for the extraordinary second harmonic wave varies continuously between the values of $n_O(2\omega)$ and $n_E(2\omega)$ as a function of the propagation angle. Therefore there must be an angle at which this index of $n_e(2\omega, \theta) = n_O(\omega)$ (Fig. 2.10). This is the desired angle at which phase matching takes place.

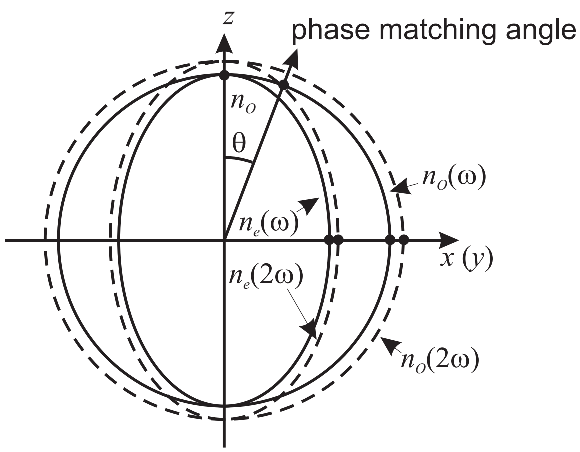


Figure 2.10: Critical phase matching in a BBO crystal. The phase matching angle corresponds to the intersection of $n_e(2\omega, \theta)$ and $n_O(\omega)$.

Problem 2.23 Find the phase matching angle for the configuration described above, both in the symbolic and numeric form.

For a nonlinear process in a given crystal of length L cut for optimal phase matching, the following additional important parameters can be defined.

- *Group velocity mismatch* GVM $= 1/v_{gr}(\omega) - 1/v_{gr}(2\omega)$. This parameter is relevant when the laser is pulsed. It determines the delay that the fundamental and second harmonic pulses will experience with respect to each other when propagating through the crystal: $\Delta t = L \times \mathrm{GVM}$. If this delay significantly exceeds the duration of the pulse, the second harmonic waves generated at different parts of the crystal will not be synchronized. Unlike phase mismatch, this will not lead to destructive interference; however, it may bring about temporal extension of the second harmonic pulse.

- *Phase matching bandwidth*, i.e. the range of wavelengths over which the phase-matching condition holds for a constant propagation angle. It is normally defined in terms of the full width at half-maximum (FWHM) intensity range, i.e. we are looking for a range of Δk such that

$$\mathrm{sinc}^2 \frac{\Delta k L}{2} \geq \frac{1}{2} \tag{2.40}$$

[*cf.* Eq. (2.30)]. This is approximately equivalent to $-1.39 \leq \Delta k L/2 \leq 1.39$.

- *Acceptance angle*, i.e. the range of angles over which the phase-matching condition holds for a constant wavelength. It is also defined in accordance with Eq. (2.40).

Problem 2.24 Show that the phase-matching bandwidth and the group velocity mismatch are related according to

$$\Delta\lambda \approx 0.44 \frac{\lambda^2}{cL} \frac{1}{\mathrm{GVM}}. \tag{2.41}$$

Problem 2.25 Derive the expression for the acceptance angle for SHG in a uniaxial crystal, in a configuration where the fundamental wave is ordinary and the second harmonic extraordinary. The ordinary (n_o) and extraordinary (n_E) refractive indices are known for both frequencies.

Now we have learned how to calculate the salient parameters of any second-order frequency conversion arrangement: effective nonlinear susceptibility, output power, phase matching angle, group velocity mismatch, walk-off, acceptance angle, phase-matching bandwidth. This knowledge comes in handy when one is planning to implement such conversion in a laboratory.

Suppose you are setting up an experiment which requires a nonlinear optical arrangement. What to begin with? An obvious starting point is the set of "boundary conditions" imposed by the experiment: wavelength, pulse duration, available input power, requirements for the output power, geometric restrictions. Based on this information, you can go ahead and choose the crystal to work with. To that end, you equip yourself with the *Handbook of nonlinear optical crystals* or a similar book containing crystal data; nowadays a lot of information is also available on the Internet.

The first question you ask yourself in deciding whether a particular crystal is suitable for your experiment is whether it is transparent for the wavelengths involved. This will allow you to select several crystals. For each of these crystals, you proceed by choosing the suitable phase matching configuration.

In order to evaluate the phase matching angle, you require precise data on the crystal's indices of refraction for the relevant wavelengths and temperatures. For some wavelengths and temperatures, this information is provided in the reference literature. If your conditions are different, you may need to interpolate the data you have handy or make use of empirical *Sellmeyer equations* which do this job for you. For example, for BBO at room temperature the Sellmeyer equations are as follows:

$$n_O^2 = 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2 \tag{2.42}$$

$$n_E^2 = 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01516\lambda^2. \tag{2.43}$$

For each possible phase matching configuration, you can determine the effective nonlinear susceptibility.

A high nonlinearity is an important criterion, but not the only one. If you are working with pulsed lasers, you need to make sure your group velocity dispersion is sufficiently small so your output pulse does not smear in time. You also need to check the limitation on the beam width imposed by the walk-off angle and the acceptance angle. All these considerations will determine the limit on the length of your crystal. Once the length is decided, you can estimate the output power you can expect in the chosen configuration. If the answer is satisfactory, you may want to look into some original research literature to learn about the experiences of other scientists who used a similar crystal. Finally, you can go ahead and order the crystal from a manufacturer.

Let us now practice this procedure (except for the last step) for a few different settings.

Problem 2.26 For SHG in a BBO crystal (see discussion in the beginning of the section) of length $L = 5$ mm, perform the following calculations.

a) Find the group velocity mismatch.

b) Find the FWHM of the phase matching band (in units of wavelength).

c) Find the acceptance angle. What limitation does it impose on the beam width?

d) Find the walk-off angle for both beams. What limitation does it impose on the beam width?

e) The power of the fundamental continuous-wave beam is $P = 100$ mW. Estimate the power of generated second harmonic assuming optimal beam geometry if $d_{\text{eff}} = 1.4$ pm/V.

Problem 2.27 The optimal beam configuration according to Boyd and Kleinman (page 23) implies that the beam should be focused into the crystal with a certain angular aperture, which is proportional to $L^{-1/2}$ (check this!). On the other hand, the phase-matching acceptance angle is proportional to L^{-1}. Therefore, for the crystal length exceeding a certain critical value, the focusing strength will be limited by phase matching rather than the optimal intensity. Estimate this critical length for the SHG in BBO in the setting of Problem 2.26.

Problem 2.28 Lithium triborate (LBO) is a biaxial crystal with the following parameters:

λ, μm	n_x	n_y	n_z
0.8	1.570	1.596	1.611
0.4	1.590	1.619	1.636

You need to generate the second harmonic of laser light at wavelength $\lambda = 800$ nm. Analyze all three principal planes of this crystal and determine all possible Type I phase matching configurations in terms of polarization combinations and beam angles (θ and φ).

Note 2.9 Second-order nonlinear processes in which the two lower energy waves have the same polarization (e.g. both ordinary or both extraordinary) are called *Type I* processes. If the polarizations are orthogonal, the nonlinear process is *Type II*.

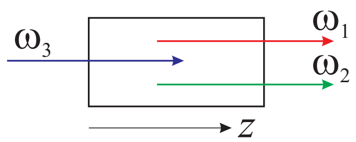


Figure 2.11: Spontaneous parametric down-conversion. The sum of energies of the generated photons must equal the energy of input photons, i.e. $\omega_3 = \omega_1 + \omega_2$.

A very important second-order nonlinear process which we have not yet discussed is *(spontaneous) parametric down-conversion (PDC or SPDC)*. In PDC, photons comprising a *pump* laser beam propagating through a nonlinear crystal can split into pairs of photons of lower energy. The quantum nature of PDC is inverse to that of SFG (Fig. 2.11), but unlike SFG, PDC is not possible within the framework of classical theory of light we used so far. To see this, let us recall our discussion in Sec. 2.1: classical nonlinear response occurs at frequencies that are linear combinations of the frequencies of the input fields. If we pump our crystal at frequency ω_3, we can expect emission at frequencies zero, $2\omega_3$, $3\omega_3$, etc., but not, say, $\omega_3/2$.

Parametric down-conversion is entirely due to the quantum nature of light. As we shall see in Sec. 4, the quantum description of an electromagnetic field mode resembles that of the harmonic oscillator. Even if a mode is in the vacuum state (zero photons), its electric and magnetic field amplitudes do not completely vanish. Such "vacuum amplitude" is present in all field modes at all frequencies. If the frequency of the mode is less than that of the pump field, it may enter into nonlinear interaction with the pump, resulting in difference-frequency emission, accompanied by the amplification of the light in that mode. This manifests itself as emission of photon pairs.

Similarly to to other nonlinear processes, effective PDC requires phase matching. That is, strong emission occurs into pairs of modes that, in addition to $\omega_3 = \omega_1 + \omega_2$, also have $\vec{k}_3 = \vec{k}_1 + \vec{k}_2$. If $\omega_1 = \omega_2$ or $\vec{k}_1 = \vec{k}_2$, PDC is called, respectively, *spectrally* or *spatially degenerate*.

Problem 2.29 You need to implement parametric down-conversion in frequency-degenerate, but spatially non-degenerate type I $e \to o + o$ configuration with a pump wavelength of 532 nm in a

BBO crystal. The three waves are located in the same meridional plane (same φ), but propagate at different polar angles ($\theta_1 < \theta_3 < \theta_2$). The generated photons must be emitted at angle $\theta_2 - \theta_1 = 6°$. Find the polar angle θ_3 of the pump wave vector which enables phase matching for such a process.

Note 2.10 Because the ordinary index of refraction does not change significantly around the direction of the pump, the photons will be emitted not only inside the meridional plane, but along the surface of the *cone* with the axis along the pump wavevector and a 6° apex angle. In each pair, the two photons will be emitted along diametrally opposite lines on the side surface of the cone. This situation is typical in type-I frequency-degenerate PDC settings with emitted photons of ordinary polarization. In other cases the emission typically occurs along the surfaces of two cones which may or may not intersect with each other.

A relatively recent and elegant technique of achieving phase matching in a second-order nonlinear material is *periodic poling*. In a periodically poled crystal, the orientation of the $\chi^{(2)}$ tensor is periodically inverted. Technically this can be achieved in a number of ways, the most common one being ferroelectric domain engineering. A periodic pattern of electrodes is attached to the crystal by means of a photolithographic process, to which a spatially alternating high voltage is applied which causes domain reorientation (Fig. 2.12). This technology has so far been developed only for a few nonlinear materials and there are limitations on the thickness of the crystals that can be manufactured. Furthermore, periodically poled crystals are substantially more expensive than regular ones. All these factors limit the application of periodically poled materials, in spite of their many advantages that we discuss below.

In order to understand the function of periodic poling, let us recall the mechanism of phase mismatch as discussed in Sec. 2.4. The sum frequency wave is generated with phase $k_1 + k_2$ equal to the sum wavevector of the fundamental waves, and propagates with its own wavevector k_3. Because of that, the SFG wavelets generated at different points within the crystal have different phases and may not add constructively.

For example, the wavelet generated at point $z = \pi/\Delta k$ will have phase $(k_1+k_2)\pi/\Delta k$ and the wave that was generated at $z' = 0$ and propagated to $z = \pi/\Delta k$ will have phase $k_3\pi/\Delta k$. The difference of these two phases is π, so the waves interfere destructively.

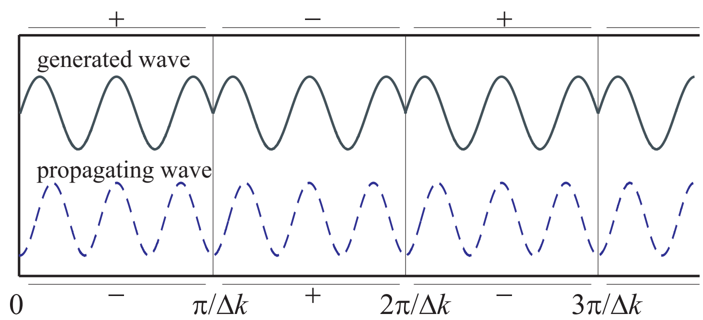


Figure 2.12: Mechanism of quasi phase matching.

Now let us suppose that $z = \pi/\Delta k$ is the point where domain is reversed, so the sum-frequency wavelets generated at z' between $\pi/\Delta k$ and $2\pi/\Delta k$ are shifted in phase by π with respect to those generated between 0 and $\pi/\Delta k$. The mutual phase shift that has accumulated between the two waves within the first domain is now compensated, so the waves again interfere constructively. At point $z = 2\pi/\Delta k$, when another π phase shift has accumulated, the domain is reversed again, and so on. As seen in Fig. 2.12, the two waves remain almost in phase for the entire length of the crystal, thereby exhibiting constructive interference in spite of $\Delta k \neq 0$. This phenomenon is called *quasi phase matching* and the quantity

$$\Lambda = \frac{2\pi}{\Delta k} \tag{2.44}$$

the *poling period*.

Quasi phase matching has many advantages compared to critical phase matching:

- polarizations and direction of the waves can be chosen to utilize the highest tensor element of $\chi^{(2)}$ in the given crystal;
- QPM allows generation of almost any wavelength by choosing the proper combination of the reversal period and crystal temperature;
- weaker dependence on the orientation greatly increases the phase matching bandwidth and the acceptance angle, which permits stronger focusing (*cf.* Problem 2.27).

Note that the phase matching is not perfect throughout the crystal. This is because the phase shift is accumulated continuously, and the domain reversal occurs discretely. Therefore the frequency conversion is not as efficient as in a perfectly phase matched crystal with the same effective susceptibility. In order to quantify the difference, let us solve the following problem.

Problem 2.30 Solve the coupled-wave equation (2.19c) for SHG in the nondepleted regime in a periodically poled nonlinear material. Show that on average the intensity of the generated wave with propagation distance behaves similarly to that in a non-poled material with $\Delta k = 0$ and $d'_{\text{eff}} = \frac{2}{\pi} d_{\text{eff}}$.
Hint: assume that the crystal comprises an integer number of poling periods.

This result is illustrated graphically in Fig. 2.13.

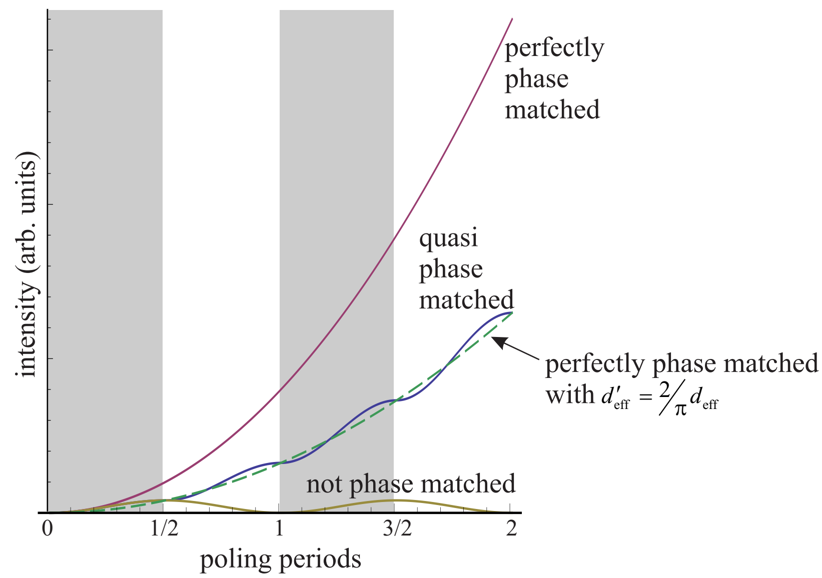


Figure 2.13: The sum-frequency intensity $I_3(z)$ as a function of the propagation distance for $0 \le z < 2\Lambda$. In the same graph, we also plot $I_3(z)$ for the same material with the same Δk, but without periodic poling; the same material, without periodic poling, with perfect phase matching; and another material with perfect phase matching and $d'_{\text{eff}} = (2/\pi) d_{\text{eff}}$.

Problem 2.31 You need to implement parametric down-conversion in periodically poled potassium titanyl phosphate (PPKTP) in (a) Type I and (b) Type II spectrally and spatially degenerate configuration. The pump light is at $\lambda = 532$ nm. The crystal has the following refractive indices:

λ, μm	n_x	n_y	n_z
1.064	1.738	1.745	1.830
0.532	1.778	1.789	1.889

It is also known that the crystal has the following non-vanishing elements of the susceptibility tensor: d_{31} = 1.4pm/V; d_{32} = 2.65pm/V; d_{33} = 10.7pm/V. For both (a) and (b), choose the configuration that would provide the highest effective nonlinearity and find the necessary poling period.

Problem 2.32 You generate second harmonic of continuous-wave laser light at wavelength λ = 1.064 nm in a PPKTP crystal of length L = 5 mm. Light propagates along the x axis and is polarized along the z axis.

a) Find the FWHM of the phase matching band. **Hint:** Eq. (2.41) cannot be used here.

b) Find the acceptance angle.

c) The power of the fundamental beam is P = 100 mW. Find the power of generated second harmonic assuming optimal beam geometry.

Chapter 3

Fundamentals of the coherence theory

3.1 Temporal coherence

Consider an electromagnetic field $E(t)$ whose amplitude is not constant, but drifts randomly in the complex plane. Such a field can be treated as a stochastic process. The primary characteristic of a stochastic process is its *correlation function*

$$\Gamma(t,\tau) = \langle E^{-}(t)E^{+}(t+\tau)\rangle, \tag{3.1}$$

where

$$E^{+}(t) = \int_0^{\infty} E_F(\omega)e^{-i\omega t}\mathrm{d}\omega \tag{3.2}$$

is the positive-frequency part of the field, $E^{-}(t) = (E^{+}(t))^{*}$ and the averaging is done over all possible realizations of the process. We assume that the stochastic process associated with the field is *stationary*, i.e. its correlation function is time-independent[1]: $\Gamma(t,\tau) \equiv \Gamma(\tau)$. The quantity

$$g^{(1)}(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} \tag{3.3}$$

is called the *degree of coherence* or *first-order coherence function.* The *coherence time* of the wave is defined as

$$\tau_c = \sqrt{\frac{\int_{-\infty}^{+\infty}\tau^2|g^{(1)}(\tau)|^2\mathrm{d}\tau}{\int_{-\infty}^{+\infty}|g^{(1)}(\tau)|^2\mathrm{d}\tau}}. \tag{3.4}$$

Problem 3.1 Show that

a) $g^{(1)}(\tau) = g^{(1)}(-\tau)^{*}$;

b) $|g^{(1)}(\tau)| \leq 1$;

Problem 3.2 Calculate the correlation function and the coherence time for the field $E(t) = \mathcal{E}(t)e^{-i\omega_0 t} + c.c.$ whose amplitude $\mathcal{E}(t)$ exhibits random jumps between values $+a$ and $-a$ (*random telegraph signal*). The probability for a jump to occur within time interval $\mathrm{d}t$ is given by $R\mathrm{d}t$, with $R \ll \omega_0$.
Hint: The number of jumps that may occur between moments t and $t+\tau$ follows Poissonian statistics (why?).

[1] This restricts our analysis to continuous fields and excludes pulsed fields.

Problem 3.3 A wave with temporal coherence function $\gamma(\tau)$ enters a fiber Mach-Zehnder interferometer (Fig. 3.1) with temporal path-length difference τ. The interference fringes are observed with a detector; the averaging time is much longer than the coherence time of the wave. Show that the visibility of the interference pattern is given by

$$\mathcal{V}(\tau) = |g^{(1)}(\tau)|. \tag{3.5}$$

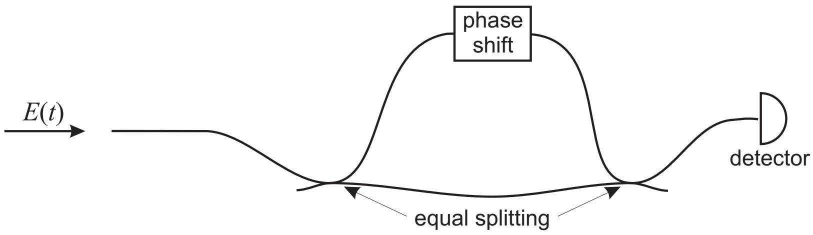


Figure 3.1: Mach-Zehnder fiber interferometer.

Problem 3.4 Verify that

$$\langle E_F^*(\omega) E_F(\omega') \rangle = \Gamma_F(\omega)\delta(\omega - \omega') \tag{3.6}$$

where $\Gamma_F(\omega)$ is the Fourier transform of the correlation function (3.1):

$$\Gamma_F(\omega) = \frac{1}{2\pi}\Gamma(\tau)e^{i\omega\tau}\mathrm{d}\tau. \tag{3.7}$$

Problem 3.5 Consider a bichromatic field $E(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t}$.

a) Find the amplitude spectrum $E_F(\omega)$ as well as the correlation function $\Gamma(\tau)$ and its Fourier image $\Gamma_F(\omega)$ for this field.

b) Verify that the results of part (a) are inconsistent with Eq. (3.6) and explain why this is the case.

Problem 3.6 For an arbitrary stationary field $E(t)$, calculate the intensity I as the absolute value of the Poynting vector averaged over one optical period:

$$I(t) = 2nc\epsilon_0 E^-(t)E^+(t). \tag{3.8}$$

Show that

$$\langle I(t) \rangle = 2nc\epsilon_0 \int\limits_0^\infty \Gamma_F(\omega)\mathrm{d}\omega. \tag{3.9}$$

Note 3.1 The result (3.9) means that the quantity $2nc\epsilon_0\Gamma_F(\omega)$ has the meaning of the spectral density of the electric field energy flow: the differential $2nc\epsilon_0\Gamma_F(\omega)\mathrm{d}\omega$ gives the intensity associated with the wave components whose frequencies lie in the interval between ω and $\omega + \mathrm{d}\omega$. The fact that the spectral power density is proportional to the Fourier transform of the correlation function is known as the *Wiener-Khinchine theorem*. It is of fundamental importance for any application that involves processing of oscillatory signal. For example, the electronic spectrum analyzer displays the spectral power of its input signal within the resolution bandwidth. This quantity is the Fourier transform of the correlation function of the input signal.

The Wiener-Khinchine theorem can also be used in the reverse way, for calculating the optical coherence from the spectral power density. An example is the following exercise.

Problem 3.7 The input to the interferometer of Fig. 3.1 is white light from an incandescent bulb that has passed through a spectral filter of width δ. Assuming the transmission of the filter to be given by $T(\omega) = e^{-(\omega-\omega_0)^2/2\delta^2}$,

a) calculate the interference visibility as a function of the path-length difference;

b) find the coherence time of the interferometer input.

Note 3.2 The result $\tau_c\delta \sim 1$ if the familiar time-energy uncertainty relation for electromagnetic waves.

3.2 Second-order coherence and thermal light

The *second-order coherence function* or *intensity correlation function* is defined as

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}, \tag{3.10}$$

where the instantaneous intensity is given by Eq. (3.8). This quantity is convenient to evaluate experimentally with a single photodiode connected to an oscilloscope which records the photocurrent as a function of time.

Problem 3.8 Show that

a) $g^{(2)}(\tau) \geq 0$;

b) $g^{(2)}(\tau) = g^{(2)}(-\tau)$;

c) $g^{(2)}(0) \geq 1$;

d) $g^{(2)}(\tau) \leq g^{(2)}(0)$;

Note 3.3 The last two properties of the second-order coherence do not remain valid if the light is treated quantum-mechanically. Experiential violations of these properties are viewed as evidence of nonclassical character of a light source.

Problem 3.9 Calculate $g^{(2)}(\tau)$ for perfectly coherent light.

Perfectly *white (chaotic)* light has energy spectrum $\Gamma_F(\omega)$ = const and hence the value of the electric field $E(t)$ at each moment in time is completely uncorrelated with the field at any other moments. Its spectral density is constant.

Perfectly white light is an abstraction. However, some of its properties are well approximated by *thermal* light emitted by a black body in a thermodynamic equilibrium. The energy spectrum of thermal light is given by Planck's formula

$$\Gamma_F(\omega, T) \propto \frac{\omega^3}{e^{\hbar\omega/kT} - 1}, \tag{3.11}$$

where T is the temperature of the source (Fig. 3.2).

Thermal light is obtained due to interference of radiation from multiple partially coherent sources of different frequencies, for example, excited atoms. According to the central limit theorem, this interference results in a Gaussian probability distribution of probabilities to observe a particular value of the field at each moment: $\mathrm{pr}[E(t)] = \exp[-(E(t)/E_0)^2]$.

Now let us suppose that thermal light is subjected to narrow spectral filtering with amplitude transmission $U_F(\omega)$ whose width δ is significantly less than the central frequency ω_0. Because the filter enacts a linear transformation we can write (*cf.* Problem 1.3):

$$E_F(\omega)_{\text{after filter}} = U_F(\omega)E_F(\omega)_{\text{before filter}} \tag{3.12}$$

and thus

$$E(t)_{\text{after filter}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\tau)E_F(t-\tau)_{\text{before filter}}\mathrm{d}\tau, \tag{3.13}$$

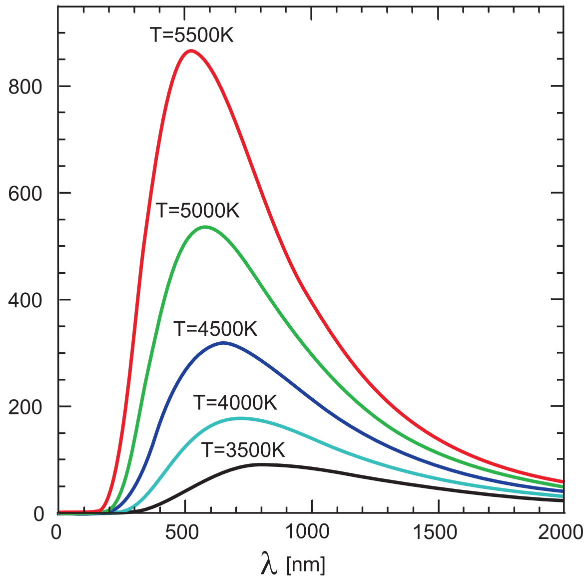


Figure 3.2: Thermal spectrum according to Planck's formula (from *Wikipedia*).

where $U(\tau)$ is the inverse Fourier transform of $U_F(\omega)$. Due to causality, $U(\tau)$ vanishes at $\tau < 0$. In other words, the field after the filter is a linear function of the field before the filter at all preceding moments.

The point of the story is that the left-hand side of Eq. (3.13) is a sum of Gaussian random variables, so it must be a Gaussian variable itself. In addition (see Problem 3.10), the filter introduces correlations between fields at different moments in time. Careful analysis shows that these correlations are also of Gaussian form. This is important because it allows us to apply the *Gaussian moment theorem (Isserlis theorem)*, according to which all ≥ 2 moments of a multivariate Gaussian distribution can be expressed through the second-order moments. In particular,

$$\langle E^-(t_1)E^-(t_2)E^+(t_3)E^+(t_4)\rangle = \langle E^-(t_1)E^+(t_3)\rangle\langle E^-(t_2)E^+(t_4)\rangle + \langle E^-(t_1)E^+(t_4)\rangle\langle E^-(t_2)E^+(t_3)\rangle. \tag{3.14}$$

Problem 3.10 Using Eq. (3.13), express the degree of coherence of the field after the filter through $U(\tau)$. Verify consistency with the Wiener-Khinchine theorem (**Hint:** Remember that $U_F(\omega)$ is the filter transmission for the amplitude, not intensity).

Problem 3.11 Show that the probability distribution of the *intensity* of the thermal field after the filter follows thermal statistics:

$$\mathrm{pr}(I) = \frac{1}{I_0}e^{-I/I_0}. \tag{3.15}$$

Express I_0 through $\Gamma(0)$.
Hint: After a narrow filter, the amplitude of $E^+(t)$ (and $E^-(t)$) is slow varying with respect to one optical period, so it can be assumed constant when using Eq. (3.8). The real and imaginary parts of this amplitude are *independent*, and both of them obey Gaussian statistics.

Problem 3.12 Show that for the thermal light after the filter,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \tag{3.16}$$

Note 3.4 According to Eq. (3.16) the intensity correlation function of thermal light is typically of the form shown in Fig. 3.3(a). At the same time, the intensity must obey thermal statistics (3.15). A typical behavior of the thermal light intensity as a function of time is shown in Fig. 3.3(b). We

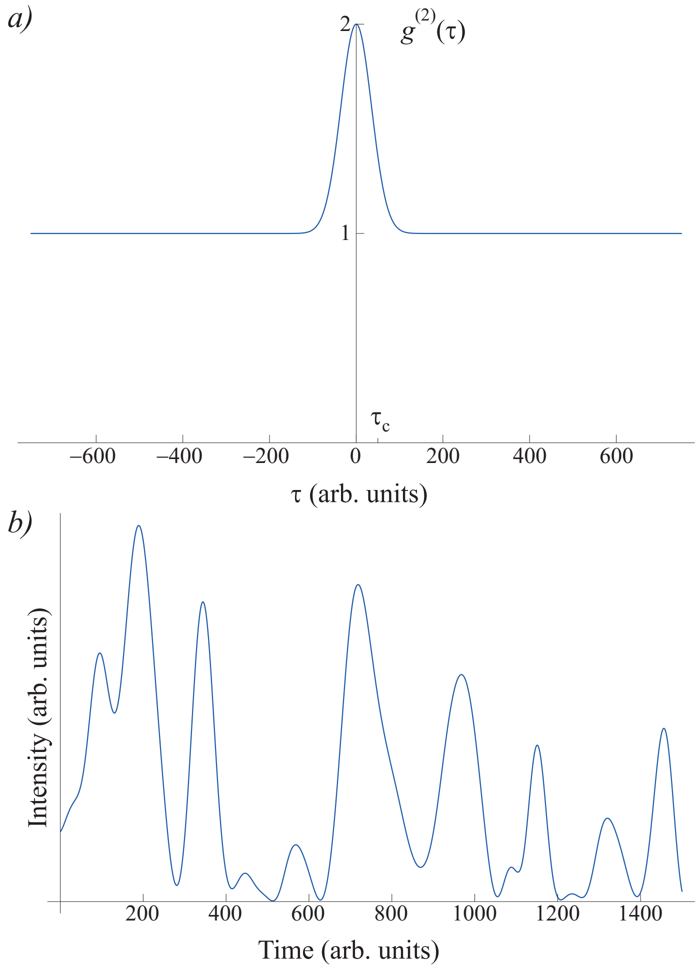


Figure 3.3: Thermal light. a) Intensity correlation function $g^{(2)}(\tau)$. b) Typical behavior of intensity as a function of time. The time scales are the same in (a) and (b).

see that the intensity fluctuates on a time scale of the coherence time. This is because the thermal light obtains as interference of light of multiple sources. Because these sources are incoherent, the interference pattern is *nonstationary*: its lifetime is restricted by the coherence time imposed by the filter.

Problem 3.13 Simulate thermal light on a computer using MatLab or Mathematica.

a) Generate two arrays of 100,000 random numbers from a Gaussian distribution to simulate perfectly chaotic light.

b) Subject these arrays to spectral filtering. To this end, convolve them with a Gaussian function of $1/e$ full width of 100 (the time scale is given by the array index). As we know, convolution corresponds to multiplication in the Fourier domain. Find the $1/e$ width of the corresponding frequency filter. The filtered arrays correspond to the real and imaginary parts of the complex field amplitude.

c) Calculate $g^{(1)}(\tau)$ from your data. Plot it for $-500 \le \tau \le 500$ together with the theoretical prediction obtained from the Wiener-Khinchine theorem.

d) Calculate the intensity $I(t)$. Plot $I(t)$ for any interval of length 3000. Plot the histogram of the intensity and verify its consistency with the thermal distribution in accordance with Problem 3.11.

e) Calculate $g^{(2)}(\tau)$ from your intensity data. Plot it for $-500 \le \tau \le 500$ together with the theoretical prediction obtained from Eq. (3.16).

Problem 3.14 The result shown in Fig. 3.3(a) appears surprising. Indeed, we expect the thermal light for $\tau < \tau_c$ to be indistinguishable from coherent light. Nevertheless, for thermal light we find $g^{(2)}(\tau < \tau_c) = 2$ while for coherent light $g^{(2)}(\tau) \equiv 1$ (Problem 3.9).

How can you explain this discrepancy? Confirm your explanation using the data from Problem 3.13.

3.3 Spatial coherence

The notion of temporal coherence can be straightforwardly generalized to the spatial domain. Instead of Eq. (3.1), for example, we can write

$$\Gamma(\vec{r}_1, \vec{r}_2, t, \tau) = \langle E^-(\vec{r}_1, \vec{r}_2, t) E^+(\vec{r}_1, \vec{r}_2, t+\tau)\rangle, \tag{3.17}$$

where $\vec{r}_1$ and $\vec{r}_2$ denote two spatial locations. For simplicity, when analyzing spatial coherence, we assume that the light is quasimonochromatic so the time dependence can be neglected. Assuming that the light is spatially uniform ("stationary" in space) we can assume that the correlation function depends only on the spatial separation $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. Therefore, $\Gamma(\vec{r}_1, \vec{r}_2, t, \tau) \equiv \Gamma(\Delta\vec{r})$. The properties of the spatial correlation function are largely similar to those of its temporal counterpart.

Furthermore, it is convenient to assume that all components of the optical wave propagate at a small angle with respect to a specific direction, which we define as the z axis. In this *paraxial approximation*, we have $k_x, k_y \ll k_z$ and therefore the absolute value of the wavevector

$$k = c\sqrt{k_z^2 + k_x^2 + k_y^2} \approx k_z + \frac{k_x^2}{2k_z} + \frac{k_y^2}{2k_z} \tag{3.18}$$

is to the first order independent of its transverse component.

In the paraxial approximation, we can define the Fourier transform

$$E^+(\vec{r}) = e^{ikz} \int E_{\vec{k}_\perp} e^{i\vec{k}_\perp \cdot \vec{r}_\perp} \mathrm{d}^2\vec{k}_\perp, \tag{3.19}$$

where $\vec{r}_\perp = (x, y)$ and $\vec{k}_\perp = (k_x, k_y)$. Similarly to the Wiener-Khinchine theorem for temporal case, we have, under the condition of spatial uniformity,

$$E^*_{\vec{k}_\perp} E_{\vec{k}'_\perp} = \Gamma_{\vec{k}_\perp} \delta^2(\vec{k}_\perp - \vec{k}'_\perp) \tag{3.20}$$

where $\Gamma_{\vec{k}_\perp}$ is the Fourier image of the spatial correlation function and the energy flow density associated with the direction defined by $k_\perp$. This result is called the *van Cittert-Zernike theorem.* It can be used, for example, to measure diameters of stars.

Although each atom of a star emits a spherical wave, an observer on the Earth can assume it to be a plane wave (see Problem 3.18), so we can associate each point on the surface of the star with a specific wave vector $\vec{k}_\perp$. The angular size of the star determines the directional energy flow density $I_{\vec{k}_\perp} = 2c\epsilon_0\Gamma_{\vec{k}_\perp}$, which is related, through the Fourier transform, to the experimentally measurable correlation function $\Gamma(\Delta\vec{r}_\perp)$ (Fig. 3.4(a)).

Because the light emitted by stars obeys thermal statistics, one can also measure their diameter by looking at intensity correlations rather than direct interferometry. For these we have, similarly to the previous section,

$$g^{(2)}(\Delta\vec{r}) = \frac{\langle I(\vec{r}) I(\vec{r} + \Delta\vec{r})\rangle}{\langle I(\vec{r})\rangle\langle I(\vec{r} + \Delta\vec{r})\rangle} = 1 + |g^{(1)}(\Delta\vec{r})|^2. \tag{3.21}$$

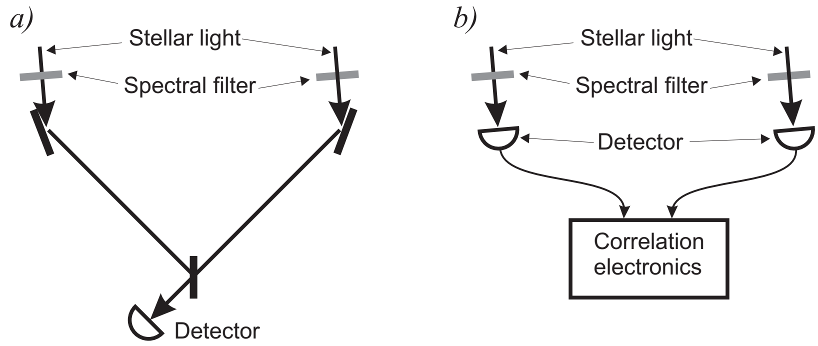


Figure 3.4: Measuring stellar diameter via (a) classic interferometry; (b) intensity (Hanbury Brown-Twiss) interferometry.

The measurement is performed by observing correlations between photocurrents from two spatially separated detectors (*Hanbury Brown-Twiss experiment* — Fig. 3.4(b)). In fact, Hanbury Brown-Twiss measurements are preferred to classic interferometry because they are robust with respect to atmospheric fluctuations and easier to implement technically.

Problem 3.15 The course web site shows the image of a speckle pattern observed on an office wall when a green laser pointer of wavelength of wavelength $\lambda = 532$ nm was pointed at the opposite wall. The width of the image is $d = 12$ cm, the distance between walls $L = 4$ m.

a) Download the image and plot the histogram of the intensity (take only the green component into account). Explain possible reasons for deviations from the expected exponential shape.

b) Estimate the beam diameter of the laser pointer. **Hint:** Rather than calculating the intensity correlation function directly from the image data, use the Fast Fourier Transform to determine $I_{\vec{k}_\perp}$. Square its absolute value to find $\Gamma_{\vec{k}_\perp}$.

Problem 3.16 From the data shown in Fig. 3.5, estimate the diameter of Sirius given that its distance from the Earth is 8.6 light years. The wavelength can be assumed to be $\lambda = 320$ nm.

Problem 3.17 While we assumed the light to be monochromatic, a realistic monochromator has a finite linewidth.

a) What requirement does this bandwidth impose on the maximum allowed deviation of the position of the star from zenith in the measurement shown in Fig. 3.4(a)? Assume that the two mirrors are in the same horizontal plane and the two interferometer arms have the same length.

b) What requirement does this bandwidth impose on the response time of the photodetector in the Hanbury Brown-Twiss experiment? Explain your answer.

Problem 3.18 For Problem 3.16, estimate whether the plane wave approximation of a spherical wave emitted from each element of the star surface is valid. **Hint:** The difference between the wavefronts must be much less than a wavelength.

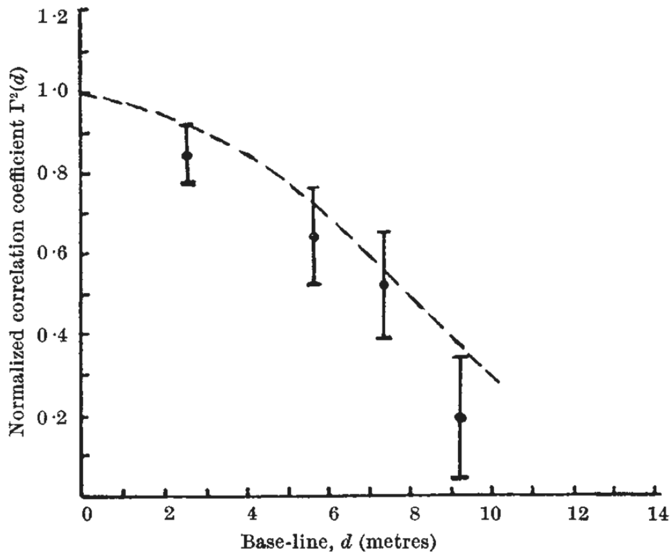


Figure 3.5: Experimental intensity correlation data for Hanbury Brown—Twiss interferometry of Sirius [From R. Hanbury Brown and R. Q. Twiss, Nature **178**, 1046 (1956)].

Chapter 4

Quantum light

4.1 Quantization of the electromagnetic field

The next two chapters are dedicated to quantum technology of optical fields: their creation, manipulation and characterization. We begin by describing the procedure of quantization of the electromagnetic field. Precise treatment of this matter requires delving deep into the intricacies of quantum electrodynamics, which is beyond the scope of this text. Therefore we offer a somewhat simplified version of this derivation.

The main principle behind the quantization of the electromagnetic field is to observe that the mathematical description of its evolution in time very similar to that of a harmonic oscillator. Some features of this analogy are self-evident: the field in electromagnetic wave does oscillate, and, just as in a mechanical oscillator, the energy quantum of a field mode oscillating at frequency ω is a photon of energy $\hbar\omega$. Our goal is to extend this analogy deeper, to the level of the Hamiltonian and the canonical equations of motion.

Let us consider electromagnetic field in a large evacuated box with volume $V = L_x \times L_y \times L_z$. The field inside the box can be decomposed into Fourier series

$$\vec{E}(\vec{r},t) = \sum_{\vec{k},s} e^{i\vec{k}\vec{r}} \vec{u}_{\vec{k},s}(t) + c.c; \tag{4.1a}$$

$$\vec{B}(\vec{r},t) = \sum_{\vec{k},s} e^{i\vec{k}\vec{r}} \vec{w}_{\vec{k},s}(t) + c.c, \tag{4.1b}$$

where $s = 1,2$ denotes one of the two possible polarization directions, $\vec{u}_{\vec{k},s}(t)$ and $\vec{w}_{\vec{k},s}(t)$ are the time-dependent amplitudes of the electric and magnetic fields, respectively, for a given set $(\vec{k},s)$ [which defines a *field (plane wave) mode*].

Any arbitrary electromagnetic field configuration inside the box can be decomposed in the form (4.1). Because the box is of a finite size, it is sufficient to use a series over a discrete set of wavevectors $\vec{k} = (2\pi n_x/L_x, 2\pi n_y/L_y, 2\pi n_z/L_z)$ (where n_x, n_y, n_z are arbitrary natural numbers), rather than an integral.

Before we proceed with quantizing, let us perform an auxiliary calculation.

Problem 4.1 a) Show that $|w_{\vec{k},s}(t)| = |u_{\vec{k},s}(t)|/c$.

b) Given that the electromagnetic energy density in vacuum equals $(\epsilon_0 E^2 + B^2/\mu_0)/2$, show that the total energy of the electromagnetic field in the box is given by

$$H_{\text{total}} = 2\epsilon_0 V \sum_{\vec{k},s} |u_{\vec{k},s}(t)|^2. \tag{4.2}$$

The latter result is important: it shows that the total energy (i.e., the Hamiltonian) of the field is a sum of energies in individual modes. This means that the motion of each mode is not influenced by others, and hence can be considered separately. This is what we will do. In what follows, we will omit subscripts $\vec{k}, s$ keeping in mind that we are dealing with only one specific mode.

The Hamiltonian of a mode is

$$H = 2\epsilon_0 V u^\star(t) u(t). \tag{4.3}$$

In order to quantize the field, we need to define, in terms of $u(t)$, the *position* x and *momentum* p that satisfy the classical *canonical equations of motion*

$$\dot{p} = -\frac{\partial H}{\partial x}; \tag{4.4a}$$

$$\dot{x} = \frac{\partial H}{\partial p}. \tag{4.4b}$$

Once we succeed, we follow the steps used in transition from classical to quantum mechanics. We say that x and p behave like the canonical position and momentum of a mechanical particle, and hence they can be replaced by operators whose commutator is equal to $[\hat{x}, \hat{p}] = i\hbar$.

The logic here is different from that one follows when solving a problem in mechanics. In the latter case, one starts with well-defined x and p and solves the canonical equations to find the evolution of these variables with time. Here, we already know the evolution of the system (i.e. that $u(t) = u(0)e^{-i\omega t}$ with $\omega_{\vec{k}} = c|k|$) and we use this knowledge to define x and p. The last thing we need to do before we answer the above question is to introduce a convention. In quantum optics, the position and momentum are defined a bit differently compared to mechanics, namely, $X = x/\sqrt{\hbar}, P = p/\sqrt{\hbar}$. Accordingly, the canonical equations of motion take the form

$$\dot{P} = -\frac{1}{\hbar}\frac{\partial H}{\partial X}; \tag{4.5a}$$

$$\dot{X} = \frac{1}{\hbar}\frac{\partial H}{\partial P}. \tag{4.5b}$$

and the commutator $[\hat{X}, \hat{P}] = i$.

Problem 4.2 Obtain Eqs. (4.5) from (4.4).

Problem 4.3 Show that, if we define

$$X(t) = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega}}\frac{u(t) + u^*(t)}{\sqrt{2}} \tag{4.6a}$$

$$P(t) = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega}}\frac{u(t) - u^*(t)}{\sqrt{2}i} \tag{4.6b}$$

(where $\omega_{\vec{k}} = ck$ and the $\vec{u}$'s are treated as scalars) then

a) the Hamiltonian takes the form

$$H_{\vec{k},s} = \frac{\hbar\omega}{2}[X(t)^2 + P(t)^2]; \tag{4.7}$$

b) the canonical equations (4.5) are satisfied.

We will now treat the position and momentum associated with an electromagnetic mode as quantum operators with $[\hat{X}, \hat{P}] = i$. We see that the quantum treatment of light in vacuum is identical to that of a harmonic oscillator. In the following sections, we rederive the basic properties of important quantum states of a harmonic oscillator that are known from undergraduate quantum mechanics.

Initially, we will be working in the Schrödinger picture. This means that the operators are assumed time-independent $[\hat{X} \equiv \hat{X}(t=0), \hat{P} \equiv \hat{P}(t=0)]$, while quantum states are time-dependent. Later (in Sec. 4.7) we will switch to the Heisenberg picture, which utilizes the opposite convention.

The position and momentum operators have eigenstates, $|X\rangle$ and $|P\rangle$, respectively, which are related according to[1]

$$\langle X|\,X'\rangle = \delta(X - X'); \tag{4.8a}$$
$$\langle P|\,P'\rangle = \delta(P - P'). \tag{4.8b}$$

The position and momentum eigenstates are related to each other via the *de Broglie wave*:

$$\langle X|\,P\rangle = \frac{1}{\sqrt{2\pi}} e^{iPX} \tag{4.9}$$

and thus, for an arbitrary state $|\psi\rangle$

$$\langle X|\,\psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \langle P|\,\psi\rangle\, e^{iPX} \mathrm{d}P; \tag{4.10a}$$
$$\langle P|\,\psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \langle X|\,\psi\rangle\, e^{-iPX} \mathrm{d}X. \tag{4.10b}$$

In other words, the wave functions of a given state $|\psi\rangle$ in the position and momentum representations are direct and inverse Fourier transforms of each other.

Problem 4.4 Show that

$$\langle X|\,\hat{P}\,|\psi\rangle = -i\frac{d}{dX}\psi(X); \quad \langle P|\,\hat{X}\,|\psi\rangle = i\frac{d}{dP}\psi(P). \tag{4.11}$$

Hint: $\hat{X} = \int_{-\infty}^{+\infty} X\,|X\rangle\langle X|\,\mathrm{d}X$; $\hat{P} = \int_{-\infty}^{+\infty} P\,|P\rangle\langle P|\,\mathrm{d}P$.

Problem 4.5 The *annihilation* operator is defined as follows:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right); \tag{4.12}$$

The operator $\hat{a}^\dagger$ is called the *creation* operator. Show that:

a)
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right); \tag{4.13}$$

b) the creation and annihilation operators are not Hermitian;

c) their commutator is
$$[\hat{a}, \hat{a}^\dagger] = 1; \tag{4.14}$$

d) position and momentum can be expressed as
$$\hat{X} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right); \quad \hat{P} = \frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right); \tag{4.15}$$

e) the Hamiltonian can be written as
$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right); \tag{4.16}$$

f) the following commutator relations hold:
$$[\hat{a}, \hat{a}^\dagger\hat{a}] = \hat{a}; \quad [\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] = -\hat{a}^\dagger. \tag{4.17}$$

[1] Because these states are not normalizable, they, strictly speaking, belong not to the Hilbert space, but to the so-called *rigged Hilbert space*. See R. de la Madrid, Eur. J. Phys. **26**, 287-312 (2005) for details.

4.2 Fock states

Our next goal is to find the eigenvalues and eigenstates of the Hamiltonian. Because of Eq. (4.16) the latter are also eigenstates of $\hat{a}^\dagger\hat{a}$.

Problem 4.6 Suppose some state $|n\rangle$ is an eigenstate of the operator $\hat{n} = \hat{a}^\dagger\hat{a}$ [called the *(photon) number operator*] with eigenvalue n. Then

a) the state $\hat{a}|n\rangle$ is also an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n-1$;

b) the state $\hat{a}^\dagger|n\rangle$ is also an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n+1$.

Hint: Use Eq. (4.17).

The above exercise shows that the states $\hat{a}|n\rangle$ and $\hat{a}^\dagger|n\rangle$ are proportional to normalized states $|n-1\rangle$ and $|n+1\rangle$, respectively. In the following, we find the proportionality coefficient.

Problem 4.7 Using $\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$, show that

a)
$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle\,; \tag{4.18}$$

b)
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,; \tag{4.19}$$

We found that, unless $n=0$, if the state $|n\rangle$ with energy $\hbar\omega(n+1/2)$ exists (i.e. is an element of the Hilbert space), so does the state $|n-1\rangle$ with energy $\hbar\omega(n-1/2)$. Similarly, states $|n-2\rangle$, $|n-3\rangle$ etc. must also exist. On the other hand, negative energy states are not allowed. The only way to resolve this contradiction is to assume that n *must be nonnegative integer* so the chain is broken at $n=0$ (in which case $\hat{a}|0\rangle = |\text{zero}\rangle$). Furthermore, if we prove the existence of the state $|n=0\rangle$, we automatically prove the existence of all higher energy states because

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{4.20}$$

Energy eigenstates of a harmonic oscillator are called *Fock* or *number* states. The state $|0\rangle$ is called the *vacuum state*. From the above results, we conclude that the Fock states have energy $\hbar\omega(n+1/2)$, where n is a nonnegative integer.

Problem 4.8 Using $\hat{a}|0\rangle = 0$, calculate the wavefunction of the vacuum state in the position and momentum representations. **Hint:** use Eq. (4.11).
Answer:
$$\psi_0(X) = \frac{1}{\pi^{1/4}}e^{-X^2/2}; \quad \tilde{\psi}_0(P) = \frac{1}{\pi^{1/4}}e^{-P^2/2}. \tag{4.21}$$

Energy eigenvalues are nondegenerate. Indeed, as we show in Problem 4.8, the equation $\hat{a}|0\rangle = 0$ has only one normalizable solution, therefore there is only one vacuum state. The higher energy Fock states are obtained from the vacuum state by applying the creation operator, and hence are also unique. As eigenstates of a Hermitian operator, Fock states form a basis.

Problem 4.9 a) By applying Eq. (4.20), calculate the wavefunctions of Fock states $|1\rangle$ and $|2\rangle$.

b)* Calculate the wavefunction of an arbitrary Fock state $|n\rangle$.

Answer:
$$\psi_n(X) = \frac{H_n(X)}{\pi^{1/4}\sqrt{2^n n!}}e^{-X^2/2}, \tag{4.22}$$

where $H_n(X)$ are the Hermite polynomials.

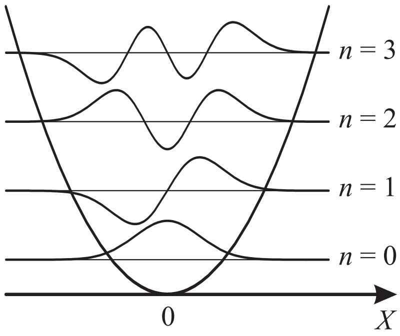


Figure 4.1: Wavefunctions of the first three energy levels of a harmonic oscillator.

Problem 4.10 For an arbitrary $|n\rangle$, calculate $\langle X\rangle$, $\langle \Delta X^2\rangle$, $\langle P\rangle$, $\langle \Delta P^2\rangle$ and verify the uncertainty principle. (**Hint:** do *not* use wavefunctions!)

Note 4.1 Heisenberg's uncertainty principle states that for any state we must have $\left\langle \Delta X^2\right\rangle \left\langle \Delta P^2\right\rangle \geq 1/4$. From the above calculation we see that the vacuum state is a minimum-uncertainty state, and it is the only such state of all Fock states.

Problem 4.11 Find the evolution of the state $\alpha|0\rangle + \beta|1\rangle$; calculate the time dependence of $\langle X\rangle$, $\langle P\rangle$ and plot the trajectory in the phase space[2].

4.3 Coherent states

A *coherent (Glauber) state* $|\alpha\rangle$ is an eigenstate of the annihilation operator with eigenvalue α (known as the coherent state's *amplitude*:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{4.23}$$

Coherent states comprise an important family of optical states, as these, under some approximations, are the states emitted by lasers. Accordingly, their properties for large amplitudes are quite reminiscent of classical light.

We begin our study of coherent states by calculating their energy distribution.

Problem 4.12 Find the decomposition of the coherent state into the number basis (make sure your answer is properly normalized).
Answer:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{4.24}$$

Note 4.2 This shows that there exists a coherent state for each complex α.

If one performs an energy measurement on a coherent state, the probability to project onto a particular Fock state is given by the *Poissonian distribution* (Fig. 4.2):

$$\mathrm{pr}_n = |\langle n|\ \alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{4.25}$$

Problem 4.13 Find $\langle n\rangle$, $\langle \Delta n^2\rangle$ for a coherent state.
Answer: $\langle n\rangle = \langle \Delta n^2\rangle = |\alpha|^2$.

[2]The *phase space* is the two-dimensional space whose axes are X and P.

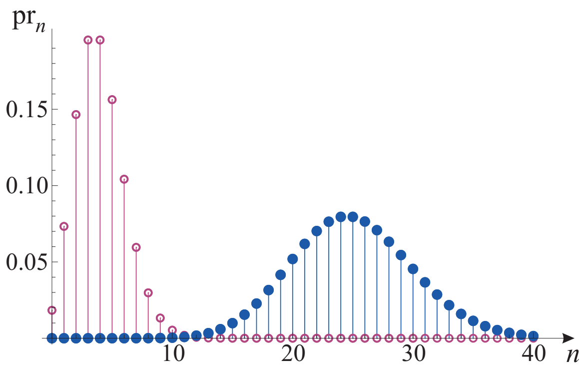


Figure 4.2: the Poisson distribution with $\langle n \rangle = 4$ (empty circles) and $\langle n \rangle = 25$ (filled circles).

We see that, in the coherent state, the standard deviation of the number of photons equals the square root of the mean photon number: $\sqrt{\langle \Delta n^2 \rangle} = \sqrt{\langle n \rangle}$. The relative uncertainty is then $\sqrt{\langle \Delta n^2 \rangle}/\langle n \rangle = 1/\sqrt{\langle n \rangle}$.

Consider a pulsed laser. Even if we maintain its temperature, pump energy, etc., perfectly constant, we will see that the pulse energy varies from pulse to pulse. This phenomenon is known as the *shot noise*. For example, in a laser pulse with one million photons on average, the uncertainty is 1000 photons, which is 1/1000 of the mean photon numbers. Thus, in this macroscopic pulsed emission, the energy per pulse is almost constant. The quantum shot noise is negligible, so the behavior is largely classical. On the other hand, if we attenuate this pulse to an average of one photon, the uncertainty will also become one photon. Although the absolute value of this uncertainty is smaller, the relative per-pulse photon number variation, associated with the quantum nature of light, is now significant.

Problem 4.14 Show that in the limit of large α, the Poissonian distribution approaches Gaussian.

Problem 4.15 Find the overlap $\langle \alpha | \, \alpha' \rangle$.
Answer:

$$\langle \alpha | \, \alpha' \rangle = e^{-|\alpha|^2/2 - |\alpha'|^2/2 + \alpha'^* \alpha}. \tag{4.26}$$

This result shows that coherent states associated with different eigenvalues of the annihilation operator ar not orthogonal. This is not surprising if we remember that, according to linear algebra, only eigenstates of a Hermitian operator must be orthogonal, and the annihilation operator is not Hermitian. On the other hand, a thorough analysis shows that coherent states form a spanning set in the optical Hilbert space: any optical state can be written as a linear combination of coherent states. This spanning set is *overcomplete*: the decomposition of a given state into coherent states is not unique.

Problem 4.16 Do eigenstates of the creation operator exist and if so, what is their decomposition into the number basis?

Let us now calculate and investigate the wavefunctions of coherent states.

Problem 4.17 Find the wavefunctions of the coherent state in the position and momentum basis. Find $\langle X \rangle$, $\langle \Delta X^2 \rangle$, $\langle P \rangle$, $\langle \Delta P^2 \rangle$.
Answer:

$$\psi_{|\alpha\rangle}(X) = \psi_0(X - X_0) e^{iP_0 X} \text{ with } \frac{X_0 + iP_0}{\sqrt{2}} = \alpha. \tag{4.27}$$

Note 4.3 In coherent states, similarly to the vacuum state, the position-momentum uncertainty product takes the minimum possible value.

Problem 4.18 Find the action of the evolution operator $\exp(-i\hat{H}t/\hbar)$ upon the state $\hat{\alpha}$. Find $\langle X\rangle$ and $\langle P\rangle$ as functions of time. Plot the trajectory in the phase space.
Answer: A coherent state evolves into another coherent state with a different eigenvalue: $\exp(i\hat{H}t/\hbar)\left|\alpha\right\rangle = e^{-i\omega t/2}\left|e^{-i\omega t}\alpha\right\rangle$.

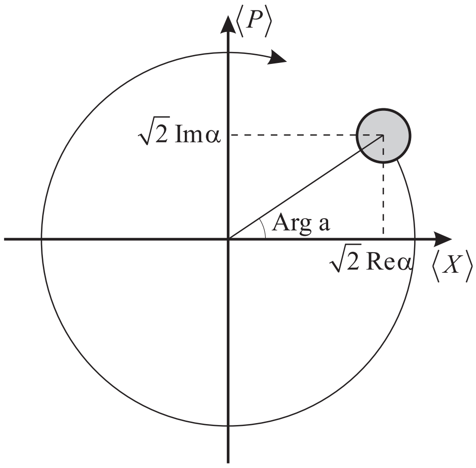


Figure 4.3: Time evolution of the position and momentum observables' expectation values in the coherent state. The circle represents the position and momentum uncertainties.

These results again illustrate the classical analogy of large-amplitude coherent states. Fig. 4.3 shows the expectation values of X and P in the phase space, along with their uncertainties and evolution. This evolution is identical to that expected from a classical field, except that a classical field would exhibit no uncertainties. If the amplitude of the coherent state is macroscopic, the relative uncertainties are negligible, so the coherent state is well approximated by classical light. In contrast, for microscopic amplitudes, the uncertainties' role becomes significant and the classical approximation fails.

4.4 Wigner function

Consider a classical particle that is prepared multiple times with random values of position X and momentum P that follow a certain probability distribution — the *phase-space probability density* $W(X,P)$. The probability that the particle will have some certain values of position and momentum (with some tolerances) is proportional to $W(X,P)$.

Suppose now, that every time the particle is prepared, we measure the observable

$$\hat{X}_\theta \equiv \hat{X}\cos\theta + \hat{P}\sin\theta. \tag{4.28}$$

Based on the data obtained in this measurement, we can construct a *histogram* of the experimental results, a.k.a. *marginal distribution* $\mathrm{pr}(X_\theta)$. This marginal distribution is the integral projection of the phase-space probability density on the vertical plane oriented at angle θ with respect to the vertical axis (Fig. 4.4):

$$\mathrm{pr}(X_\theta) = \int_{-\infty}^{+\infty} W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta)\mathrm{d}P. \tag{4.29}$$

Now let us suppose the same measurement of $\hat{X}_\theta$ is performed on a quantum particle in state with density matrix $\hat{\rho}$ that is prepared anew after each measurement. Again, we can construct the histogram, which is related to the state according to

$$\mathrm{pr}(X_\theta) = \left\langle X_\theta\right|\hat{\rho}\left|X_\theta\right\rangle, \tag{4.30}$$

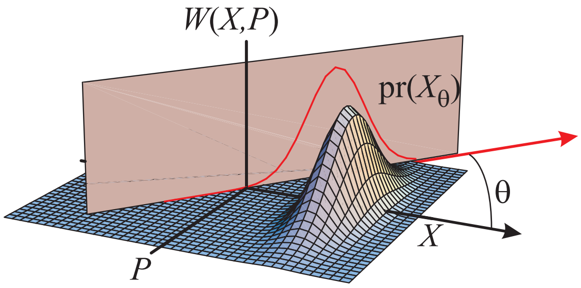


Figure 4.4: The phase-space probability density and the marginal distribution.

where $|X_\theta\rangle$ is an eigenstate of $\hat{X}_\theta$. In the quantum domain, there can exist no phase-space probability density because, according to the uncertainty principle, the particle cannot have certain values of position and momentum at the same time. However, for every quantum state there exists a phase-space *quasi*probability density — a function $W_{\hat\rho}(X,P)$ for which Eq. (4.29) holds for all θ's. Without derivation, the expression for this quasiprobability density (now known as the *Wigner function*) is as follows:

$$W_{\hat\rho}(X,P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|\hat\rho\left|X+\frac{Q}{2}\right\rangle \mathrm{d}Q. \tag{4.31}$$

This equation is called the *Wigner formula*.

Problem 4.19 Prove Eq. (4.29) for the classical case.

Problem 4.20 Use a mathematical software package to calculate the Wigner function and verify that Eq. (4.29) holds for $\theta = 0$ and $\theta = \pi/2$ for the following states:

a) vacuum state;

b) coherent state with $\alpha = 2$;

c) coherent state with $\alpha = 2e^{i\pi/6}$;

d) the single-photon state;

e) the ten-photon state;

f) $(|0\rangle + |1\rangle)/\sqrt{2}$;

g) $(|0\rangle + i\,|1\rangle)/\sqrt{2}$;

h) state with the density matrix $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$;

i) squeezed vacuum state $\psi_r(X) = \frac{1}{\pi^{1/4}\sqrt{r}}e^{-x^2/2r^2}$ with $r = 2$;

j) Schrödinger cat state $|\alpha\rangle \pm |-\alpha\rangle$ with $\alpha = 3$ and $\alpha = 1$ (please also calculate the norm for these states);

k) state with the density matrix $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$ with $\alpha = 3$.

Problem 4.21 Verify the following properties of the Wigner function:

a)

$$\iint\limits_{-\infty}^{+\infty} W_{\hat\rho}(X,P)\mathrm{d}X\mathrm{d}P = 1; \tag{4.32}$$

b)

$$W_{(\alpha\hat\rho_1+\beta\hat\rho_2)}(X,P)\mathrm{d}X\mathrm{d}P = \alpha W_{\hat\rho_1}(X,P) + \beta W_{\hat\rho_2}(X,P); \tag{4.33}$$

c) the state is uniquely defined by its Wigner function;

d) Wigner function of a state is real.

Note 4.4 We see that the quantum phase-space quasiprobability density inherits most of its properties from its classical counterpart. One difference of the quantum case is that the Wigner function is allowed to take on negative values (e.g. with the one-photon state). This is because the Wigner function no longer has the meaning of a probability distribution. The marginals of the Wigner function, however, do have a meaning of measurable probability densities even in the quantum case. So whenever the Wigner function has a negative "well", it must be surrounded by a positive "hill" so all its projections are nonnegative.

Note 4.5 One can define the Wigner function for any operator, rather than just density operators, by analogy with Eq. (4.31). However, the Wigner function of an arbitrary operator is not necessarily real and normalized.

Problem 4.22 Show that for any two operators $\hat{A}$ and $\hat{B}$,

$$\mathrm{Tr}(\hat{A}\hat{B}) = 2\pi \iint_{-\infty}^{+\infty} W_{\hat{A}}(X,P) W_{\hat{B}}(X,P) \mathrm{d}X \mathrm{d}P \tag{4.34}$$

Note 4.6 An important consequence of the above result is that, if we know the Wigner function of a state, we can calculate its density matrix in any basis. For example, in the Fock basis:

$$\rho_{mn} = \mathrm{Tr}(\hat{\rho}\,|m\rangle\langle n|) = 2\pi \iint_{-\infty}^{+\infty} W_{\hat{\rho}}(X,P) W_{|m\rangle\langle n|}(X,P) \mathrm{d}X \mathrm{d}P, \tag{4.35}$$

where $W_{|m\rangle\langle n|}(X,P)$, i.e. the Wigner function of the operator $|m\rangle\langle n|$, is easily calculated.

Problem 4.23 Show that

$$\forall X, P, \hat{\rho} : W_{\hat{\rho}}(X,P) \leq 1/\pi. \tag{4.36}$$

Hint: Develop the proof for pure states first. Use the Wigner formula and the Cauchy-Schwarz inequality. In order to generalize the result to mixed states, use Eq. (4.33).

A few interesting results can be obtained by studying the Wigner function of the so-called *parity operator*, defined as

$$\hat{\Pi}|X\rangle = |-X\rangle \tag{4.37}$$

Problem 4.24 Show that

a)

$$W_{\hat{\rho}}(0,0) = \frac{1}{\pi}\mathrm{Tr}(\hat{\rho}\hat{\Pi}), \tag{4.38}$$

b)

$$\hat{\Pi} = (-1)^{\hat{n}} \tag{4.39}$$

Hint: Calculate the wave function of the state $\hat{\Pi}\,|n\rangle$ in the position basis.

Note 4.7 This result is widely employed in experimental physics in state tomography. After measuring the photon number statistics $\mathrm{pr}(n)$ of a quantum state, one finds the mean value of the parity operator $\langle\hat{\Pi}\rangle = \sum_{n=0}^{\infty}(-1)^n \mathrm{pr}(n)$ and thus the value of the Wigner function at the phase space origin. In order to determine the Wigner function at other points, one needs to perform the same measurement after applying the phase space displacement operator (see below) to the state being examined.

Note 4.8 It follows from Eq. (4.38) that for number states $W_{|n\rangle\langle n|} = (-1)^n/\pi$, which is consistent with Eq. (4.36).

4.5 Other phase-space distributions

The *Husimi function* (*Q function*) of state $\hat{\rho}$ is the convolution of the Wigner function of that state with the vacuum state Wigner function.

$$Q_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} W_{\hat{\rho}}(X',P') W_{|0\rangle\langle 0|}(X'-X,P'-P)\,\mathrm{d}X'\mathrm{d}P' \tag{4.40}$$

Problem 4.25 Prove the following properties of the Q function:

a)

$$Q_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} W_{\hat{\rho}}(X',P') W_{|\alpha\rangle\langle\alpha|}(X',P')\,\mathrm{d}X'\mathrm{d}P' \text{ with } \alpha = \frac{X+iP}{\sqrt{2}}; \tag{4.41}$$

b)

$$Q_{\hat{\rho}}(X,P) = \frac{1}{2\pi}\langle\alpha|\,\hat{\rho}\,|\alpha\rangle \text{ with } \alpha = \frac{X+iP}{\sqrt{2}}; \tag{4.42}$$

[**Hint:** use Eq. (4.34)]

c)

$$Q_{\hat{\rho}}(X,P) \geq 0; \tag{4.43}$$

d)

$$\iint_{-\infty}^{+\infty} Q_{\hat{\rho}}(X,P)\mathrm{d}X\mathrm{d}P = 1. \tag{4.44}$$

Problem 4.26 Calculate the Q function of the single-photon state. Compare it with the Wigner function of an equal mixture of the single photon and vacuum.

Note 4.9 This result can be generalized: the Wigner function of a quantum state that has undergone a 50% loss is identical to the state's rescaled Q function.

The *Glauber-Sudarshan function* (*P function*) of state $\hat{\rho}$ is the *de*convolution of the Wigner function of that state with the vacuum state Wigner function.

$$W_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} P_{\hat{\rho}}(X',P') W_{|0\rangle\langle 0|}(X-X',P-P')\,\mathrm{d}X'\mathrm{d}P' \tag{4.45}$$

Note 4.10 For many states, the P function only exists as a highly singular generalized function. However, any state can be approximated, with arbitrary high fidelity, with a state that has a regular P function[3].

Problem 4.27 a) Show that the P function of a coherent state $|\alpha\rangle$ is

$$P_{|\alpha\rangle\langle\alpha|}(X',P') = \delta(X'-X,P'-P) \tag{4.46}$$

with $\alpha = (X+iP)/\sqrt{2}$.

b) Calculate the Fourier transform of the P function of the squeezed vacuum state with $\psi_r(X) = \pi^{-1/4}r^{-1/2}e^{-x^2/2r^2}$. Verify that the P function of this state does not exist as a regular function. **Hint:** Fourier transformation transforms a convolution into a product.

Problem 4.28 Prove the *optical equivalence theorem:*

$$\hat{\rho} = \iint_{-\infty}^{+\infty} P_{\hat{\rho}}(X',P')\,|\alpha\rangle\langle\alpha|\,\mathrm{d}X'\mathrm{d}P' \text{ with } \alpha = \frac{X'+iP'}{\sqrt{2}}. \tag{4.47}$$

Hint: Possible steps towards the solution are as follows:

[3] J. R. Klauder, Phys. Rev. Lett. **16**, 534 (1966).

a) express both sides of Eq. (4.47) in the coordinate basis, i.e. calculate $\langle X_1 | \hat{\rho} | X_2 \rangle$;

b) substitute the Wigner formula into the left-hand side of Eq. (4.45) and apply the inverse Fourier transform with respect to P to both sides of that equation;

c) verify that the results of (a) and (b) are identical.

Problem 4.29 It may appear from the optical equivalence theorem and Note 4.10 that any quantum state can be arbitrarily well approximated by a statistical mixture of coherent states. Is this correct?

4.6 Nonclassicality criteria

A quantum state is called *classical* if it can be written as a coherent state or their statistical mixture. Otherwise it is called *nonclassical.* As follows from the optical equivalence theorem, classicality of a state is equivalent to its P function being positive definite[4].

If we know a quantum state exactly, we can find out if it is nonclassical by a simple calculation. In an experiment, however, only partial information about a state is often available. In such cases one can use *nonclassicality criteria* — sufficient, but not necessary conditions of nonclassical character of a state that can be verified by a measurement that is simpler than full state tomography.

Problem 4.30 The *squeezing* criterion states that a state is nonclassical if its quadrature variance for at least one phase is below that of the vacuum state (*standard quantum limit*): $\langle \Delta X_\theta^2 \rangle < 1/2$.

a) Verify that classical states do not satisfy the squeezing criterion.
Hint: Remember the intuitive meaning of a statistical mixture of states: each term in the mixture occurs with some probability. Show that the criterion is not satisfied by each term in the mixture, and then argue that it cannot be satisfied by the entire mixture.

b) Give an example of a state that does not satisfy the squeezing criterion, yet is nonclassical.

The *antibunching* criterion states that a state is nonclassical if its second-order coherence function is less then one:

$$g^{(2)}(0) < 1 \tag{4.48}$$

(*cf.* Problem 3.8). It is implied that each moment in time is associated with an electromagnetic mode, and all these modes have identical quantum states. In determining $g^{(2)}$, the so-called *normal ordering* of operators is used, i.e. all creation operators must precede all annihilation operators. Treating the intensity as a quantum operator, we write $I(t) \propto \hat{a}^\dagger(t)\,\hat{a}(t)$ and thus the classical formula (3.10) takes the following form in the normal order:

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\,\hat{a}^\dagger(t+\tau)\,\hat{a}(t)\,\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\,\hat{a}(t)\rangle^2}, \tag{4.49}$$

where averaging is in the quantum sense.

An equivalent criterion is the negativity of the *Mandel parameter*

$$Q = \frac{\langle \Delta n^2 \rangle - \langle n \rangle}{\langle n \rangle} < 0. \tag{4.50}$$

Problem 4.31 For the antibunching and Mandel parameter criteria,

a) verify that these criteria are equivalent;

b) verify that these criteria are not satisfied by classical states;

c) give an example of a state that does not satisfy these criteria, yet is nonclassical.

[4] A simple delta function is considered positive definite for the purposes of this analysis.

Problem 4.32 Show that any state that has zero probability to contain exactly n photons ($\langle n|\,\hat{\rho}\,|n\rangle = 0$), for any $n \neq 0$, is nonclassical.

Note 4.11 This implies that any state with a finite number of terms in its decomposition into the Fock basis is nonclassical.

Problem 4.33 Another important nonclassicality criterion is the state's Wigner function taking on negative values somewhere in the phase space.

a) Verify that the Wigner function negativity criterion is not satisfied by classical states;

b) Give an example of a state that does not satisfy this criterion, yet is nonclassical.

Problem 4.34 Consider the following criterion: the state is nonclassical if the probability for it to contain an odd number of photons is higher than 1/2. Show that any state that satisfies this condition also satisfies the Wigner function negativity criterion.

Problem 4.35 Which of the states in Problem 4.20 are nonclassical?

Problem 4.36 The *Vogel criterion*[5] is as follows: the state is nonclassical if the Fourier transform of any of its marginal distributions decays slower than that of the vacuum state, i.e.

$$\exists\theta, k_\theta : \ \mathrm{pr}_F(k_\theta) > e^{-k_\theta^2/4}, \tag{4.51}$$

where k_θ is the Fourier variable.

a) Prove the Vogel criterion.

b)* Verify that all nonclassical states in Problem 4.20 satisfy this criterion.

c)* Find a nonclassical state that does not satisfy the Vogel criterion.

Problem 4.37 The *thermal state* is the state with Boltzmann photon number statistics

$$\hat{\rho} = (1 - e^{-\beta}) \sum_{n=0}^{\infty} |n\rangle\langle n|\, e^{-\beta n} \tag{4.52}$$

where $\beta = \hbar\omega/k_B T$ for k_B being the Boltzmann constant and T the temperature associated with the state.

a) Calculate $\langle n\rangle$ for the thermal state and verify it to obey the Bose statistics.

b) Calculate $g^{(2)}(\tau)$ for the thermal state for $\tau = 0$ and $\tau \gg \tau_c$ (i.e. the moments t and $t + \tau$ representing separate electromagnetic modes).

4.7 A few important operators

In this section, we will employ the Heisenberg picture of quantum evolution, which assumes that all operators evolve according to

$$\hat{A}(t) = e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t} \tag{4.53}$$

while all quantum states stay constant. This is in contrast to the Schrödinger picture, which assumes that quantum states evolve:

$$|\psi(t)\rangle = e^{-i(\hat{H}/\hbar)t}\,|\psi_0\rangle\,, \tag{4.54}$$

while operators are constant.

Problem 4.38 For the Heisenberg picture,

[5] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000).

a) show that the behavior of operator expectation values $\langle\psi|\, A\, |\psi\rangle$ as a function of time is the same as in the Schrödinger picture;

b) show that the operator evolution can be written in the form

$$\partial_t \hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}(t)]. \tag{4.55}$$

Note 4.12 The right-hand sign of the differential equation for the evolution of the density operator in the Schrödinger picture has the opposite sign compared to Eq. (4.55): $\partial_t \hat{\rho}(t) = -i/\hbar[\hat{H}, \hat{\rho}(t)]$.

Problem 4.39 Consider a classical mechanical harmonic oscillator with the Hamiltonian $\hat{H} = \hat{p}^2/2m + \kappa\hat{x}^2/2$. Show that the Heisenberg equations of motion (4.55) for the position and momentum is exactly the same as the classical (Hamiltonian) equations of motion (4.4) under Hamiltonian equations.

Note 4.13 If the Hamiltonian contains only terms up to the second order in the position and momentum, one can show[6] that each point in the phase space evolves according to the classical equations of motion. So if the position and momentum transform according to $(\hat{X}(t), \hat{P}(t)) = F_t(\hat{X}_0, \hat{P}_0)$, (where $F_t(\cdot,\cdot)$ is an affine transformation), the Wigner function evolves in an intuitive fashion, i.e. as follows:

$$W(X, P, t) = W(F_t^{-1}(X, P), 0). \tag{4.56}$$

Problem 4.40 The *phase-space displacement* operator is given by

$$\hat{D}(X_0, P_0) = \exp(iP_0\hat{X} - iX_0\hat{P}), \tag{4.57}$$

where X_0, P_0 are real numbers.

a) White the Hamiltonian $\hat{H}$ such that the evolution $e^{-i(\hat{H}/\hbar)t_0}$ under $\hat{H}$ for time t_0 is equal to $\hat{D}(X_0, P_0)$.

b) Write the differential Heisenberg equations for the position and momentum operators under $\hat{H}$ and verify that the action of this Hamiltonian results in the transformation $\hat{X} \to \hat{X} + X_0$, $\hat{P} \to \hat{P} + P_0$.

Note 4.14 The displacement operator transforms the vacuum state into a coherent state $|\alpha\rangle$ with $\alpha = (X_0 + iP_0)/\sqrt{2}$.

Problem 4.41 The optical *phase-shift operator* is given by

$$\hat{U}(\varphi) = \exp(-i\varphi\hat{n}), \tag{4.58}$$

where φ is a real number.

a) Using the same approach as in the previous problem, show that the phase shift transforms the field operators as follows:

$$\hat{a} \to \hat{a}e^{-i\varphi} \tag{4.59}$$
$$\hat{a}^\dagger \to \hat{a}^\dagger e^{i\varphi} \tag{4.60}$$

b) Show that applying the phase shift operator leads to clockwise rotation of the phase space by angle φ around the origin point.

Note 4.15 The optical phase-shift operator is not the same as the quantum phase shift, given by multiplication of the state by a phase factor $e^{-i\varphi}$. The former, in contrast to the latter, has observable physical meaning. For Fock states, however, the action of the optical phase shift is equivalent to multiplication by phase factor $e^{-i\varphi n}$, i.e. does not bring about any observable modification of the state. This is the reason why Fock states have an undefined optical phase and their Wigner functions are axially symmetric.

[6]See, for example, W. Schleich, *Quantum Optics in Phase Space* (Wiley, 2001), Sec. 3.3.

Problem 4.42 The *single-mode (position) squeezing* operator is given by

$$\hat{S}(\zeta) = \exp[\zeta(\hat{a}^2 - \hat{a}^{\dagger 2})/2], \tag{4.61}$$

where ζ is a real number.

a) Show that the squeezing operator is associated with the following transformation:

$$\begin{aligned} \hat{a} &\to \hat{a}\cosh\zeta - \hat{a}^\dagger \sinh\zeta && (4.62) \\ \hat{a}^\dagger &\to \hat{a}^\dagger \cosh\zeta - \hat{a}\sinh\zeta && (4.63) \\ \hat{X} &\to \hat{X}e^{-\zeta} && (4.64) \\ \hat{P} &\to \hat{P}e^{\zeta}. && (4.65) \end{aligned}$$

b) Calculate the Wigner function of the squeezed vacuum state $\hat{S}(\zeta)|0\rangle$.

Problem 4.43 Consider the evolution under the squeezing Hamiltonian

$$\hat{H} = i\alpha[\hat{a}^2 - (\hat{a}^\dagger)^2]/2 \tag{4.66}$$

with a real, positive α in the Schrödinger picture. Verify that the wavefunction

$$\psi(X) = \frac{1}{\pi^{1/4}\sqrt{r}} e^{-X^2/2r^2} \tag{4.67}$$

with $r = e^{-\alpha t}$ satisfies the Schrödinger equation.

Problem 4.44 The *two-mode squeezing* operator, acting on two electromagnetic modes with operators $\hat{a}_1, \hat{a}_2$ is given by

$$\hat{S}_2(\zeta) = \exp[\zeta(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger)], \tag{4.68}$$

where ζ is a real number.

a) Show that this operator is associated with the following transformation:

$$\begin{aligned} \hat{a}_1 &\to \hat{a}_1\cosh\zeta - \hat{a}_2^\dagger \sinh\zeta; && (4.69) \\ \hat{a}_2 &\to \hat{a}_2\cosh\zeta - \hat{a}_1^\dagger \sinh\zeta; && (4.70) \\ \hat{X}_1 \pm \hat{X}_2 &\to (\hat{X}_1 \pm \hat{X}_2)e^{\mp\zeta}; && (4.71) \\ \hat{P}_1 \pm \hat{P}_2 &\to (\hat{P}_1 \pm \hat{P}_2)e^{\pm\zeta}. && (4.72) \\ & && (4.73) \end{aligned}$$

b) Calculate the Wigner function of the two-mode squeezed vacuum state $\hat{S}_2(\zeta)|0,0\rangle$. **Hint:** write the Wigner function in variables $(\hat{X}_1 \pm \hat{X}_2)/\sqrt{2}, (\hat{P}_1 \pm \hat{P}_2)/\sqrt{2}$.

Note 4.16 The mode operator transformation given by Eqs. (4.62), (4.63) or Eqs. (4.69), (4.70) is called the *Bogoliubov* transformation.

Note 4.17 Single-and two-mode squeezed vacuum states are generated, respectively, by degenerate and non-degenerate spontaneous parametric down-conversion. One can show[7], via a relatively lengthy calculation, that the decomposition of the two-mode vacuum state into the Fock basis is

$$\left\langle n_1, n_2 \right| \hat{S}_2(\zeta) \left| 0, 0 \right\rangle = \frac{\delta_{n_1 n_2}}{\cosh\zeta}(\tanh\zeta)^{n_1}, \tag{4.74}$$

where $\delta_{n_1 n_2}$ is the Kronecker symbol, which accounts for the fact that the down-conversion photons are always generated in pairs.

[7] See, for example, W. Vogel, D.-G. Welsch, *Quantum Optics* (Wiley, 2006), Sec. 3.3.

Problem 4.45 Let us consider the two-mode squeezed state in the extreme case $\zeta \to \infty$. In this case we obtain the original Einstein-Podolsky-Rosen (EPR) state[8]. Suppose the modes $\hat{a}_1$ and $\hat{a}_2$ are given to two observers, Alice and Bob. Based on the results of Problem 4.44, visualize the Wigner function of the EPR state and answer the following questions.

a) Suppose Alice performs a measurement of her mode's position and obtains some result X_0. Onto which state will Bob's particle project?

b) Suppose Alice instead performs a measurement of her mode's momentum and obtains some result P_0. Onto which state will Bob's particle project?

Note 4.18 We see that by choosing to measure in the position or momentum basis, Alice can create one of two mutually incompatible physical realities associated with Bob's mode [(certain position, uncertain momentum) or (certain momentum, uncertain position)].

4.8 The beam splitter

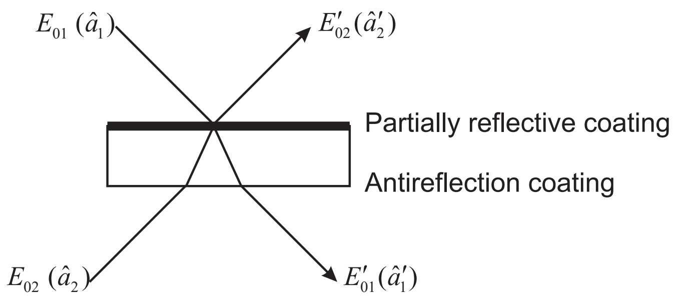


Figure 4.5: The beam splitter.

Consider two waves with amplitudes E_{01} and E_{02} overlapped on a beam splitter in matching optical modes (i.e. the reflected wave 1 is in the same mode as the transmitted wave 2 and vice versa) and generating waves E'_{01} and E'_{02}. Because the beam splitter is a linear optical device, the transformation experienced by the waves is also linear:

$$\begin{pmatrix} E'^{(+)}_{01} \\ E'^{(+)}_{02} \end{pmatrix} = \underline{B} \begin{pmatrix} E^{(+)}_{01} \\ E^{(+)}_{02} \end{pmatrix}, \tag{4.75}$$

where

$$\underline{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{4.76}$$

is a 2×2 matrix.

Problem 4.46 Show that $\underline{B}$ is a unitary matrix.

Problem 4.47 Show that any unitary 2×2 matrix can be written in the form

$$\underline{B} = e^{i\Lambda/2} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \tag{4.77}$$

where all the parameters are real numbers.

Note 4.19 In Eq. (4.77),

[8]A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).

- the first factor corresponds to a common optical (not quantum!) phase shift;
- the second factor corresponds to equal but opposite phase shifts of the output fields;
- the third factor corresponds to transmission and reflection with $r^2 + t^2$;
- the second factor corresponds to equal but opposite phase shifts of the input fields.

We see that the third factor is the only one that brings about the mixing of the modes. The remaining factors enact single-mode phase shifts and are hereafter neglected:

$$\underline{B} = \begin{pmatrix} t & -r \\ r & t \end{pmatrix}, \tag{4.78}$$

The quantities t^2 and r^2 are, respectively, the intensity *transmissivity* and *reflectivity* of the beam splitter.

If we switch to the quantum domain, the transformation of the wave amplitudes is replaced by the transformation of the field annihilation operators in the Heisenberg picture, i.e. Eq. (4.75) takes the form

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \underline{B} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}. \tag{4.79}$$

Problem 4.48 Using rule (4.56) for the Winger function transformation in the Heisenberg picture show that

a) overlapping the two modes of a two-mode squeezed vacuum state on a symmetric ($r = t = 1/\sqrt{2}$) beam splitter will result in a separable state of two single-mode squeezed vacua with the same degree of squeezing, one of which is position-squeezed and the other momentum-squeezed;

b) the reverse statement is also valid.

Our next task is to write the beam splitter transformation in the Fock basis, i.e. calculate the beam splitter output $\hat{U}|n_1, n_2\rangle$ if two Fock states $|n\rangle_1$, $|n\rangle_2$ are present in the input. This problem must be solved in the Schrödinger picture, whereas all our calculations so far were performed in the Heisenberg picture.

In order to switch to the Schrödinger picture, let us treat the beam splitter transformation as evolution $\hat{U} = e^{-i(\hat{H}/\hbar)t}$ that takes place under fictitious Hamiltonian $\hat{H}$ for time t. Note that matrix $\underline{B}$ is not identical to $\hat{U}$, because the form of Eq. (4.79) is clearly different from Eq. (4.53). However, we can rely on these two equations to write

$$\hat{a}_1' = \hat{U}^\dagger \hat{a}_1 \hat{U} = t\hat{a}_1 - r\hat{a}_2; \tag{4.80a}$$
$$\hat{a}_2' = \hat{U}^\dagger \hat{a}_2 \hat{U} = r\hat{a}_1 + t\hat{a}_2. \tag{4.80b}$$

Problem 4.49 Show that the beam splitter (4.78) can be associated with the Hamiltonian

$$\hat{H}_{\rm BS} = \hbar\Omega(i\hat{a}_1\hat{a}_2^\dagger - i\hat{a}_1^\dagger\hat{a}_2), \tag{4.81}$$

with $\sin\Omega t = r$.

Note 4.20 The interpretation of Hamiltonian (4.81) is that the beam splitter exchanges photons between modes 1 and 2, but preserves the total number of photons.

Problem 4.50 Calculate the Fock decomposition of the beam splitter output with input $|n_1, n_2\rangle$ as described below.

a) Show that

$$\langle m_1, m_2|\hat{U}|n_1, n_2\rangle = \frac{1}{\sqrt{m_1! m_2!}} \langle 0, 0|\hat{U}^\dagger (\hat{a}_1)^{m_1} (\hat{a}_2)^{m_2} \hat{U}|n_1, n_2\rangle. \tag{4.82}$$

Hint: use the fact that the beam splitter acting upon the double vacuum state generates double vacuum: $\hat{U}|0, 0\rangle = |0, 0\rangle$

b) Use Eqs. (4.80) to verify the following:

$$\begin{aligned}&\langle m_1, m_2|\hat{U}|n_1, n_2\rangle \\ &= \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\delta_{k_1+k_2,n_1}\delta_{n_1+n_2,m_2+m_2}\frac{(-1)^{m_1-k_1}\sqrt{n_1!n_2!m_1!m_2!}}{k_1!(m_1-k_1)!k_2!(m_2-k_2)!}t^{m_2+k_1-k_2}r^{m_1-k_1+k_2}.\end{aligned} \tag{4.83}$$

Problem 4.51 Show that if one of the beam splitter inputs is in the vacuum state, Eq. (4.83) takes a simple form

$$\hat{U}|n,0\rangle = \sum_{k=0}^{n} A_{nk}|n-k,k\rangle, \tag{4.84}$$

where

$$A_{nk} = \sqrt{\begin{pmatrix} n \\ k \end{pmatrix}} t^{n-k} r^k \tag{4.85}$$

Verify that the probability to detect $(k, n-k)$ photons in the two outputs could be calculated by treating photons as classical particles that have probabilities t^2 and r^2, respectively, to be transmitted or reflected.

Problem 4.52 Calculate $\hat{U}|1,1\rangle$. Verify the *Hong-Ou-Mandel effect*: for a symmetric beam splitter, the probability to detect one photon in each output vanishes.

4.9 The beam splitter model of absorption

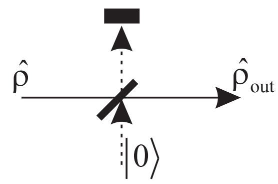


Figure 4.6: The beam splitter model of absorption.

Problem 4.53 The *beam splitter model of absorption* (Fig. 4.6) provides the means to determine the quantum state after it has undergone optical loss. The model consists of replacing the lossy channel with a beam splitter that has the same transmissivity and the vacuum state entering its other channel. The reflected mode of the beam splitter is assumed to be lost.

a) Show that if a quantum state with density matrix

$$\hat{\rho} = \sum_{m,n} \rho_{mn}|m\rangle\langle n| \tag{4.86}$$

will generate two-mode output with density matrix

$$\hat{\mathcal{P}} = \sum_{m,n}\sum_{j=0}^{m}\sum_{k=0}^{n}\rho_{mn}A_{mj}A_{nk}|m-j,j\rangle\langle n-k,k|. \tag{4.87}$$

b) By taking the partial trace over the reflected mode, calculate the density matrix of the transmitted mode:

$$\hat{\rho}_{\text{out}} = \text{Tr}_2\hat{\mathcal{P}} = \sum_{m,n}\sum_{k=0}^{\min(m,n)}\rho_{mn}A_{mk}A_{nk}|m-k\rangle\langle n-k|. \tag{4.88}$$

Note 4.21 Result (4.88) is called the *generalized Bernoulli transformation.*

Problem 4.54 Show that two coherent states with amplitudes α and β entering the beam splitter will produce two coherent states with amplitudes $B_{11}\alpha + B_{12}\beta$ and $B_{21}\alpha + B_{22}\beta$. **Hint:** Show that the state $\hat{U}|\alpha\rangle|\beta\rangle$ is an eigenstate of both $\hat{a}_1$ and $\hat{a}_2$.

Problem 4.55 Show that in order to obtain a nonclassical state at a beam splitter output, the input state must also be nonclassical.

Problem 4.56 Show that a coherent state $|\alpha\rangle$, after propagation through a loss channel with transmissivity t^2, becomes $|t\alpha\rangle$ (neglecting the phase shift).

Problem 4.57 Show that the Glauber-Sudarshan function of a state that has propagated through a loss channel with transmissivity t^2, rescales as follows:

$$P_{\text{out}}(X,P) = \frac{1}{t^2}P_{\text{in}}\left(\frac{X}{t},\frac{P}{t}\right). \tag{4.89}$$

Hint: use the optical equivalence theorem.

Problem 4.58 Show that the Wigner function of a state that has propagated through a loss channel with transmissivity t^2, transforms as follows:

$$W_{\text{out}}(X,P) = \frac{1}{\pi t^2(1-t^2)}W_{\text{in}}\left(\frac{X}{t},\frac{P}{t}\right) * \exp\left(-\frac{X^2+P^2}{1-t^2}\right). \tag{4.90}$$

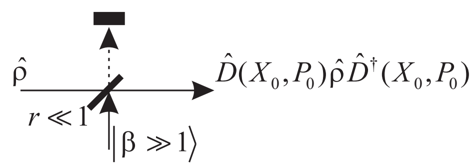


Figure 4.7: Implementation of the phase-space displacement operator with a beam splitter and a coherent state.

Note 4.22 The latter result can be interpreted as follows. The loss leads to "shrinkage" of the initial phase space, i.e. each point transforms according to $X \to tX, P \to tP$. In addition, random noise of variance $(1-t^2)/2$ (the second factor in the convolution) is added to both quadratures due to admixture of the reflected portion of the vacuum state.

Note 4.23 If $t^2 = 1/2$, Eq. (4.90) confirms the statement of Note 4.9.

Problem 4.59 Show that the phase-space displacement operator can be implemented as shown in Fig. 4.7, i.e. by transmitting the quantum state through a low-reflectivity beam splitter and shining a strong coherent state $|\beta\rangle$ into the other input channel. Show that the displacement is given by $(X_0 + iP_0)/\sqrt{2} = -\beta r$.
Hint: Calculate the P function of the output state.

4.10 Homodyne tomography

Consider a task of determining the complete information about a quantum state $\hat{\rho}$ of an optical mode by making measurements on multiple available copies of this state. To accomplish this task, it is not sufficient to perform measurements, however multiple, in a single basis, e.g. the position basis. Such measurements will provide us only with a set of probabilities $\text{pr}(X) = \langle X|\,\hat{\rho}\,|X\rangle$ (i.e. the diagonal elements of the density matrix in the position basis), but no information about the phases of the wave function (the off-diagonal elements). In order to fully reconstruct the state, we need to perform measurements in multiple bases, acquiring sufficient statistics in each basis. This procedure is called *quantum state tomography*. Its general principles are applicable to any quantum system, not necessarily optical.

Problem 4.60 Show that

$$\mathrm{pr}_{F,\theta}(k_X) = 2\pi W_F(k_X \cos\theta, k_X \sin\theta), \tag{4.91}$$

where $W_F(k_X, k_P)$ is the Fourier transform of the Wigner function $W(X,P)$ of a state $\hat{\rho}$ and $\mathrm{pr}_{F,\theta}(k_X)$ is the Fourier transform of its marginal $\mathrm{pr}_\theta(X)$.

Note 4.24 We see that if we can measure the histograms $\mathrm{pr}_\theta(X)$ for all θ's, we can reconstruct the state's Wigner function and thus its density matrix (see Problem 4.21). In order to implement this in practice, we need to learn how to measure the field quadrature $X_\theta = X\cos\theta + P\sin\theta$.

This requires phase-sensitive measurement of the electromagnetic field amplitude. Given that the light field oscillates at hundreds of Terahertz, i.e. much faster than the bandwidth of any electronic circuit, such measurements cannot be performed directly. Instead, one utilizes an interference-based scheme known as *(balanced) homodyne detection* (Fig. 4.8). Here a signal mode, defined by annihilation operator $\hat{a}$ (whose state we need to measure) is overlapped with a *local oscillator* mode $\hat{a}_{\mathrm{LO}}$ containing a high-amplitude coherent state $|\alpha_{\mathrm{LO}}\rangle$ on a symmetric beam splitter. The two beam splitter output channels are detected by highly-efficient photodiodes, so each photon is converted into a photoelectron. The photocurrents are integrated in time, so the total number of photoelectrons produced by each diode corresponds to a measurement of the photon number operator $\hat{n}_1$ and $\hat{n}_2$ in the respective mode. Finally, the measurements are subtracted from each other, yielding $\hat{n}_- = \hat{n}_1 - \hat{n}_2$.

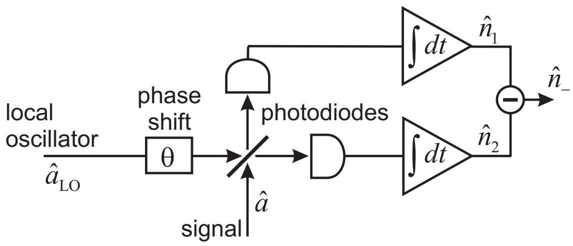


Figure 4.8: Balanced homodyne detection.

Problem 4.61 Show that

$$\hat{n}_- = \hat{a}_{\mathrm{LO}}^\dagger \hat{a} e^{i\theta} + \hat{a}_{\mathrm{LO}} \hat{a}^\dagger e^{-i\theta}, \tag{4.92}$$

where θ is the relative optical phase between the signal and the local oscillator.

Problem 4.62 One can show[9] that, for the local oscillator in a high-amplitude coherent state, the annihilation operator of the local oscillator can be approximated by its eigenvalue: $\hat{\alpha} \to \alpha_{\mathrm{LO}}$ and thus

$$\hat{n}_- = \alpha_{\mathrm{LO}}^* \hat{a} e^{i\theta} + \alpha_{\mathrm{LO}} \hat{a}^\dagger e^{-i\theta}. \tag{4.93}$$

Assuming α_{LO} to be real, we find that the homodyne detector measures the observable that is proportional to $\hat{X}_\theta$.

Note 4.25 We see from Eq. (4.93) that the subtraction charge scales as the local oscillator amplitude, or the square root of the local oscillator intensity:

$$n_- \sim \alpha_{\mathrm{LO}} = \sqrt{N_{\mathrm{LO}}}, \tag{4.94}$$

where N_{LO} is the number of photons in the local oscillator. This is a macroscopic quantity, so homodyne tomography does not require detectors with single-photon sensitivity.

[9] W. Vogel, D.-G. Welsch, *Quantum Optics* (Wiley, 2006), Sec. 6.5.4.

Problem 4.63 Suppose the beam splitter is slightly asymmetric, i.e. $t^2 - r^2 = \varepsilon \ll 1$.

a) Write the approximate expression for $\hat{n}_-$ in this situation.

b) Show that proper functioning of the detector requires $\varepsilon \ll N_{\rm LO}^{-1/2}$.

Problem 4.64 Realistic photodiodes are imperfect: they convert each photon into a photoelectron with non-unitary probability η (this quantity is called the detector's *quantum efficiency*). Such a photodiode can be modeled by a perfect photodiode preceded by an attenuator of intensity transmissivity η [Fig. 4.9(a)].

a) Write the expression for $\hat{n}_-$.

b) Show that this imperfect detector is equivalent to a detector in which the photodiodes are perfect, but the signal and local oscillator are both attenuated before the beam splitter [Fig. 4.9(b)].

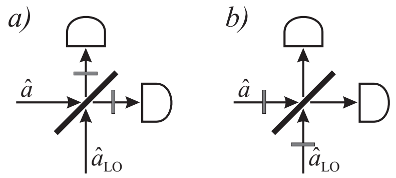


Figure 4.9: Illustration to Problem 4.64. The schemes in (a) and (b) are equivalent if both attenuators have the same transmissivity η.

So far, we have been dealing with plane-wave electromagnetic modes. This treatment is easier mathematically, but largely unpractical because plane waves have infinite extent in space and time. In order to study practically relevant modes, we remember that, according to the rules of field quantization,

$$\hat{E}(\vec{r},t) = \sum_j \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}\hat{a}_j e^{i\vec{k}_j\vec{r}-i\omega_j t} + H.c, \tag{4.95}$$

where we neglected the vector character of the field. As evident from this equation, a field of any arbitrary spatiotemporal shape can be obtained, using Fourier decomposition, as a linear combination of plane-wave modes:

$$\hat{A} = \sum_j \beta_j \hat{a}_j. \tag{4.96}$$

If we require that $\sum_j |\beta_j|^2 = 1$, then we have $[\hat{A},\hat{A}^\dagger] = 1$ so annihilation operator $\hat{A}$ defines a valid optical mode.

In order to formalize this procedure, we can define a unitary transformation

$$\hat{A}_i = \sum_j U_{ij}\hat{a}_j \tag{4.97}$$

with $\hat{A}_1 = \hat{A}$, $U_{1j} = \beta_j$ and other $\hat{A}_i$'s completing the orthonormal set (obtained e.g. via the Gram-Schmidt procedure). The set $\{\hat{A}_i\}$ forms then a new basis such that one of its elements is the mode of interest, and transformation U can be treated similarly to a beam splitter.

Problem 4.65 Suppose each of the plane wave modes $\hat{a}_i$ is prepared in a coherent state $|\alpha_i\rangle$ with $\alpha_i = \alpha\beta_i$. Show that mode $\hat{A}$ is then in a coherent state $|\alpha\rangle$ while all other $\hat{A}_i$ (with $i \geq 2$) are in vacuum states.

Problem 4.66 Consider a homodyne detection experiment with the local oscillator prepared in a coherent state $|\alpha_{\rm LO}\rangle$ of mode $\hat{A}_{\rm LO} = \sum_i \beta_i \hat{a}_{{\rm LO},i}$ while all other basis modes $\hat{A}_i$ (with $i \geq 2$) are in vacuum states. Show that the subtraction charge generated by the homodyne detector is then given by

$$\hat{n}_- = \alpha^*_{\rm LO}\hat{A}e^{i\theta} + \alpha_{\rm LO}\hat{A}^\dagger e^{-i\theta}. \tag{4.98}$$

where $\hat{A} = \sum_j \beta_j \hat{a}_j$, with each $\hat{a}_j$ being associated with the mode that matches the corresponding local oscillator mode $\hat{a}_{{\rm LO},i}$.

We conclude that balanced homodyne detection measures the state of the optical mode matching that of the local oscillator.

Chapter 5

The single-photon qubit

5.1 Encoding and tomography

In this chapter, we study some phenomena and procedures associated with the two-dimensional subspace $\mathbb{V}$ of the two-mode optical Hilbert space spanned by states $|0_{\text{logic}}\rangle \equiv |01\rangle$ and $|1_{\text{logic}}\rangle \equiv |01\rangle$, where the right-hand side is written in the Fock basis. This subspace represents a *single-photon dual-rail qubit*.

The two modes could be two separate spatial modes, or two pulsed temporal modes in the same spatial mode, or the two polarizations of a single spatiotemporal mode. In this chapter, we will be dealing primarily with the latter case. We call the associated Hilbert space the *single-photon polarization qubit*. A few important states of this qubit are listed in the preamble to the Appendix.

We also define the four *Bell states* that are maximally entangled states of two polarization qubits:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \quad (5.1a)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) \quad (5.1b)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle) \quad (5.1c)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle) \quad (5.1d)$$

Problem 5.1 Show that any state in $\mathbb{V}$ is an eigenstate of the operator $\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$ of the total number of photons in both modes with eigenvalue 1.

Problem 5.2 Write the four Bell states in the Fock basis of a four-mode optical Hilbert space.

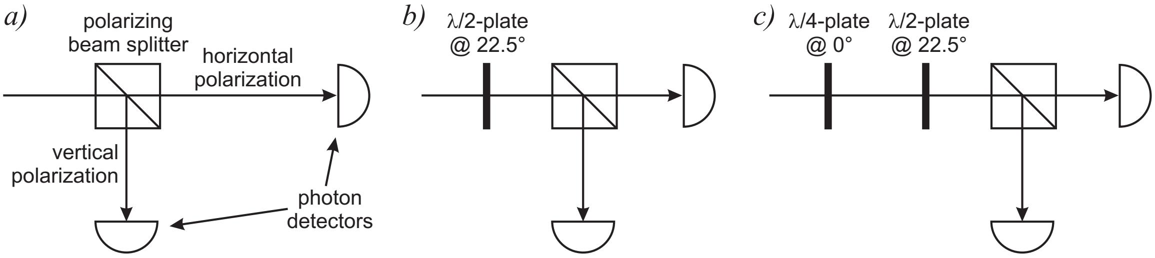


Figure 5.1: Photon polarization measurements in the $\{|H\rangle, |V\rangle\}$ (a), $\{|+\rangle, |-\rangle\}$ (b) and $\{|R\rangle, |L\rangle\}$ (c) bases.

Problem 5.3 Show that the four Bell states form a basis in the Hilbert space of two polarization qubits.

Problem 5.4 Verify that the apparata shown in Fig. 5.1 perform photon polarization measurements as stated in the caption.

Problem 5.5 Invent a scheme for the $\{|R\rangle, |L\rangle\}$ basis that would use just one waveplate.

Problem 5.6 Show that for complete tomography of a polarization qubit state $\hat\rho$ it is sufficient to acquire the photon count statistics from the three measurements shown in Fig. 5.1. Express the four matrix elements of $\hat\rho$ through the experimentally observed count rates $N_H, N_V, N_+, N_-, N_R, N_L$.

Under a *quantum process* we understand a "black box" that performs some kind of processing on quantum states (Fig. 5.2). The process can involve interaction with the environment and thus does not have to be represented by a unitary operator. Under *quantum process tomography* we understand a procedure in which one measures the effect of the process on some set of *probe* states and uses this information to predict the effect of the process on any other state.

Figure 5.2: A quantum process.

Problem 5.7 Give an example of a process that is not represented by a linear operator, i.e.

$$|\psi\rangle \overset{\mathbf{E}}{\mapsto} |\psi'\rangle\,,\ |\varphi\rangle \overset{\mathbf{E}}{\mapsto} |\varphi'\rangle \ \text{ but } \alpha|\psi\rangle + \beta|\varphi\rangle \overset{\mathbf{E}}{\not\mapsto} \alpha|\psi'\rangle + \beta|\varphi'\rangle\,. \tag{5.2}$$

Problem 5.8 Show that any process must be linear with respect to density matrices, i.e.

$$\mathbf{E}(\alpha\hat\rho_1 + \beta\hat\rho_2) = \mathbf{E}(\alpha\hat\rho_1) + \mathbf{E}(\alpha\hat\rho_1). \tag{5.3}$$

Hint: use the probabilistic nature of the density operator.

Problem 5.9 Suppose $\{\hat\rho_i\}$ is a a basis in the space of *density operators* over the Hilbert space and one knows the effect $\mathbf{E}(\hat\rho_i)$ of the process on each of these states. Show that the effect of the process on an arbitrary state is given by

$$\mathbf{E}(\hat\rho) = \sum_i \lambda_i \mathbf{E}(\hat\rho_i), \tag{5.4}$$

where λ_i's are the coefficients of the decomposition of the density operator $\hat\rho$ into this basis:

$$\hat\rho = \sum_i \lambda_i \hat\rho_i, \tag{5.5}$$

Note 5.1 Problem 5.9 provides a technique for quantum process tomography. One applies the process to each of the $\hat\rho_i$ and performs quantum state tomography on the output state $\mathbf{E}(\hat\rho_i)$. This information can then be used to predict the effect of the process on any other state.

Problem 5.10 Propose a set of probe states for quantum process tomography on a single-photon polarization qubit.

Problem 5.11 Consider a process on the Hilbert space associated with a single optical mode. Suppose the effect of the process $\mathbf{E}(|\alpha\rangle\langle\alpha|)$ is known for all coherent states. Based on this information, determine $\mathbf{E}(\hat\rho)$ for an arbitrary state $\hat\rho$.

Note 5.2 This shows that coherent states can be used as probe states for the tomography of processes on a single optical mode[1].

[1] M. Lobino *et al.*, Science **322**, 563 (2008).

5.2 Quantum cryptography

Suppose a communication party, Alice, wishes to send a secret message to her distant partner, Bob. Classically, this task would require a secure communication channel. Quantum mechanics offers an alternative: Alice and Bob can use the destructive nature of quantum measurement to ensure secure transfer of messages even through an insecure channel.

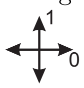
Before we proceed to describing the protocol, we notice that it is not necessary to transfer the message itself through the secret channel. Instead, Alice can transfer a random sequence of bits (the "secret key"). She can then encode her message by XOR'ing it with the key and transfer it via an insecure public channel. Even if this message is intercepted, no one would be able to read it without the key. Bob does possess the key so he can XOR it with the encoded message to get the original one.

The protocol, invented in 1984 by C.H. Bennett and G. Brassard[2] ("BB84"), runs as follows:

a) Alice chooses randomly either the H-V or the ±45° basis.

b) Alice chooses randomly if she wants to send "0" or "1".

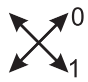
c) Alice generates a photon and encodes her bit via polarization:



$0 \to |H\rangle$
$1 \to |V\rangle$

OR

$0 \to |{+45^\circ}\rangle$
$1 \to |{-45^\circ}\rangle$



d) Alice sends the photon to Bob.

e) Bob chooses randomly in which basis he will measure.

f) Bob measures the arriving photon in the chosen basis:
→ If he chose the same basis as Alice, he will detect the correct bit value.
→ If he chose not the same basis as Alice, he will detect a random bit value.

g) Repeat (a) through (f) many times.

h) Alice and Bob communicate (via a classical, insecure channel) their choice of bases in each run, but *not* the bit values.

i) Alice and Bob discard the data for those instances in which different bases were used or photons were lost.

j) Alice and Bob now share a secret string of identical bits.

k) Alice and Bob exchange, via an insecure channel, a part of this string to verify that they are indeed identical. Errors indicate eavesdropping (see Problem 5.12 below).

l) The remainder of the secret string can be used as the secret key.

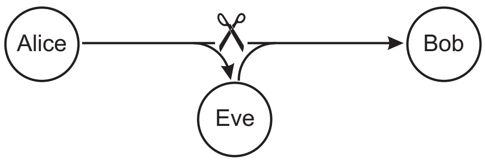


Figure 5.3: Eavesdropping in quantum cryptography.

[2]C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing*, (Bangalore, 1984), p. 175.

Problem 5.12 Suppose an eavesdropper (Eve) cuts the transmission line, intercepts Alice's photons and measures them in a randomly chosen basis (Fig. 5.3). She then resents the bit she measured in the same basis. What error rate will Alice and Bob register, i.e. what fraction of their data bits will come out different?

Note 5.3 By using a more sophisticated strategy, Eve can reduce this error rate down to about 11%.

When the communication distance increases, the secure bit error rate decreases due to optical losses. At some point, the rate of "clicks" in Bob's detector due to the photons sent by Alice becomes comparable with the rate of the detector's dark count events. In this situation, bit errors will arise and the communication can no longer be considered completely secure. This is quantified in the following problem.

Problem 5.13 Assuming that Alice has a perfect single-photon source, sketch the secure bit transfer rate and the quantum bit error rate as functions of the distance and estimate the maximum possible secure communication distance given the following parameters:

- photon loss in the fiber communication line: $\beta = 0.2$ dB/km;
- emission rate of Alice's source: $n_{in} 10^6$ photons per second;
- quantum efficiency of the photon detectors: $\eta = 0.1$;
- dark count rate of the photon detectors: $f_d = 10^4$ s^{-1};
- time resolution of the photon detectors: $\tau = 1$ ns.

Hint: Keep in mind that one can reduce the error rate due to the dark counts by discarding those "clicks" that are not synchronized (within the resolution time) with Alice's photon pulses.

5.3 Bell inequality

5.3.1 The Einstein-Podolsky-Rosen paradox

The paradox of quantum nonlocality is based on two primary notions.

Physical reality. An obervable is defined to be an *element of physical reality* when its value can be correctly predicted prior to measurement.

Locality principle (Local realism). If two parties are far away from each other and/or do not interact with each other, then no action by one party can change physical reality at the other.

Suppose Alice and Bob (two remote, non-interacting parties) share two photons in the state $|\Psi^-\rangle = |HV\rangle - |VH\rangle$. Let Alice measure the polarization of her photon in the (H, V) basis. She will then remotely prepare the state $|H\rangle$ or $|V\rangle$ at Bob's. After Alice's measurement, the physical reality at Bob's station has the following features:

- If Bob chooses to measure his photon in the (H, V) basis, his measurement result can be predicted with certainty.
- If, on the other hand, Bob decides to measure in the $\pm 45°$ basis, his result would be fundamentally uncertain[3]

If Alice instead measures in the $\pm 45°$ basis, Bob's reality changes.

- If Bob chooses to measure his photon in the $\pm 45°$ basis, his measurement result can be predicted with certainty.

[3] "Fundamentally uncertain" means that Bob's future measurement result is not only unknown, but that quantum mechanics *prohibits* it from being known. Assuming that quantum mechanics is correct, this "lack of knowledge" is thus an integral part of physical reality associated with Bob's particle.

- If Bob decides to measure in the (H, V) basis, his result would be completely uncertain.

We see that the two physical realities described above are incompatible with each other. Therefore, according to quantum mechanics, Alice can change Bob's physical reality instantaneously and without interaction — thus violating the locality principle!

This result is known as the Einstein-Podolsky-Rosen (EPR) paradox, first described in 1935 for a continuous-variable system (see Problem 4.45). Quantum mechanics appears to be in contradiction with the fundamental laws of nature that follow from the most simple common sense. On the other hand, quantum mechanics is known to predict experimental results very well, so one cannot just say it is plain wrong. Therefore, EPR made a more careful statement, saying that that "quantum-mechanical description of reality . . . is not *complete*". According to EPR, a theory can be developed which predicts experimental results as well as QM, but does uphold local realism.

Because this "new" theory is postulated to predict the same experimental results as quantum mechanics, for the next 30 years there appeared to be no possibility to verify if EPR were correct in their hypothesis. However, in 1964 *J. Bell* proposed[4] an experiment in which *any* local realistic theory would predict a result which is different from that predicted by QM. Specifically, he proposed an inequality *("Bell inequality")* that would hold in any local realistic theory, but is violated according to quantum mechanics.

The groups of *J.F. Clauser* (1972)[5] and *A. Aspect* (1981-1982)[6] performed this experiment and verified that QM is correct. Since then, experiments have been improving and always showed violation of the Bell inequality. However, all so far existing Bell inequality experiments contain loopholes.

There are two primary types of loopholes in modern Bell inequality tests.

- *Locality loophole:* the two parties are not sufficiently far apart to ensure their space-like separation.

- *Detection loophole:* Some of the particles being communicated are lost, leading to no detection events either at Alice's or Bob's stations. The details of this loophole are reviewed in Problem 5.18.

To date, there exist experiments that close either of the two loopholes, but not both at the same time[7]. A loophole-free test can be expected in the next few years.

Bell's argument consists of two parts. In the first part, we describe an experiment in terms of its superficial features, not specifying its actual construction. We will analyze this experiment using only very general physical principles, such as causality and local realism, and derive an inequality that the results of this experiment must obey. In the second part, we study a specific setup whose superficial features are consistent with those described in the first part. Using quantum mechanics, we will predict the results that turn out to violate the inequality obtained *ab initio* in the first part.

5.3.2 Local realistic argument

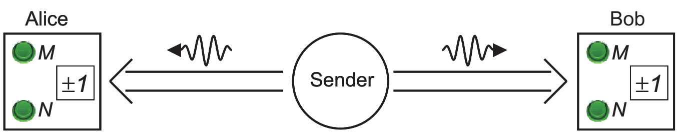


Figure 5.4: Scheme of Bell's experiment

[4]J. S. Bell, Physics **1**, 195 (1964).

[5]S. J. Freedman and J. F. Clauser, Phys. Rev. Lett. **28**, 938 (1972)

[6]A. Aspect it et al., Phys. Rev. Lett. **47**, 460 (1981); Phys. Rev. Lett. **49**, 91 (1982); Phys. Rev. Lett. **49**, 1804 (1982)

[7]Locality loophole closed in G. Weihs *et al.,* Phys. Rev. Lett. **81**, 5039 (1998). Detection loophole closed in M. Rowe *et al.,* Nature **409**, 791 (2001).

Description of the experiment

- Each of the two remote observers, Alice and Bob, operates an apparatus shown in Fig. 5.5. The construction of the apparata is unimportant, but it is known that their front pannels have the following, identical appearance: each has two buttons marked M and N, and a display that can show either "+1" or "-1".
- Alice and Bob have no possibility to communicate with each other.
- A "sender" located about halfway between Alice and Bob sends them a pair of particles of unknown nature.
- Alice and Bob receive the particle and insert them into their "black boxes".
- Alice and Bob simultaneously push one of the buttons on their black boxes. Each black box will then display a value of ± 1 related to the properties of the particle received. We refer to this procedure as a measurement of M_A or N_A by Alice and measurement of M_B or N_B by Bob.
- After making measurements on many particle pairs, the parties meet and discuss the results.

Derivation of the Bell inequality

Local realism implies that the measurement result obtained by each party is not affected by the button pushed by the other party. Each black box determines the value displayed for each button based on the local information at hand (i.e. the properties of the particles) and some algorithm. Therefore, the quantities M_A, N_A, M_B and N_B are elements of reality for each pair of particles once they are distributed.

Consider the quantity $X = M_A(M_B - N_B) + N_A(M_B + N_B)$ for a given pair of particles. Because both M_B and N_B has a definite value of +1 or −1, either $(M_B - N_B)$ or $(M_B + N_B)$ must be equal to zero. Because both M_A and N_A is either +1 or −1, we find that X must be either +2 or −2: $|X| = 2$.

The measurement is repeated many times. Since, in each run, $|X| = 2$, the average value of X over a series of experiments must obey $|\langle X\rangle| \le 2$, or

$$|\langle M_A M_B - M_A N_B + N_A M_B + N_A N_B\rangle| \le 2. \tag{5.6}$$

This is the *Bell inequality.* Note again that it does not rely on any assumption about the physics of the particles distributed or the measurement apparata, but only on very general principles (causality + local realism). Therefore, it should hold for any experiment whose superficial description falls under that given above.

Problem 5.14 When deriving Eq. (5.6), we assumed that the dependence of M's and N's on the particles' properties is deterministic. Modify the above argument to account for possible probabilistic behavior (i.e. assuming that for a particle with a given set of properties, results +1 and −1 appear randomly with some probabilities).

5.3.3 Quantum argument

We now describe a specific setup that is consistent with the above description yet violates the Bell inequality. The two particles received by Alice and Bob are two photons in the Bell state $|\Psi^-\rangle$. When Alice and Bob press their buttons, their apparata measure the following observables on their photons:
$\hat{M}_A$: eigenvalues ± 1, eigenstates $(|H\rangle, |V\rangle)$;
$\hat{M}_B$: eigenvalues ± 1, eigenstates $(|\pi/8\rangle, |\pi/8 + \pi/2\rangle)$;
$\hat{N}_A$: eigenvalues ± 1, eigenstates $(|\pi/4\rangle, |\pi/4 + \pi/2\rangle)$;
$\hat{N}_B$: eigenvalues ± 1, eigenstates $(|3\pi/8\rangle, |3\pi/8 + \pi/2\rangle)$,
where $|\theta\rangle$ denotes linear polarization at angle θ to horizontal[8].

[8]Obviously, $\hat{M}_A = \hat{\sigma}_z$, $\hat{N}_A = \hat{\sigma}_x$.

The measurement result, which is one of the measured observables' eigenvalues (±1), is displayed.

Our next goal is to make a quantum mechanical prediction for the measurement outcomes' statistics, so we can determine the expectation value of observable X. This is done in the following exercise.

Problem 5.15 Express the observables defined above as operators. Calculate the expectation values of the following operators in the state $|\Psi^-\rangle$:

a) $\hat{M}_A \otimes \hat{M}_B$;

b) $\hat{M}_A \otimes \hat{N}_B$;

c) $\hat{N}_A \otimes \hat{M}_B$;

d) $\hat{N}_A \otimes \hat{N}_B$.

Hint: To reduce calculations, verify and use the isotropicity of $|\Psi^-\rangle$: for any θ, the state $|\Psi^-\rangle = |H_A V_B\rangle - |V_A H_B\rangle$ can be expressed as $|\Psi^-\rangle = \left|\theta_A(\frac{\pi}{2}+\theta)_B\right\rangle - \left|(\frac{\pi}{2}+\theta)_A\theta_B\right\rangle$.
Answer: $-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$.

We now find that according to quantum mechanics, the expectation value of X is

$$\langle X\rangle = \langle \hat{M}_A\hat{M}_B + \hat{M}_A\hat{N}_B + \hat{N}_A\hat{M}_B + \hat{N}_A\hat{N}_B\rangle = -2\sqrt{2}, \tag{5.7}$$

which violates the Bell inequality (5.6).

Problem 5.16 Show that no matter what buttons are pressed, in a large number of measurements each individual party will get an approximately equal number of results +1 and −1.

Problem 5.17 Reproduce Bell's argument for other Bell states.

Problem 5.18 Suppose a Bell inequality experiment is performed with photons. Both Alice's and Bob's channels are lossy, and detectors are imperfect, so the probability for each photon to be detected is 10%. Show that such an experiment does not refute local realism, i.e. there exists a local realistic model for the particles and detectors such that the correlations between the detection events follow the predictions of the quantum theory.

5.4 Greenberger-Horne-Zeilinger nonlocality

In this section, we will study an argument[9] that is similar to Bell's, but contains no inequalities.

5.4.1 Local realistic argument

- A "sender" sends simultaneously three particles to three remote observers, Alice, Bob, and Charley.
- Alice, Bob, and Charley insert their particles into their "black boxes". Each box has two buttons marked "σ_x" and "σ_y", and a display that can show either "+1" or "-1".
- Alice, Bob, and Charley simultaneously push one of the buttons on their black boxes. Each black box will then display a value of ±1 related to the state of the particle. We refer to this as a "measurement of σ_x or σ_y".
- After many "experiments", the parties meet and discuss the results.

[9]Theoretical proposal: D. M. Greenberger, M. A. Horne, A. Shimony, A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (M. Kafatos, ed.), p. 73 (Kluwer Academic, Dordrecht, 1989). First experiment: J. W. Pan *et. al.*, Nature 403, 515 (2000).

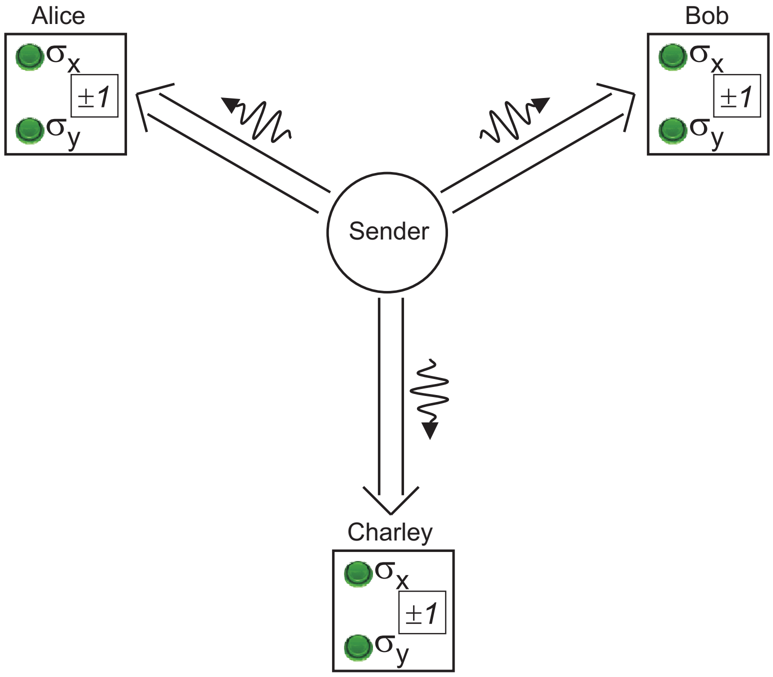


Figure 5.5: Scheme of the Greenberger-Horne-Zeilinger experiment

- They find that whenever any two of them pressed the σ_y button, and the third one the σ_x button, the product of the three results is always -1.
- Local realism implies that the measurement result obtained by each party is not affected by the buttons pushed by other parties. Each black box determines the value displayed for each button based on the local information at hand (i.e. the contents of the envelope) and some algorithm. Therefore, the quantities $\sigma_{xA}, \sigma_{yA}, \sigma_{xB}, \sigma_{yB}, \sigma_{xC}, \sigma_{yC}$ are elements of reality for each set of envelopes once they are distributed.
- Based on observation (e) above, we conclude that the following is valid for every triplet of particles:

$$\sigma_{x_A}\sigma_{y_B}\sigma_{y_C} = -1 \tag{5.8a}$$
$$\sigma_{y_A}\sigma_{x_B}\sigma_{y_C} = -1 \tag{5.8b}$$
$$\sigma_{y_A}\sigma_{y_B}\sigma_{x_C} = -1 \tag{5.8c}$$

- Multiplying these equations together and recalling that each set of the σ's is either ± 1, we find:

$$\sigma_{x_A}\sigma_{x_B}\sigma_{x_C} = -1 \tag{5.9}$$

  In other words, any local realistic theory predicts that whenever all three observers push the "σ_x" button, the product of the displayed values will be -1.

Again, his prediction is not based on any assumption about the physics of the system aside from very general ones (causality + local realism).

5.4.2 Quantum argument

Consider the state $|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|HHV\rangle + |VVV\rangle)$ distributed among Alice, Bob, and Charley.

Problem 5.19 Show that $|\Psi_{GHZ}\rangle$ is an eigenstate of operators $\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B} \otimes \hat{\sigma}_{y_C}$, $\hat{\sigma}_{y_A} \otimes \hat{\sigma}_{x_B} \otimes \hat{\sigma}_{y_C}$, $\hat{\sigma}_{y_A} \otimes \hat{\sigma}_{y_B} \otimes \hat{\sigma}_{x_C}$ with the eigenvalue (-1).

Note 5.4 This proves that state $|\Psi_{GHZ}\rangle$ fits the description in Section 5.4.1, particularly Eqs. (5.8)

Problem 5.20 Show that $|\Psi_{GHZ}\rangle$ is also an eigenstate of the operator $\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{x_B} \otimes \hat{\sigma}_{x_C}$ with the eigenvalue (+1).

Note 5.5 This proves that whenever Alice, Bob, and Charley all press the σ_x button, the product of their emasurements will be +1. This result is in direct contradiction with the local realistic prediction (5.9)!

Problem 5.21 Reproduce the GHZ argument for

$$\left|\Psi'_{GHZ}\right\rangle = \frac{1}{2}(|HHH\rangle + |HVV\rangle + |VVH\rangle + |VHV\rangle)$$

and operators

$$\begin{array}{c} \hat{\sigma}_z \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_z \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_z \\ \hline \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z \end{array}$$

5.5 Cloning and remote preparation of quantum states

Problem 5.22 State $|\Psi^-\rangle$ is shared between Alice and Bob. Alice measures her portion of this state in basis $\{|\psi\rangle, |\psi_\perp\rangle\}$, where $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$ and $|\psi_\perp\rangle$ is the state orthogonal to $|\psi\rangle$.

a) What are the probabilities for Alice to obtain each of the two results?

b) What state will be prepared at Bob's location after Alice's measurement and her communicating the result to Bob?

Note 5.6 This procedure is called *remote state preparation.* By setting up her measurement apparatus in a certain way, Alice can remotely prepare any quantum state at Bob's station.

Problem 5.23 In the context of the previous problem, suppose Alice has no communication with Bob. Which state will Bob obtain after Alice's measurements?

Quantum cloning is a map in the Hilbert space $\mathbb{V}_1 \otimes \mathbb{V}_2$ ($\mathbb{V}_1 = \mathbb{V}_2$) such that, for some $|0\rangle \in \mathbb{V}_2$, $\forall\ |a\rangle \in \mathbb{V}_1$

$$|a\rangle \otimes |0\rangle \to |a\rangle \otimes |a\rangle \tag{5.10}$$

Problem 5.24 Show that quantum cloning, as defined above, is impossible.
Hint: use Eq. (5.3)

Problem 5.25 Show that, if quantum cloning were possible, superluminal communication would also be possible.
Hint: use remote preparation and quantum tomography.

Note 5.7 We find that quantum mechanics and special relativity are compatible to each other. This compatibility is surprising because these theories are developed based on completely different first principles, and is not yet completely understood.

5.6 Quantum teleportation

Consider two remote parties, Alice and Bob. Alice possesses one copy of a qubit state $|\chi\rangle = \alpha |H\rangle + \beta |V\rangle$ that she wishes to transfer to Bob. However, there is no direct quantum communication link along which Alice could send the qubit, and neither Alice nor Bob have any classical information about $|\chi\rangle$. Instead, Alice and Bob share an entangled resource — one copy of the Bell state $|\Psi^-\rangle$. In addition, there exists a direct classical communication channel between the two parties. It turns out that Alice and Bob can utilize these resources to "teleport" $|\chi\rangle$, i.e. make an exact copy of this state emerge at Bob's station while collapsing Alice's original.

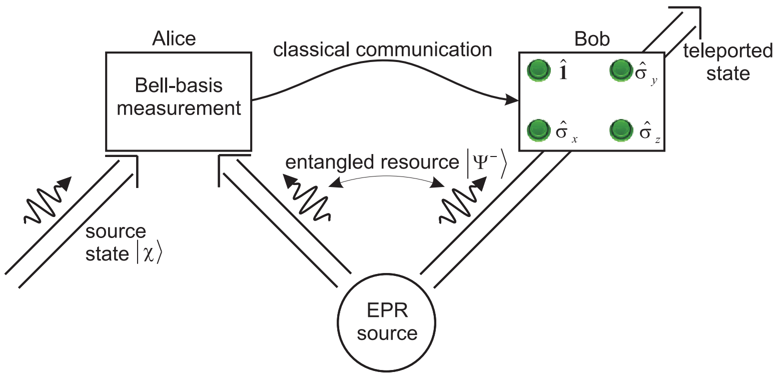


Figure 5.6: Quantum teleportation

The scheme of the quantum teleportation protocol is shown in Fig. 5.6. We say that the state $|\chi\rangle$ lives in the qubit space $\mathbb{V}_1$, and $|\Psi^-\rangle$ in $\mathbb{V}_2 \otimes \mathbb{V}_3$. The spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ are localized with Alice, and $\mathbb{V}_3$ with Bob.

Problem 5.26 Express the initial state $|\chi\rangle \otimes |\Psi^-\rangle$ in the canonical basis of $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$. (i.e. $\{|HHH\rangle, \ldots, |VVV\rangle\}$)

Problem 5.27 Express state $|\chi\rangle \otimes |\Psi^-\rangle$ in the basis which is the direct product of the Bell basis in $\mathbb{V}_1 \otimes \mathbb{V}_2$ and the canonical basis in $\mathbb{V}_3$.

Problem 5.28 Suppose Alice performs a local measurement of her fraction of state $|\chi\rangle \otimes |\Psi^-\rangle$ in the Bell basis. Calculate the probability of each measurement outcome and the state onto which space $\mathbb{V}_3$ is projected.

Problem 5.29 Alice communicates her measurement result to Bob via the classical channel. Show that with this information, Bob can convert the state of $\mathbb{V}_3$ into $|\chi\rangle$ via a local operation.

Note 5.8 In this protocol, neither Alice nor Bob obtain any classical information about the state $|\chi\rangle$ (see Problem 5.28). This ensures its perfect transfer.

Note 5.9 Both teleportation and remote state preparation are quantum communication protocols that allow disembodied transfer of quantum information by means of an entangled state and a classical communication channel. The difference between them is that while in teleportation, Alice possesses a copy of the state she wishes she transfer, in remote state preparation she instead has full classical information about this state.

Problem 5.30 Will teleportation work with Alice and Bob sharing

a) $|\Psi^+\rangle$

b) $|\Phi^+\rangle$

c) $|\Phi^-\rangle$

For each positive answer, determine the local operations Bob would need to perform on $\mathbb{V}_3$ after receiving classical communication from Alice.

How can we perform the measurement in the Bell basis? It has been proven[10] that perfect discrimination of all four Bell states of two single-photon dual-rail qubits using only linear optics is impossible. However, one can set up a scheme that would be able distinguish Bell states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ from each other and from $|\Phi^\pm\rangle$.

Problem 5.31 Suppose two modes in one of the four Bell states are overlapped on a symmetric ($r = t = 1/\sqrt{2}$) beam splitter.

a) Show that if the input state is $|\Psi^-\rangle$, the two photons will leave the beam splitter through the opposite output ports, but if it is one of the other Bell states, the photons will come out in the same spatiotemporal mode.

b) If the two photons come out in the same spatiotemporal mode, can one distinguish whether the input state was $|\Psi^+\rangle$ or $|\Phi^\pm\rangle$? If your answer is positive, design and sketch the detection setup.

Problem 5.32 *(Entanglement swapping).* Consider 4 photons A, B, C, D prepared in a partially entangled state with $|\Psi^-_{AB}\rangle \otimes |\Psi^-_{CD}\rangle$. A measurement is performed on photons B and C in the Bell basis (Fig. 5.7). Determine the state of photon pair (A, D) for each possible measurement result.

Note 5.10 Entanglement swapping creates entanglement between two particles without any interaction between them.

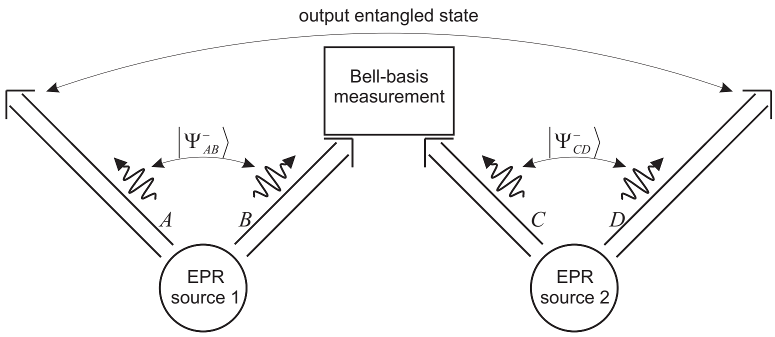


Figure 5.7: Entanglement swapping

5.7 Quantum computing with single-photon qubits

Problem 5.33 Consider a setup shown in Fig. 5.8(a). The "Kerr box" is a nonlinear optical device such that the optical phase experienced by the light propagating through it depends on its intensity:

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |10\rangle \\ |11\rangle &\to -|11\rangle, \end{aligned} \tag{5.11}$$

where the states are written in the Fock basis. In other words, the phase shift is present only if the photons from both qubits are propagating through the box. Show that this setup implements a

[10] L. Vaidman and N. Yoran, Phys. Rev. A, **59**, 116 (1999); N. Lütkenhaus, J. Calsamiglia and K. A. Suominen, Phys. Rev. A **59**, 3295 (1999).

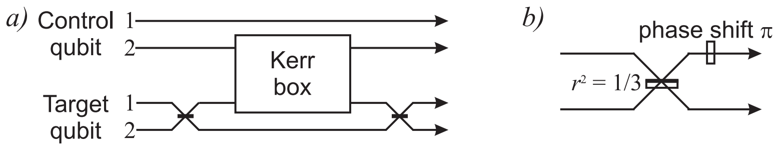


Figure 5.8: Quantum computing with with dual-rail single-photon qubits. a) CNOT gate. The Kerr box performs a nonlinear operation according to Eq. (5.11). b) Linear-optical simulation of the Kerr box. The scheme performs operation (5.11) contingent on the number of photons in both modes being the same at the input and output.

quantum conditional-not (CNOT) gate, i.e. the logical value of the target qubit will flip dependent on the logical value of the control qubit. The beam splitters are symmetric.

Note 5.11 Operation (5.11) is the conditional phase (CPHASE) gate.

Problem 5.34 Estimate the order of magnitude of the nonlinear susceptibility that a bulk material would need to possess in order to enable operation (5.11). Compare this quantity with the result of Problem 2.7.
Hint: Assume that the length of the nonlinear medium is $L = 1$ cm and the beam configuration is close to optimal (see Problem **??**). Write the third-order nonlinear susceptibility at the input wave frequency as

$$P^{(3)}(t) = 3\epsilon_0\chi^{(3)}[E^+(t)]^2E^-(t). \tag{5.12}$$

Now notice that $E^+(t)E^-(t)$ is proportional to the field intensity, so the third-order nonlinearity can be interpreted as an intensity-dependent index of refraction: $n = n^{(1)} + n^{(3)}I$. Now express $n^{(3)}$ in terms of $\chi^{(3)}$ and calculate I corresponding to a single-photon pulse of spatial extent L. Then estimate the order of magnitude of $\chi^{(3)}$ from which one obtains a nonlinear refractive index change which would result in a phase shift of π.

The optical nonlinearities required for quantum computing are beyond the reach of modern technology. Fortunately, it turns out that one can utilize the entangling power of the beam splitter to implement *conditional* CNOT gates. These gates operate with non-unitary probability and are heralded by certain detection events in ancilla modes. They turn out to be sufficient to implement efficient optical quantum computing[11].

Problem 5.35 Verify that the scheme shown in Fig. 5.8(b)[12] implements operation (5.11) contingent on the number of photons in both modes being the same at the input and output. Calculate the probability of the gate's successful operation as a function of the input state. The beam splitter is described by Eq. (4.78), with the $-r$ matrix element corresponding to the mode that reflects from the top surface of the beam splitter.

Note 5.12 In order to compensate for the non-unitary probability of the linear-optical Kerr box operation, absorbers with transmissivities 1/3 are inserted into those modes [Fig. 5.8(a)] that do not propagate through this box. Also note that the scheme of Fig. 5.8(b) is not useful for linear-optical quantum computing because the success of the gate can only be determined by measuring the number of photons in the output qubits. This measurement will destroy the qubits. However, more sophisticated schemes, such as the one in the original Knill-Laflamme-Milbutn paper, do not suffer from this shortcoming.

[11] E. Knill, R. Laflamme and G. J. Milburn, Nature **409**, 46 (2001)
[12] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **66**, 024308 (2002); T. C. Ralph, N. K. Langford, T. B. Bell, and A. G. White, Phys. Rev. A **65**, 062324 (2002).

Chapter 6

Elements of atomic physics

6.1 Interaction picture

In the Schrödinger picture, operators are constant and states evolve:

$$|\psi_S(t)\rangle = e^{-i(\hat{H}/\hbar)t} |\psi_0\rangle . \tag{6.1}$$

In the Heisenberg picture, states are constant and operators evolve:

$$\hat{A}_H(t) = e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t}. \tag{6.2}$$

The *interaction (Dirac) picture* is halfway between the two. Suppose the Hamiltonian consists of two terms, the (typically large) *unperturbed Hamiltonian* and the (typically small) *perturbation Hamiltonian*:

$$\hat{H} = \hat{H}_0 + \hat{H}_I. \tag{6.3}$$

States in the interaction picture evolve according to

$$|\psi_I(t)\rangle = e^{i(\hat{H}_0/\hbar)t} |\psi_S(t)\rangle , \tag{6.4}$$

and operators according to

$$\hat{A}_I(t) = e^{i(\hat{H}_0/\hbar)t} \hat{A}_0 e^{-i(\hat{H}_0/\hbar)t}. \tag{6.5}$$

Note 6.1 It may appear that Eq. (6.4) can be rewritten as $|\psi_I(t)\rangle = e^{-i(\hat{H}_I/\hbar)t} |\psi_0\rangle$. However, this is true only if $\hat{H}_0$ and $\hat{H}_I$ commute.

Problem 6.1 Show

a) for any state:

$$\partial_t |\psi_I(t)\rangle = -\frac{i}{\hbar} \hat{V}(t) |\psi_I(t)\rangle , \tag{6.6}$$

where

$$\hat{V}(t) = e^{i(\hat{H}_0/\hbar)t} \hat{H}_I e^{-i(\hat{H}_0/\hbar)t}; \tag{6.7}$$

b) for any operator,

$$\left\langle \psi_S(t) \right| \hat{A}_0 \left| \psi_S(t) \right\rangle = \left\langle \psi_I(t) \right| \hat{A}_I(t) \left| \psi_I(t) \right\rangle . \tag{6.8}$$

Note 6.2 We see that the interaction picture is identical to the Schrödinger picture if $\hat{H}_0 = 0$ and to the Heisenberg picture if $\hat{H}_I = 0$.

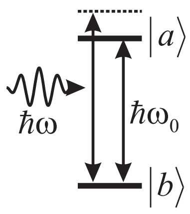


Figure 6.1: Two-level atom excited by a laser of frequency ω.

6.2 Two-level atom

6.2.1 The rotating-wave approximation

Consider a two-level atom (Fig. 6.1) excited by a laser field

$$\vec{E}(t) = \vec{E}_0 e^{-i\omega t} + \vec{E}_0^* e^{i\omega t} \tag{6.9}$$

Assuming that the ground state $|b\rangle$ has zero energy, the Hamiltonian of this system is given by

$$\hat{H} = \hbar\omega_0 |a\rangle\langle a| - \hat{\vec{E}}(t)\hat{\vec{d}}, \tag{6.10}$$

where

$$\hat{\vec{d}} = \begin{pmatrix} \vec{d}_{aa} & \vec{d}_{ab} \\ \vec{d}_{ba} & \vec{d}_{bb} \end{pmatrix} \tag{6.11}$$

is the dipole moment operator in the $\{|a\rangle, |b\rangle\}$ basis. It can be written as $\hat{\vec{d}} = e\hat{\vec{x}}$, where $\hat{\vec{x}}$ is the position of the electron.

Problem 6.2 Show that for an atom positioned at the origin of the reference frame, $\vec{d}_{aa} = \vec{d}_{bb} = 0$.

The dipole moment operator can thus be rewritten as

$$\hat{\vec{d}} = \begin{pmatrix} 0 & \vec{d} \\ \vec{d}^* & 0 \end{pmatrix}, \tag{6.12}$$

so we can thus rewrite the two-level atom Hamiltonian as

$$\hat{H} = \begin{pmatrix} \hbar\omega_0 & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix}, \tag{6.13}$$

where $\vec{d} = \vec{d}_{ab}$.

Problem 6.3 Show that the perturbation Hamiltonian in the interaction picture with[1]

$$\hat{H}_0 = \begin{pmatrix} \hbar\omega & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.14}$$

equals

$$\hat{V}(t) = \begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d}e^{i\omega t} \\ -\vec{E}\vec{d}^* e^{-i\omega t} & 0 \end{pmatrix}, \tag{6.15}$$

where $\Delta = \omega - \omega_0$.

[1]It appears more natural to define $\hat{H}_0 = \begin{pmatrix} \hbar\omega_0 & 0 \\ 0 & 0 \end{pmatrix}$. However, convention (6.14) is more convenient because it leads to a constant interaction Hamiltonian (6.16) under the rotating-wave approvimation.

Problem 6.4 We now apply the *rotating-wave approximation (RWA)*: we neglect those terms in Eq. (6.15) that oscillate at an optical frequency. Show that under RWA, Eq. (6.15) takes the form

$$\hat{V} = \hat{H} = \hbar \begin{pmatrix} -\Delta & -\Omega \\ -\Omega^* & 0 \end{pmatrix}, \tag{6.16}$$

where $\Omega = \vec{E}_0 \vec{d}/\hbar$ is the *Rabi frequency.*

Note 6.3 We have thus reduced the problem involving a fast oscillating Hamiltonian to one with a constant (or slowly varying) Hamiltonian (6.16). This significantly simplifies the calculations.

Problem 6.5 Show that the evolution of the atomic state $|\psi(t)\rangle = \begin{pmatrix} \psi_a(t) \\ \psi_b(t) \end{pmatrix}$, if the atom is initially in the ground state, is given by

$$\psi_a(t) = \left(i\frac{\Omega}{W}\sin Wt\right)e^{i\Delta t/2}; \tag{6.17a}$$

$$\psi_b(t) = \left(\cos Wt - i\frac{\Delta}{2W}\sin Wt\right)e^{i\Delta t/2} \tag{6.17b}$$

with $W = \sqrt{\Delta^2/4 + |\Omega|^2}$ (Fig. 6.2).

Note 6.4 This sinusoidal behavior of the state populations is referred to as *Rabi oscillations.* On resonance ($\Delta = 0$), for real Ω, Eqs. (6.17) become

$$\psi_a(t) = i\sin\Omega t; \tag{6.18a}$$

$$\psi_b(t) = \cos\Omega t. \tag{6.18b}$$

The quantity $2\Omega t$ is called the *pulse area.* A pulse of area $\pi/2$ excites the ground state into an equal superposition of the ground and excited states. A pulse of area π leads to inversion of the ground and excited state populations. Pulse area 2π corresponds to a complete cycle of the atomic population, but the quantum state of the atom acquires an overall phase π.

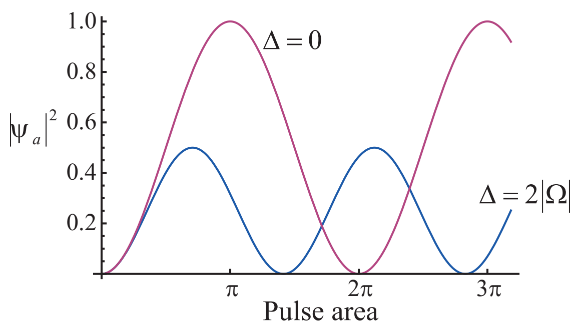


Figure 6.2: Population of the excited level as a function of the pulse area. The cases of resonant and off-resonant excitation are displayed.

6.2.2 Bloch sphere

The dynamics of a two-level atom, or any qubit system, is often convenient to visualize using a geometric tool called the *Bloch sphere*[2]. This is a sphere of radius 1, with each point on or inside its surface corresponding to one and only one state of the atom. Specifically, any state $\hat{\rho}$ of the atom corresponds to a vector with coordinates

$$R_i = \langle\hat{\sigma}_i\rangle = \mathrm{Tr}\hat{\rho}\hat{\sigma}_i, \tag{6.19}$$

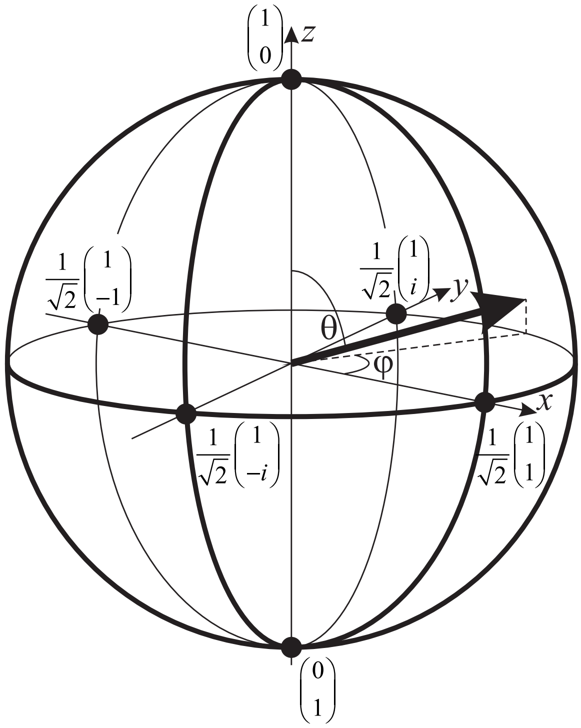


Figure 6.3: The Bloch sphere.

where $\hat{\sigma}_i$ are the Pauli operators, index i corresponding to x, y, or z.

Problem 6.6 Verify that the six specific basic states correspond to points in the Bloch sphere as shown in Fig. 6.3.

Problem 6.7 Show that a point with polar coordinates (θ, φ) on the surface of the Bloch sphere corresponds to the pure state $\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix}$.

Problem 6.8 Show that mixed states correspond to points inside the Bloch sphere. What point does the fully mixed state correspond to?

Problem 6.9 Consider the evolution of an atomic state under the Hamiltonian $\hat{H} = \hbar(v_0\hat{\mathbf{1}} + \vec{v}\cdot\hat{\vec{\sigma}}) = \hbar(v_0\hat{\mathbf{1}} + \sum_{i=1}^{3} v_i\hat{\sigma}_i)$, where v_0 is an arbitrary real number, v is an arbitrary real vector and $\hat{\vec{\sigma}}$ is the "vector" consisting of the three Pauli operators.

a) Using the Heisenberg picture, show that the evolution of the Pauli operators follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\vec{\sigma}} = 2\vec{v} \times \hat{\vec{\sigma}}. \tag{6.20}$$

b) Show that this evolution corresponds to precession of the Bloch vector around vector $\vec{v}$.

Taking the expectation value of both sides of Eq. (6.20) using the definition (6.19) of the Bloch vector, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{R} = 2\vec{v} \times \vec{R}. \tag{6.21}$$

[2]For optical polarization states, the Bloch sphere is called the *Poincaré sphere*.

This result shows the power of the Bloch sphere as a visualization tool. Instead of abstract quantum states, we can deal with vectors, and instead of the evolution operator — with rotation. Furthermore, the evolution of the Bloch vector has a direct analogue: Larmor precession of the magnetic moment in a magnetic moment in a magnetic field.

Let us now discuss specific examples where the Bloch sphere is helpful.

Problem 6.10 Verify that the evolution of the Bloch vector predicted by Eq. (6.20) is consistent with

a) Eqs. (6.18) under conditions $\Delta = 0, \Omega = \Omega^*$ for arbitrary Ωt.

b) Eqs. (6.17) under general conditions, for specific cases $Wt = 0, \pi/2, \pi$.

Problem 6.11 Suppose the optical frequency is slightly off resonance: $\Delta \ll \Omega$. Consider the following procedure.

- A pulse of area $\pi/2$ is applied to an atom initially in the ground state. Because $\Delta \ll \Omega$, we can neglect the detuning and use Eq. (6.18) to calculate the evolution.
- The laser field is turned off for time t, so the atom evolves freely.
- Another pulse of area $\pi/2$ is applied.

Plot the evolution on the Bloch sphere and show that the final population of the excited state behaves as $|\psi_a|^2 = \cos^2 \Delta t/2$.

This phenomenon, known as *Ramsey fringes*, is used to build *a*tomic clocks. The atomic clock employs a narrowband, stable and reproducible atomic transition as the "pendulum". While the idea of using atomic transitions to measure time was first proposed as early as 1979 by Lord Kelvin, the implementation of this idea is not straightforward. One problem is as follows. An arbitrary atomic state $\psi_a |a\rangle + \psi_b |b\rangle$ will evolve, in time t, into $\psi_a |a\rangle e^{-i\omega t} + \psi_b |b\rangle$. The quantity that needs to be measured is the acquired phase $e^{-i\omega t}$. But how can we measure a quantum phase?

This is where the Ramsey procedure comes in handy. After the atom (or an atomic ensemble) is subjected to it, the phase information is transformed into the populations of the two levels, and these populations are easily measurable — for example, by observing the amount of fluorescence from $|a\rangle$ to $|b\rangle$. These measurements are used to provide feedback to the laser to set $\Delta = 0$, i.e. $\omega = \omega_0$. In this way, one obtains a highly stable electromagnetic wave source.

Problem 6.12 The principle of Ramsey fringes is apparently paradoxical. As we found out, the final population of the excited state is $\cos^2 \Delta t/2$. But the process that leads to the dependence on Δt is the free evolution of the atom, while the laser is off. How can the detuning of a laser that is turned off have any effect on an experimentally measurable quantity?

6.2.3 Eigenstates of a two-level Hamiltonian

Problem 6.13 Show that the eigenvalues of Hamiltonian (6.16) are given by

$$U_{1,2} = \hbar \frac{-\Delta \pm \sqrt{\Delta^2 + 4\Omega^2}}{2}. \tag{6.22}$$

Find the eigenstates (they are called the *dressed states*). Verify that for high detunings, the dressed state energies are approximated by $(\Omega^2/\Delta, -\Delta - \Omega^2/\Delta)$. Show that the dressed state whose energy is close to zero consists primarily of the ground state (Fig. 6.4), with the population of the excited state given by $|\psi_a|^2 \approx \Omega^2/\Delta^2$.

We see that if the laser is red detuned ($\Delta < 0$), the energy of the ground dressed state decreases with increasing Ω. This means that the atom will experience a potential well of depth $\sim \Omega^2/|\Delta|$ inside the laser beam, which is known as the *ac Stark shift* or *light shift.* It is the principle behind *dipole traps* and *optical tweezers.*

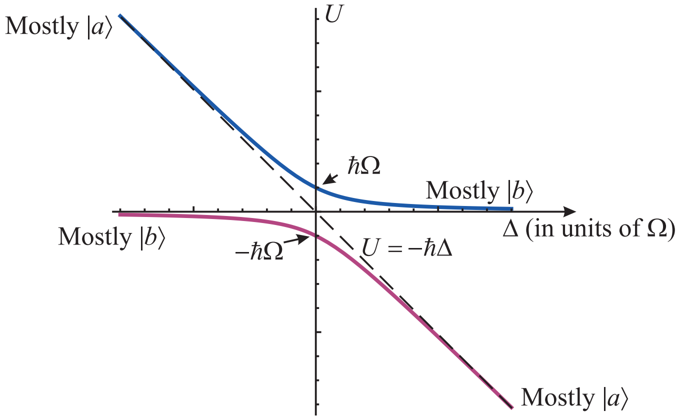


Figure 6.4: Eigenvalues of the RWA Hamiltonian.

Compared to other traps for neutral atoms, the dipole trap has the advantage that it generates minimal disturbance to the atoms' quantum state. There are no resonant optical fields, nor any dc magnetic fields. On the other hand, the dipole trap requires a relatively high laser power and does not cool the atoms. Furthermore, because the trapped state contains a fraction of excited state, the atom has a chance to scatter a photon and gain kinetic energy. This occurs at a rate of

$$\Gamma_{\rm sc} = \Gamma|\psi_a|^2 \approx \Gamma\Omega^2/\Delta^2, \tag{6.23}$$

where Γ is the spontaneous decay rate.

Problem 6.14 A dipole trap for rubidium atoms (resonance wavelength $\lambda_0 = 780$ nm) is formed by a laser of wavelength $\lambda = 1064$ nm of power $P = 50$ W focused to a spot of $r = 100$ μm radius.

a) Estimate the depth of the dipole trap (in Hz).

b) Estimate the photon scattering rate given that $\Gamma = 3.6 \times 10^7$ s^{-1}.

Hint: Use the Bohr radius to estimate the dipole moment.

The properties of the eigenstates of the RWA Hamiltonian also give rise to a technique called *adiabatic rapid passage*. Suppose the atom is initially in the ground state. The laser is turned on far below the resonance and then slowly tuned to far above the resonance. The atom will adiabatically follow, remaining in the lower energy eigenstate (Fig. 6.4), which will change its nature from almost $|b\rangle$ to almost $|a\rangle$. When the laser is turned off, the atom will likely be in the excited state.

Of course, an atom can also be transferred from $|b\rangle$ to $|a\rangle$ by a π pulse. However, this would require precise setting of the pulse duration and intensity, which can be challenging. Furthermore, the intensity of a realistic laser beam depends on the transverse position, which may become an issue if an ensemble of atoms is to be excited. Adiabatic rapid passage does not suffer from any of these shortcomings.

Problem 6.15 Estimate the limitations on the transition times imposed by the spontaneous emission and the requirement of adiabaticity.

6.2.4 Master equations

Problem 6.16 Show that the evolution of the atomic density matrix $\hat\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$ in the interaction picture, in the absence of spontaneous decay, assuming the Rabi frequency to be real, is given by

$$(\partial_t\hat\rho)_{\rm int} = \begin{pmatrix} i\Omega(\rho_{ba}-\rho_{ab}) & i\Delta\rho_{ab}+i\Omega(\rho_{bb}-\rho_{aa}) \\ -i\Delta\rho_{ba}+i\Omega(\rho_{aa}-\rho_{bb}) & i\Omega(\rho_{ab}-\rho_{ba}) \end{pmatrix}. \tag{6.24}$$

Hint: use Eq. (A.4).

Problem 6.17 In order to derive the effect of the spontaneous emission on the density matrix, suppose the atom in a pure state $\psi_a |a\rangle + \psi_b |b\rangle$ interacts with the "reservoir" — a set of electromagnetic modes initially in the vacuum state. The interaction for a short time δt results in the following state evolution:

$$(\psi_a |a\rangle + \psi_b |b\rangle)_{\text{atom}} \otimes |0\rangle_{\text{res}} \;\rightarrow\; \psi_a(\sqrt{1-\delta p}\,|a\rangle_{\text{atom}} \otimes |0\rangle_{\text{res}} + \sqrt{\delta p}\,|b\rangle_{\text{atom}} \otimes |1\rangle_{\text{res}}) + \psi_b |b\rangle_{\text{atom}} \otimes |0\rangle_{\text{res}}, \tag{6.25}$$

where $|n\rangle_{\text{res}}$ denotes the state of the reservoir containing n photons (distributed over multiple modes) and p is a small number which gives the probability that the excited state will emit the photon during δt.

a) Take a partial trace over the reservoir and write the density matrix $\hat\rho(\delta t)$ of the atom after the interaction with the reservoir.

b) Show that if the atom's initial density matrix is given by $\hat\rho(0) = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$ then

$$\hat\rho(\delta t) = \begin{pmatrix} (1-\delta p)\rho_{aa} & (1-\delta p/2)\rho_{ab} \\ (1-\delta p/2)\rho_{ba} & \delta p\rho_{aa} + \rho_{bb} \end{pmatrix} \tag{6.26}$$

c) Show that, if the probability for the excited state to emit the photon is given by $\delta p = \Gamma \delta t$, then the time derivative of the density matrix due to spontaneous emission is given by[3]

$$(\partial_t \hat\rho)_{\text{spont}} = \begin{pmatrix} -\Gamma\rho_{aa} & -\Gamma/2\rho_{ab} \\ -\Gamma/2\rho_{ba} & \Gamma\rho_{aa} \end{pmatrix} \tag{6.27}$$

Problem 6.18 Find the *steady state* density matrix of the two-level atom, which satisfies the equation

$$\partial_t \hat\rho = (\partial_t \hat\rho)_{\text{int}} + (\partial_t \hat\rho)_{\text{spont}} = 0. \tag{6.28}$$

Hint: the easiest way to solve this system of four linear equations is by expressing everything through $\rho_{bb} - \rho_{aa}$.
Answer:

$$\hat\rho = \begin{pmatrix} \frac{\Omega^2}{\Gamma^2/4+\Delta^2+2\Omega^2} & \frac{i\Omega\Gamma/2-\Omega\Delta}{\Gamma^2/4+\Delta^2+2\Omega^2} \\ \frac{-i\Omega\Gamma/2-\Omega\Delta}{\Gamma^2/4+\Delta^2+2\Omega^2} & \frac{\Gamma^2/4+\Delta^2+\Omega^2}{\Gamma^2/4+\Delta^2+2\Omega^2} \end{pmatrix} \tag{6.29}$$

Problem 6.19 Show that in the limit of weak excitation ($\Omega \ll \Gamma$), to the first order in Ω, Eq. (6.29) takes the form

$$\hat\rho = \begin{pmatrix} 0 & \frac{i\Omega}{\Gamma/2-i\Delta} \\ \frac{-i\Omega}{\Gamma/2+i\Delta} & 1 \end{pmatrix} \tag{6.30}$$

Problem 6.20 This result can be obtained in an easy way using the semi-rigorous *stochastic wavefunction approach*, which is typically applicable when most of the atomic population is in a single energy eigenstate. The idea is to approximate, to the second order in Ω, the state of the atom as a pure state: $\hat\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$ with $\psi_b \equiv 1$. We then look for the steady state solution of the equation

$$\partial_t |\psi\rangle = -\frac{i}{\hbar}\hat V(t)|\psi\rangle - \frac{\Gamma}{2}\begin{pmatrix} \psi_a \\ 0 \end{pmatrix}, \tag{6.31}$$

where the second term is responsible for the spontaneous decay.

[3] This result can be obtained more rigorously using the *Lindblad formalism*. For details, see Preskill lecture notes, `http://theory.caltech.edu/~preskill/ph219/`, Sec. 3.5.

a) Find the steady state solution of Eq. (6.31) taking into account only terms up to the first order in Ω.

b) Verify that your solution is consistent with density matrix (6.30) up to the first order in Ω.

Problem 6.21 Consider an atomic gas of number density N excited by electromagnetic wave (6.9). Calculate the expectation value of the dipole moment of an individual atom (assume $\vec{d}$ real). Calculate the polarization of the gas as a function of time. Show that the first-order susceptibility is given by

$$\chi(\omega) = \frac{Nd^2}{\hbar\epsilon_0}\frac{\rho_{ab}}{\Omega} \overset{\Omega \ll \Gamma}{=} \frac{Nd^2}{\hbar\epsilon_0}\frac{i}{\Gamma/2 - i\Delta}. \tag{6.32}$$

Hint: do not forget that the dipole moment operator (6.12) is written in the Schrödinger picture. Since we are working in the interaction picture, we must account for its evolution.

Problem 6.22 Compare Eq. (6.32) with classical susceptibility (1.29) obtained from the classical theory of dispersion for the hydrogen atom. To estimate d, use $|\langle\vec{x}\rangle| = a_0$, where $a_0 = 4\pi\epsilon_0\hbar^2/(e^2m)$ is the Bohr radius, and use the Rydberg energy $\mathrm{Ry} = \hbar^2/(2ma_0^2)$ to estimate the frequency.

Problem 6.23 a) Show that in the limit of low number density ($\chi \ll 1$), the absorption index is given by

$$\alpha = \frac{Nd^2}{\hbar\epsilon_0}\frac{\omega}{c}\frac{\Gamma/2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2}. \tag{6.33}$$

b) Show that Eq. (6.33) can be written in the form

$$\alpha(I) = \frac{\alpha(I=0)}{1 + I/I_{\mathrm{sat}}}, \tag{6.34}$$

where I_{sat} is the *saturation intensity* such that the corresponding Rabi frequency satisfies $\Omega_{\mathrm{sat}} = \Gamma/2\sqrt{2}$.

c) Use the *Weisskopf-Wigner formula* (derived below)

$$\Gamma = \frac{d^2\omega^3}{3\pi\epsilon_0\hbar c^3} \tag{6.35}$$

to show that on resonance, in the weak excitation limit,

$$\alpha = N\frac{3\lambda^2}{2\pi}, \tag{6.36}$$

where λ is the optical wavelength.

Note 6.5 Result (6.36) implies that the absorption cross-section of an atom on resonance is given by $\sigma = 3\lambda^2/2\pi$. This expression is remarkable because it exhibits no dependence on the atomic dipole moment.

6.2.5 Einstein coefficients

Consider an ensemble of two-level atoms in a reservoir containing background radiation of energy density $\varrho(\omega)$ [energy per unit volume per unit frequency, measured in $\mathrm{J}/(\mathrm{m}^3\mathrm{s}^{-1})$].The densities of atoms in the excited and ground state are N_a and N_b, respectively. There is no quantum coherence between the energy eigenstates. The atoms will undergo the following processes.

- Spontaneous emission, which is characterized by the following rate equation:

$$(\dot{N}_a)_{\mathrm{spont}} = -AN_a, \tag{6.37}$$

where $A \equiv \Gamma$ is the spontaneous emission rate.

- Absorption:
$$(\dot{N}_a)_{\rm abs} = B_{ba} N_b \varrho(\omega_0), \tag{6.38}$$
where B_{ba} is a proportionality coefficient, proportional to the integral of the absorption over the atomic line (we neglect the variation of $\varrho(\omega)$ over this line).

- *Stimulated emission (amplification)*, a phenomenon we have not yet studied, which is inverse to absorption: affected by an electromagnetic wave, an atom in the excited state may undergo transition to the ground state, emitting a photon coherent with this wave. This effect can be obtained by performing analysis similar to the previous section, but in the presence of incoherent "pumping" from $|b\rangle$ to $|a\rangle$, leading to a negative absorption index (6.33). The rate of the stimulated emission is given by
$$(\dot{N}_a)_{\rm stim} = -B_{ab} N_a \varrho(\omega_0), \tag{6.39}$$
where B_{ab} is another proportionality coefficient.

Quantities A, B_{ba} and B_{ab} are determined exclusively by the atomic properties and are called the *Einstein coefficients.* In the following, we establish two universal relations among these coefficients.

Problem 6.24 Let the radiation background be due to blackbody thermal radiation. It is then described by the Planck formula:
$$\varrho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}, \tag{6.40}$$
T being the temperature. Suppose the atom is of the same temperature, i.e.
$$N_a/N_b = e^{-\hbar\omega_0/k_B T}. \tag{6.41}$$
Then the atom is in thermal equilibrium with this radiation, i.e.
$$\dot{N}_a = (\dot{N}_a)_{\rm spont} + (\dot{N}_a)_{\rm abs} + (\dot{N}_a)_{\rm stim} = 0. \tag{6.42}$$

a) Assume the temperature to be very high. Show that, in order for Eq. (6.42) to satisfy, we must have
$$B_{ba} = B_{ab} \tag{6.43}$$

b) Substitute the above result into Eq. (6.42) for a finite temperature to show that
$$\frac{A}{B_{ab}} = \frac{\hbar\omega_0^3}{\pi^2 c^3}. \tag{6.44}$$

Note 6.6 Equation (6.44) is remarkable in that it is obtained using an argument from a seemingly unrelated field of physics (thermodynamics). Interestingly, stimulated emission was not known when Einstein derived his result. Einstein introduced stimulated emission (6.39) in order to satisfy the thermal equilibrium condition.

Problem 6.25 Consider weak thermal radiation of density ϱ inside a gas of atoms of number density $N_{\rm at}$.

a) Show that the photon number density [photons/m^3] in frequency interval $d\omega$ is given by $dN_{\rm phot} = \varrho d\omega/(\hbar\omega)$.

b) Show that the loss of photons [photons/(m^{3}s)] due to absorption is given by
$$(\dot{N})_{\rm abs} = \frac{1}{3} c \int_{-\infty}^{+\infty} N_{\rm phot} \alpha(\omega) d\omega \overset{(6.33)}{=} \frac{1}{3} \frac{\pi \varrho N d^2}{\epsilon_0 \hbar^2} \tag{6.45}$$

Hint: the factor of 1/3 comes because the thermal radiation is omnidirectional and randomly polarized, whereas the atomic dipole vector $\vec{d}$ associated with a particular transition has a definite direction.

c) Use Eq. (6.38) to find the Einstein B coefficient.

d) Use Eq. (6.44) to derive the Weiskopf-Wigner formula (6.35).

6.3 Three-level atom

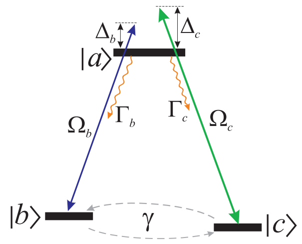


Figure 6.5: A Λ-type atom.

Consider an atom with a Λ-shaped energy level structure, as shown in Fig. 6.5. There are two ground levels $|b\rangle$, $|c\rangle$ and one excited level $|a\rangle$. Spontaneous emission rates from $|a\rangle$ into $|b\rangle$ and $|c\rangle$ are Γ_b and Γ_c, respectively. There are two electromagnetic fields: the *control* field with Rabi frequency Ω_c coupling $|c\rangle$ with $|a\rangle$ with optical frequency ω_c and detuning Δ_c and the *signal* field with Rabi frequency Ω_b coupling $|b\rangle$ with $|a\rangle$ with optical frequency ω_b and detuning Δ_b. There is ground state decoherence manifesting itself as decay of the matrix element ρ_{bc} with rate $\gamma \ll \Gamma_b, \Gamma_c$.

Problem 6.26 Taking $|b\rangle$ as the zero energy state and assuming that the unperturbed Hamiltonian is given by $\hat{H}_0 = \hbar\omega_b\,|a\rangle\langle a| + \hbar(\omega_b - \omega_c)\,|c\rangle\langle c|$, show that the perturbation Hamiltonian in the interaction picture, rotating-wave approximation is given by

$$\hat{V} = \begin{pmatrix} -\Delta_b & -\Omega_b & -\Omega_c \\ -\Omega_b^* & 0 & 0 \\ -\Omega_c^* & 0 & -\Delta_b + \Delta_c \end{pmatrix} \tag{6.46}$$

Problem 6.27 Show that if the *two-photon detuning* $\Delta_b - \Delta_c$ is zero, the *dark state*

$$|\psi_{\text{dark}}\rangle = \Omega_c\,|b\rangle - \Omega_b\,|c\rangle \tag{6.47}$$

is an eigenstate of the interaction Hamiltonian.

The dark state is interesting because it has no excited state component. If the ground state decoherence is negligible, an atom prepared in this state can remain in it permanently and exhibit no absorption. This serves as a basis for a variety of interesting phenomena.

- *Coherent population trapping (CPT).* When the two fields are tuned to a two-photon resonance, the absorption greatly reduces due to emergence of the dark state.
- Electromagnetically-induced transparency (see below).
- *Stimulated Raman adiabatic passage (STIRAP)* is a method to adiabatically transfer population between the ground levels. Suppose the atom is prepared in $|b\rangle$ and the following manipulations are performed.
  - The field coupling $|a\rangle$ and $|c\rangle$ is applied. The atom is in the dark state $|b\rangle$.
  - The second field coupling $|a\rangle$ and $|b\rangle$ is slowly turned on. The atom remains in the dark state, which adiabatically changes to superposition (6.47).
  - The field between $|a\rangle$ and $|c\rangle$ is slowly turned off. The new dark state is $|c\rangle$.

  In this way, the atom is transferred from $|b\rangle$ to $|c\rangle$. Note that the optical pulse sequence is *counterintuitive*: the field that is initially turned on does not interact with the atom.

Problem 6.28 *Electromagnetically-induced transparency (EIT)* is an effect related to CPT, but in EIT the signal field is very weak: its Rabi frequency is much smaller than all other Rabi frequencies and spontaneous decay rates. In this case, most of the atomic population is optically pumped into $|b\rangle$, so the stochastic wavefunction approximation can be used.

a) Write the Schrödinger equation for the amplitudes ψ_a and ψ_c. **Hint:** Because, in the stochastic wavefunction approximation, $\rho_{cb} = \psi_c \psi_b^* = \psi_c$, we can treat the ground state decoherence as follows: $(\dot{\psi}_c)_{\text{dec}} = -\gamma\psi_c$.

b) Find the steady state amplitudes.

c) Show that the atomic medium's susceptibility with respect to the signal field is given by

$$\chi = \frac{Nd_b^2}{\hbar\epsilon_0}\frac{\Delta_b - \Delta_c + i\gamma}{|\Omega_c|^2 - (\Delta_p - \Delta_c + i\gamma)(\Delta_b + i\frac{\Gamma_b+\Gamma_c}{2})} \tag{6.48}$$

Hint: use the first equality of Eq. (6.32).

d) Show that for $\Omega_c \ll \Gamma, \Delta_c = 0, \gamma = 0$, the absorprion spectrum of an EIT system exhibits a dip of FWHM given by

$$\text{FWHM} = 4|\Omega_c|^2/(\Gamma_b + \Gamma_c) \tag{6.49}$$

Note 6.7 The EIT window is a consequence of the dark state emerging at the two-photon resonance.

Problem 6.29 Show that the group velocity in the center of the EIT window, in the approximation $\Omega_c \ll \Gamma, \Delta_c = 0, \gamma = 0$ is given by

$$\frac{1}{v_{\text{gr}}} = \frac{1}{c} + \frac{Nd_b^2\omega}{2c\hbar\epsilon_0|\Omega_c|^2}. \tag{6.50}$$

Calculate the group velocity in rubidium ($\lambda = 795$ nm) if $\Omega_c = 10$ kHz and $N = 10^{12}$ cm^{-3}.

Problem 6.30 Show analytically that the steady state solution of Hamiltonian (6.46) in the limit $|\Delta_c| \gg \Gamma$ leads to an additional (*Raman*) absorption line [Fig. 6.6(e)] centered at the ac Stark shifted two-photon resonance, $\Delta_b = \Delta_c \pm |\Omega_c|^2/\Delta_c$ (positive and negative sign corresponding to positive and negative Δ_c, respectively) and of FWHM

$$\Gamma_{\text{Raman}} = \Gamma\frac{|\Omega_c|^2}{\Delta_c^2}. \tag{6.51}$$

The stochastic wavefunction approximation can be used.

Problem 6.31 Show that the evolution of a three-level atom under Hamiltonian (6.46) in the limit $|\Delta_c| \gg \Gamma$ is similar to evolution of a two-level system consisting of just the ground levels with Rabi frequency

$$\Omega_{\text{Raman}} = \frac{\Omega_c^*\Omega_b}{\Delta_c} \tag{6.52}$$

and the decay rate from $|c\rangle$ given by Eq. (6.51). The stochastic wavefunction approximation can be used.
Hint: one of the evolution equations is of the form

$$\dot{\psi}_a = i\Delta_b\psi_a + i\Omega_b\psi_b + i\Omega_c\psi_c - i(\Gamma/2)\psi_a \tag{6.53}$$

You can assume that ψ_a consists of a constant term $(\psi_a)_s$ and a quickly varying term $(\psi_a)_q$. We then have

$$i\Delta_b(\psi_a)_s + i\Omega_b\psi_b + i\Omega_c\psi_c - i(\Gamma/2)(\psi_a)_s = 0. \tag{6.54}$$

The quickly varying term averages to zero, so we can simply neglect it and assume that $\psi_a = (\psi_a)_s$, which can be readily found by solving Eq. (6.54). This procedure is called *adiabatic elimination* and is commonly used in atomic physics.

Problem 6.32 Suppose the Rabi frequencies vary in time, but slowly. Substitute the solution of Eq. (6.54) into Eq. (6.53) and determine the conditions on T, the characteristic time scale of the variations of the Rabi frequencies.

6.4 Heisenberg picture of atomic transitions

Problem 6.33 Define atomic transition operators

$$\hat{\sigma}_{H,ij}(t=0) = |i\rangle\langle j| \tag{6.55}$$

where i and j denote energy levels a or b. Write the differential equation (4.55) for the evolution of these operators in the Heisenberg picture in terms of these operators under Hamiltonian (6.13)

$$\hat{H} = \hbar\omega_0\hat{\sigma}_{H,aa}(0) - Ed\hat{\sigma}_{H,ab}(0) - Ed^*\hat{\sigma}_{H,ba}(0). \tag{6.56}$$

Verify consistency with Eq. (6.24) (use the fact that, e.g., $\rho_{ab} = \mathrm{Tr}[\hat{\rho}\hat{\sigma}_{H,ba}]$).
Hint: It is easy to see that, for example, $[\hat{\sigma}_{H,aa}(0), \hat{\sigma}_{H,ab}(0)] = \hat{\sigma}_{H,ab}(0)$. However, the transition operators evolve in time, so, generally speaking, $\hat{\sigma}_{H,ij} = |i\rangle\langle j|$ only at $t = 0$, and one may argue that the simple commutation relations between the $\hat{\sigma}$'s are no longer valid for $t \neq 0$. Fortunately, this is not the case. For example,

$$\begin{aligned}
&[\hat{\sigma}_{H,aa}(t), \hat{\sigma}_{H,ab}(t)] \qquad (6.57)\\
&= e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,aa}(0)e^{-i\hat{H}t/\hbar}e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,ab}(0)e^{-i\hat{H}t/\hbar} - e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,ab}(0)e^{-i\hat{H}t/\hbar}e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,aa}(0)e^{-i\hat{H}t/\hbar}\\
&= e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,aa}(0)\hat{\sigma}_{H,ab}(0)e^{-i\hat{H}t/\hbar} - e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,ab}(0)\hat{\sigma}_{H,aa}(0)e^{-i\hat{H}t/\hbar}\\
&= e^{i\hat{H}t/\hbar}[\hat{\sigma}_{H,aa}(0), \hat{\sigma}_{H,ab}(0)]e^{-i\hat{H}t/\hbar}\\
&= e^{i\hat{H}t/\hbar}\hat{\sigma}_{H,ab}(0)e^{-i\hat{H}t/\hbar} = \hat{\sigma}_{H,ab}(t)
\end{aligned}$$

Furthermore, because $e^{i\hat{H}(0)t/\hbar}\hat{H}(0)e^{-i\hat{H}(0)t/\hbar} = \hat{H}(0)$, we can write the Hamiltonian (6.56) as

$$\hat{H}(t) = \hat{H}(0) = -\hbar\omega_0\hat{\sigma}_{H,aa}(t) - Ed\hat{\sigma}_{H,ab}(t) - Ed^*\hat{\sigma}_{H,ba}(t). \tag{6.58}$$

Thus the evolution equations do not change with time.

Problem 6.34 Let us introduce the "slowly-varying operator picture", in which, in contrast to the interaction picture, operators vary slowly:

$$\hat{A}_{SV}(t) = e^{i(\hat{H}_0+\hat{H}_I)t}e^{-i\hat{H}_0 t}\hat{A}_0 e^{i\hat{H}_0 t}e^{-i(\hat{H}_0+\hat{H}_I)t}, \tag{6.59}$$

where $\hat{H}_0 = \hat{H}_0(0)$ and $\hat{H}_I = \hat{H}_I(0)$ are the unperturbed and perturbation Hamiltonians, respectively.

a) Write the explicit expression for the state $|\psi_{SV}(t)\rangle$ in the slowly-varying operator picture.

b) Show that, with the unperturbed and perturbation Hamiltonians of a two-level atom, we have

$$\begin{aligned}
\hat{\sigma}_{SV,aa} &= \hat{\sigma}_{H,aa},\\
\hat{\sigma}_{SV,ab} &= \hat{\sigma}_{H,ab}e^{-i\omega t},\\
\hat{\sigma}_{SV,ba} &= \hat{\sigma}_{H,ba}e^{i\omega t},\\
\hat{\sigma}_{SV,bb} &= \hat{\sigma}_{H,bb}.
\end{aligned} \tag{6.60}$$

c) Show that, for any arbitrary operator,

$$\partial_t\hat{A}_{SV} = \frac{i}{\hbar}[\hat{U}, \hat{A}], \tag{6.61}$$

where

$$\hat{U} = e^{i(\hat{H}_0+\hat{H}_I)t}\hat{H}_I e^{-i(\hat{H}_0+\hat{H}_I)t}. \tag{6.62}$$

d) Show that, for a two-level atom,

$$\hat{U} = -\hbar\Delta\hat{\sigma}_{SV,aa}(t) - Ed\hat{\sigma}_{SV,ab}(t)e^{i\omega t} - Ed^*\hat{\sigma}_{SV,ba}(t)e^{-i\omega t}, \tag{6.63}$$

i.e. is identical to the perturbation Hamiltonian (6.15) in the interaction picture.
Hint: observe that the perturbation Hamiltonian (6.62) is simply $\hat{H}_I(t)$ that has evolved under the Heisenberg picture. It can hence be expressed in terms of $\hat{\sigma}_{H,ij}(t)$, which, in turn, are expressed via the $\hat{\sigma}_{SV}$'s using (6.60).

e) Write the differential equation (6.61) explicitly for the four $\hat{\sigma}_{SV}$'s.

Note 6.8 The relatively simple form of Eqs. (6.60) is due to a diagonal form of the unperturbed Hamiltonian $\hat{H}_0$. In this case, the difference between the Heisenberg picture and the slowly-varying operator picture is simple phase factors.

Note 6.9 We will be using the slow-varying operator picture for the remainder of this section.

Note 6.10 Eliminating the quickly-varying terms in the perturbation Hamiltonian (6.63), we write

$$\hat{U} = -\hbar\Delta\hat{\sigma}_{SV,aa}(t) - E^+d\hat{\sigma}_{ab}(t)e^{i\omega t} - E^-d^*\hat{\sigma}_{ba}(t)e^{-i\omega t}. \tag{6.64}$$

This is interpreted as follows: the positive-frequency term of the field, which consists of photon annihilation operators, couples to σ_{ab}, a transition that excites the atom, and vice versa.

Problem 6.35 Introduce the slowly-varying position-dependent atomic operator

$$\hat{\tilde{\sigma}}_{ba}(z,t) = e^{-ikz}\frac{1}{N\delta V}\sum_{\text{atom } m\in\delta V}\hat{\sigma}_{m,ba}(t), \tag{6.65}$$

i.e. the atomic transition operator averaged over all atoms within a small volume δV with position z. N is the number density of the atoms inside the cell. The factor e^{-ikz}, with $k = \omega/c$ being the central wavevector of the optical field, has been included in Eq. (6.65) to ensure the slow variation of atomic operators as a function of position. Show that, for example,

$$[\hat{\tilde{\sigma}}_{aa}(z,t), \hat{\tilde{\sigma}}_{ba}(z',t)] = \frac{L}{NV}\delta(z-z')\hat{\tilde{\sigma}}_{ab}(z,t), \tag{6.66}$$

where V is the volume of the cell and L its length.
Hint: because our treatment is one-dimensional, write $\delta V = A\delta z$, where A is the area of the sample and δz is a short cell length interval. Calculate the commutator for finite δz and show that it approaches the delta function for $\delta z \to 0$.

Note 6.11 Other position-dependent atomic operators are defined similarly to Eq. (6.65).

Problem 6.36 We will now treat the light quantum-mechanically. Introduce the slowly-varying position-dependent optical operator

$$\hat{a}(z,t) = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega}}\hat{E}^{(+)}(z,t)e^{-ikz+i\omega t} = \sum_j \hat{a}_j(t)e^{-i(k-k_j)z+i\omega t}, \tag{6.67}$$

where j indexes the plane-wave modes and the quantization volume is taken equal to the cell volume. Show that

$$[\hat{a}(z,t), \hat{a}^\dagger(z',t)] = L\delta(z-z') \tag{6.68}$$

Hint: Because our treatment is one-dimensional, $k_j = (2\pi/L)j$. For all relevant optical modes, one can assume $\omega_j \approx \omega$. Use the relation $\sum_{j=-\infty}^{+\infty} e^{ijx} = 2\pi\delta(x)$.

Problem 6.37 Show that we can rewrite the propagation equation (1.20) for the slowly-varying envelope as follows:

$$[c\partial_z + \partial_t]\,\hat{a}(z,t) = igNV\hat{\sigma}_{ba}(z,t), \tag{6.69}$$

where

$$g = d\sqrt{\frac{\omega}{2\hbar\epsilon_0 V}} \tag{6.70}$$

is the coupling coefficient. Assume that the index of refraction is close to one.
Hint: Although (1.20) is written for the nonlinear polarization, it is also valid in the linear case provided that the linear dispersion is not already accounted for in the left-hand side (i.e. $\omega/k = c$). In the case of EIT, the index of refraction is very close to 1 but the dispersion is significant (see Problem 1.18). The polarization in the right-hand side accounts for this dispersion.

Problem 6.38 Reproduce result (6.69) using the Heisenberg equations of motion for the optical operators in the slowly-varying operator picture.

Problem 6.39 Show that the evolution for the atomic states, including the spontaneous decay terms, can be written as

$$\begin{aligned}
\partial_t\hat{\tilde{\sigma}}_{aa} &= -ig\hat{a}^\dagger\hat{\tilde{\sigma}}_{ba} + ig\hat{a}\hat{\tilde{\sigma}}_{ab} - \Gamma\hat{\tilde{\sigma}}_{aa} && (6.71\text{a})\\
\partial_t\hat{\tilde{\sigma}}_{ab} &= -i\Delta\hat{\tilde{\sigma}}_{ab} - ig\hat{a}^\dagger(\hat{\tilde{\sigma}}_{bb} - \hat{\tilde{\sigma}}_{aa}) - (\Gamma/2)\hat{\tilde{\sigma}}_{ab} && (6.71\text{b})\\
\partial_t\hat{\tilde{\sigma}}_{ba} &= i\Delta\hat{\tilde{\sigma}}_{ba} + ig\hat{a}(\hat{\tilde{\sigma}}_{bb} - \hat{\tilde{\sigma}}_{aa}) - (\Gamma/2)\hat{\tilde{\sigma}}_{ba} && (6.71\text{c})\\
\partial_t\hat{\tilde{\sigma}}_{bb} &= -ig\hat{a}\hat{\tilde{\sigma}}_{ab} + ig\hat{a}^\dagger\hat{\tilde{\sigma}}_{ba} + \Gamma\hat{\tilde{\sigma}}_{aa} && (6.71\text{d})
\end{aligned}$$

where all optical and atomic operators are understood as functions of space and time.

Note 6.12 A set of equations fully describing the evolution of the atom-light system is called the *Maxwell-Bloch equations*. An example is Eqs. (6.69) and (6.71). In Maxwell-Bloch equations, the light can be treated either classically or quantum-mechanically; the atoms are always treated quantum-mechanically.

Problem 6.40 Write the set of Maxwell-Bloch equations for an EIT system. Neglect both detunings. Treat the control field classically and the signal quantum-mechanically. The control field is externally controlled and depends on time only. For a weak signal field, the atoms remain in $|b\rangle$ and thus we can write $\hat{\tilde{\sigma}}_{bb} \equiv \hat{1}$ and neglect $\hat{\sigma}_{aa}$, $\hat{\sigma}_{cc}$ and $\hat{\sigma}_{ac}$ because they are of the second order in the signal field. Show that the propagation equation for the signal field remains of form (6.69) and for the atomic transitions we have

$$\begin{aligned}
\partial_t\hat{\tilde{\sigma}}_{ba} &= ig\hat{a} + i\Omega_c\hat{\tilde{\sigma}}_{bc} - (\Gamma/2)\hat{\tilde{\sigma}}_{ba} && (6.72\text{a})\\
\partial_t\hat{\tilde{\sigma}}_{bc} &= i\Omega_c\hat{\tilde{\sigma}}_{ba} && (6.72\text{b})
\end{aligned}$$

Problem 6.41 Show that

$$\hat{\tilde{\sigma}}_{bc}(z,t) = -\frac{g}{\Omega_c}\hat{a}(z,t) \tag{6.73}$$

and thus the propagation equation takes the form

$$[c\partial_z + \partial_t]\,\hat{a}(z,t) = -\frac{g^2NV}{\Omega_c}\partial_t\frac{\hat{a}(z,t)}{\Omega} \tag{6.74}$$

Use Eq. (1.22) to verify that the group velocity that obtains from the above is consistent with Eq. (6.50).
Hint: adiabatically eliminate $\hat{\tilde{\sigma}}_{ba}$.

Note 6.13 When Ω_c is constant, Eq. (6.74) describes the motion of the signal field to be with constant velocity v_{gr}. However, when the control field is varied, the motion becomes more complicated because Ω is a part of the derivative in the right-hand side of Eq. (6.74). Then we can describe the motion of the pulse using so-called *dark-state polaritons*[4]:

$$\hat{\Psi}(z,t) = \frac{1}{\sqrt{\Omega_c^2 + g^2 NV}}[\Omega_c \hat{a}(z,t) - gNV\hat{\tilde{\sigma}}_{bc}(z,t)]. \tag{6.75}$$

Problem 6.42 For the dark-state polariton:

a) Show that

$$[\hat{\Psi}(z,t), \hat{\Psi}^\dagger(z',t)] = L\delta(z-z') \tag{6.76}$$

if $\hat{a}(z,t)$ and $\hat{\tilde{\sigma}}_{bc}(z,t)$ are treated as independent operators.

b) Show that the motion of the dark-sate polariton is described by

$$\left[c\frac{\Omega_c^2}{g^2NV + \Omega_c^2}\partial_z + \partial_t\right]\Psi(z,t) = 0. \tag{6.77}$$

In other words, the polariton travels with velocity v_{gr} and can be arbitrarily decelerated, accelerated or stopped by varying the control field intensity.

[4]M. Fleischhauer and M. D. Lukin, Phys. Rev. A **65**, 022314 (2002).

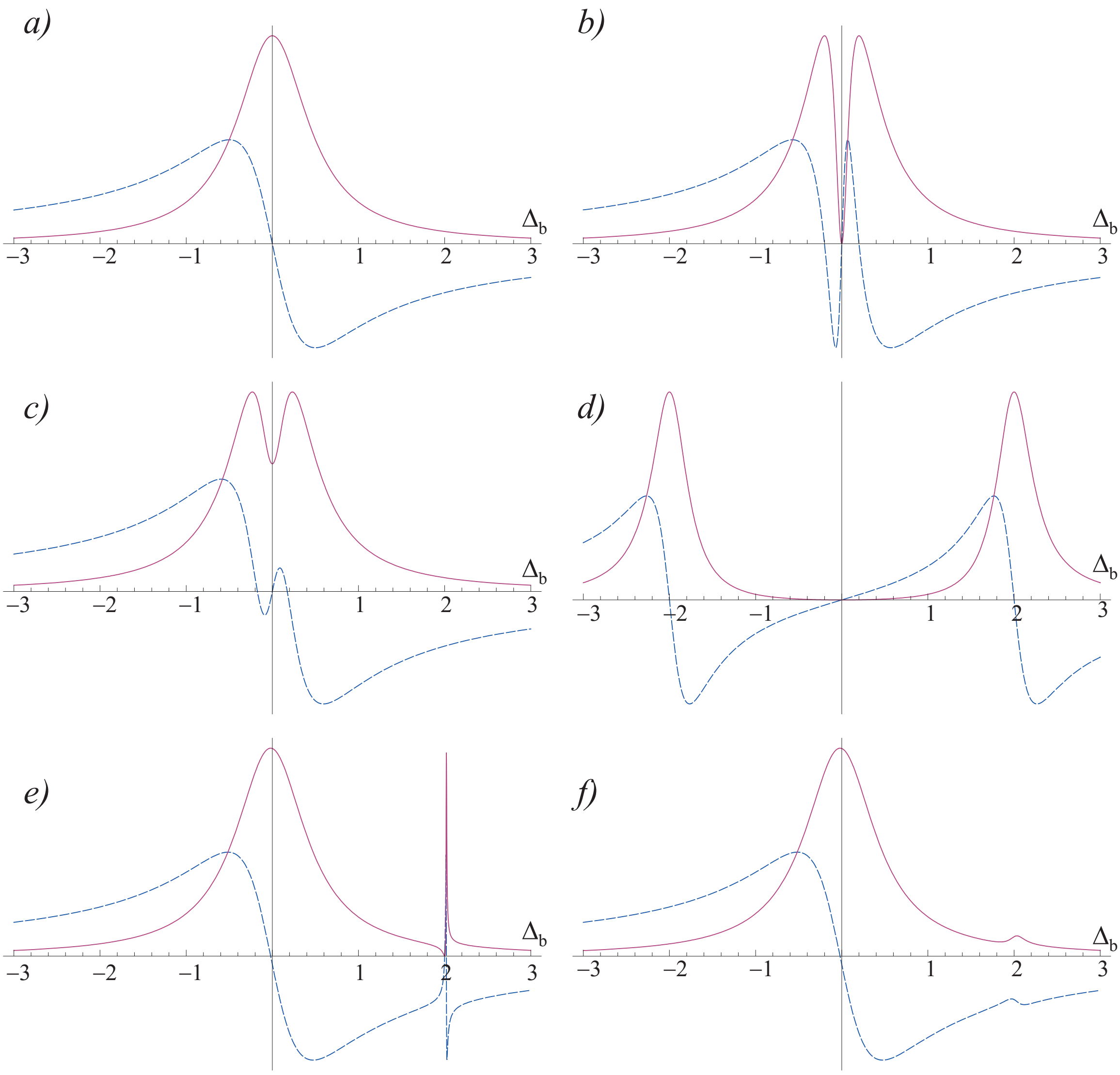


Figure 6.6: Real (dashed) and imaginary (solid) components of the susceptibility of the atomic gas with respect to the probe field under various EIT cunditions. The horizontal axis is in units of $\Gamma_1+\Gamma_2$. (a) $\Omega_c = 0$ (no EIT). (b) $\Omega_c = 0.2\Gamma, \Delta_c = 0, \gamma = 0$ (regular EIT line). (c) $\Omega_c = 0.2\Gamma, \Delta_c = 0, \gamma = 0.1$ (EIT contrast reduced in the presence of ground state decoherence) (d) $\Omega_c = 2\Gamma, \Delta_c = 0, \gamma = 0$ (large control field leads to significant ac Stark shift of the $|a\rangle - |c\rangle$ transition and, so the probe field interacts with the dressed states - the phenomenon known as *Autler-Townes splitting*). (e) $\Omega_c = 0.2\Gamma, \Delta_c = 2\Gamma, \gamma = 0$ (Raman absorption line). (f) $\Omega_c = 0.2\Gamma, \Delta_c = 2\Gamma, \gamma = 0.1$ (the Raman line is extremely sensitive to decoherence).

Appendix A

Quantum mechanics of complex systems

This tutorial is using the single-photon polarization qubit as the model system. The Hilbert space of this system is spanned by the horizontal $|H\rangle$ and vertical $|V\rangle$ polarization states, which form the primary (*canonical*) basis of the Hilbert space. Other useful polarization states are

- $+45°$ polarization state: $|+45°\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$;
- $-45°$ polarization state: $|-45°\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$;
- right circular polarization state: $|R\rangle = (|H\rangle + i\,|V\rangle)/\sqrt{2}$;
- left circular polarization state: $|L\rangle = (|H\rangle - i\,|V\rangle)/\sqrt{2}$.

A.1 The density operator

Definition A.1 Suppose our knowledge of the state of a quantum system is incomplete. We know that a system can be in state $|\psi_1\rangle$ with a probability p_1, in state $|\psi_2\rangle$ with a probability p_2, etc., with $\sum_i p_i = 1$ and the $|\psi\rangle$'s being normalized, but not necessarily orthogonal; the number of $|\psi_i\rangle$'s does not have to be equal to the dimension of the Hilbert space. Such description of the system its called its *statistical ensemble*.

Problem A.1 Suppose an ensemble is measured in basis $\{|a_m\rangle\}$ $(1 \le m \le n)$. Show that the probability of detecting $|a_m\rangle$ is given by

$$\mathrm{pr}_m = \langle a_m|\,\hat{\rho}\,|a_m\rangle\,, \tag{A.1}$$

where

$$\hat{\rho} = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|\,. \tag{A.2}$$

Definition A.2 The operator $\hat{\rho}$ in the equation above is called the *density operator* of the ensemble. The matrix of the density operator $\rho_{jk} = \langle a_j|\,\hat{\rho}\,|a_k\rangle$ in any orthonormal basis $\{|a_j\rangle\}$ is called the *density matrix*.

Note A.1 As follows from Eq. (A.1), the entire information about physical properties of an ensemble (i.e. the probabilities of any possible measurement results) is contained in its density matrix.

We can also see that the diagonal elements of the density matrix equal the probabilities of detecting the system in the corresponding basis states. This result implies that the diagonal elements of the density matrix cannot be negative and that their total is 1.

Note A.2 Frequently, the term "density matrix" is used to call the density operator.

Problem A.2 Write the density matrices (in the canonical basis) of the following ensembles in the photon polarization Hilbert stace:
a) $|H\rangle$;
b) $x|H\rangle + y|V\rangle$;
c) $|H\rangle$ with a probability 1/2, $|V\rangle$ with a probability 1/2;
c) $|+45°\rangle$ with a probability 1/2, $|-45°\rangle$ with a probability 1/2;
d) $(|H\rangle + |V\rangle)/\sqrt{2}$ with a probability 1/2, $|H\rangle$ with a probability 1/4, $|L\rangle$ with a probability 1/4.

Problem A.3 Show that the density operator is Hermitian but not necessarily unitary.

Note A.3 In a continuous-variable basis, the density matrix Eq. (A.2) becomes a two-variable function

$$\rho(x, x') = \langle x|\hat{\rho}|x'\rangle = \sum_i p_i \psi_i(x)\psi_i^*(x'), \tag{A.3}$$

where $\psi_i(x)$ are the wavefunctions of the components of the statistical ensemble.

Problem A.4 Find the representation of the density operator of the state $a|0\rangle + b|1\rangle$ of a harmonic oscillator

a) in the Fock basis;

b) in the position basis.

Definition A.3 An ensemble $\hat{\rho}$ is called *pure* if there exists a state $|\psi_0\rangle$ such that $\hat{\rho} = |\psi_0\rangle\langle\psi_0|$. Otherwise it is called *mixed.*

Problem A.5 Show that for a given density operator, there always exists a decomposition in the form (A.2), i.e. as a sum of pure state density operators. (**Hint:** If the states $|\psi_i\rangle$ in decomposition (A.2) are orthogonal, they are eigenstates of the density operators and diagonalize it).

Problem A.6 Show that such a decomposition is not always unique by using the result of Ex. A.2(d) as an example.

Note A.4 In other words, different non-pure ensembles can give rise to the same density operator. All these ensembles exhibit identical physical behavior, so by performing measurements, we cannot determine the history of how the ensemble was prepared.

Problem A.7 Show that for a pure state $|\psi_0\rangle$, there is no other pure state decomposition (A.2) of its density operator $\hat{\rho}$ aside from the trivial $\hat{\rho} = |\psi_0\rangle\langle\psi_0|$.

Problem A.8 Which of the states of Ex. A.2 are pure?

Problem A.9 Show that a diagonal density matrix with more than one non-zero element necessarily represents a non-pure ensemble.

Definition A.4 The ensemble with a density operator $\hat{\rho} = \hat{\mathbf{1}}/n$ (where n is the dimension of the Hilbert space) is called *fully mixed.* With the system described by such an ensemble, no information about the system is available.

Definition A.5 Show that the density matrix of a fully mixed state is basis independent. Interpret this result.

Problem A.10 Show that the evolution of the density matrix in the Schrödinger picture is governed by

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]. \tag{A.4}$$

and thus

$$\hat{\rho}(t) = e^{-\frac{i}{\hbar}\hat{H}t}\hat{\rho}(0)e^{\frac{i}{\hbar}\hat{H}t} \tag{A.5}$$

Note the opposite sign in comparison with Eqs. (4.53) and (4.55).

A.2 Trace

Definition A.6 The *trace* of an operator $\hat{A}$ is the sum of its matrix's diagonal elements:

$$\mathrm{Tr}\hat{A} = \sum_{m=1}^{n} \langle a_m | \hat{A} | a_m \rangle , \tag{A.6}$$

where $\{|a_m\rangle\}$ is a basis of the Hilbert space.

Note A.5 A trace is a number.

Problem A.11 Show that the trace of an operator is basis independent (even though the matrix does depend on the basis!).

Problem A.12 Show that $\mathrm{Tr}(\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{A})$.

Note A.6 It follows that $\mathrm{Tr}(\hat{A}_1 \ldots \hat{A}_k) = \mathrm{Tr}(\hat{A}_k \hat{A}_1 \ldots \hat{A}_{k-1})$ (chain rule).

Problem A.13 Propose an example showing that, generally speaking, $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) \neq \mathrm{Tr}(\hat{B}\hat{A}\hat{C})$.

Problem A.14 Show that for a density operator $\hat{\rho}$, $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ with equality if and only if $\hat{\rho}$ describes a pure state.

Note A.7 The quantity $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ is often used as a purity measure of state $\hat{\rho}$.

Problem A.15 Suppose a non-destructive projection measurement in the basis $\{|a_m\rangle\}$ is performed on an ensemble $\hat{\rho}$ and yields the result $|a_m\rangle$. Show that:
a) the (unnormalized) ensemble after the measurement is given by

$$\hat{\rho}' = \Pi_m \hat{\rho} \hat{\Pi}_m, \tag{A.7}$$

where $\hat{\Pi}_m = |a_m\rangle\langle a_m|$ is the projection operator;
b) the probability of this event is

$$\mathrm{pr}_m = \mathrm{Tr}\hat{\rho}'. \tag{A.8}$$

Note A.8 Eqs. (A.7) and (A.8) also apply to partial measurements on a tensor product system. For example, if Alice and Bob share an ensemble $\hat{\rho}_{AB}$ and Alice performs a measurement in the basis $\{|a_m\rangle_A\}$ on her part of the ensemble, the resulting state is described by Eq. (A.7) with $\hat{\Pi}_m = |a_m\rangle_A \langle a_m|_A \otimes \hat{\mathbf{1}}_B$.

Problem A.16 Apply Eq. (A.7) to determine the probability of detecting a (+45°)-polarization in each of the ensembles of Ex. A.2.

Problem A.17 Show that the expectation value of an observable $\hat{X}$ in an ensemble $\hat{\rho}$ is $\mathrm{Tr}(\hat{\rho}\hat{X})$.

Problem A.18 Suppose an ensemble is subjected to a non-destructive projective measurement, and the measurement result is not known. Show that such a procedure will set all the off-diagonal elements from the ensemble's density matrix in the measurement basis to zero, while leaving the diagonal elements intact.

Definition A.7 A *partial trace* of a bipartite ensemble $\hat{\rho}_{AB}$ over subsystem A is the density operator in Hilbert space B is defined as

$$\mathrm{Tr}_A(\hat{\rho}_{AB}) = \sum_{m=1}^{n} \langle a_m | \hat{\rho}_{AB} | a_m \rangle . \tag{A.9}$$

It is also called the *reduced density operator*.

Problem A.19 Show that the partial trace is basis independent.

Problem A.20 Show that the reduced density operator describes the quantum state of Bob's subsystem if no information about Alice's subsystem is available.

Problem A.21 Show that if the original bipartite ensemble is in a pure, separable (non-entangled) state, then both Alice's and Bob's reduced density operators are also pure states.

Problem A.22 For each of the four Bell states, find the reduced density operator for each qubit.

Problem A.23 Find the reduced density operator for Alice's qubit of the state $|\Psi_{AB}\rangle = x|00\rangle + y|11\rangle + z|01\rangle + t|10\rangle$.

Problem A.24 Show that for an entangled state of two qubits $|\Psi_{AB}\rangle = x|00\rangle + y|11\rangle$, the reduced density matrix is

$$\hat{\rho}_A = \begin{pmatrix} |x|^2 & 0 \\ 0 & |y|^2 \end{pmatrix}. \tag{A.10}$$

Note A.9 Remarkably, a reduced density operator of a pure state can be a mixed state.

A.3 Measurement, entanglement, and decoherence

How a measurement apparatus works. We would like to perform a measurement associated with the $\{|0\rangle, |1\rangle\}$ basis on a system which is initially in a pure state $|\psi\rangle_s = x|0\rangle_s + y|1\rangle_s$. We employ an apparatus which is initially in the state $|\psi\rangle_A = |0\rangle_A$. Note that the apparatus is a complex entity, and its state is the simultaneous state of the many particles composing it. We bring the apparatus into interaction with the system and it *entangles* itself with it, producing a joint state $|\Psi_{sA}\rangle = x|0\rangle_s \otimes |0\rangle_A + y|1\rangle_s \otimes |1\rangle_A$. The apparatus is then removed, after which the system is in the state (A.10), which is consistent with Ex. A.18. If the apparatus is in the state $|0\rangle$, the system is also in the state $|0\rangle$, and the associated probability is $|x|^2$. The probability of the state $|1\rangle$ is $|y|^2$.

Definition A.8 A system may interact with the environment and entangle itself with it. Since the state of the environment cannot be controlled or measured, the system's density matrix (in the entanglement basis) will lose its off-diagonal elements as a result of such interaction (see Ex. A.18 and A.24). This undesired loss of quantum information caused by such "inadvertent" measurements is called *irreversible (homogeneous) dephasing* or *decoherence*.

Problem A.25 Consider an ensemble of electrons in an initial state $|\psi(t=0)\rangle = (|m_s = +1/2\rangle + |m_s = -1/2\rangle)/\sqrt{2}$. The electron is placed in a magnetic field $\vec{B}$ pointing in the z direction.
a) Find the density matrix in the $m_s = \pm 1/2$ basis and its evolution.
b) Suppose the ensemble experiences decoherence so, in addition to the precession in the field, the off-diagonal elements of the density matrix decay according to the factor $e^{-\gamma t}$ with $\gamma \ll \mu_B B/\hbar$. Find the expectation value of the x-component of the spin and plot it as a function of time.

Note A.10 Since most physical interactions have a highly local character, the position basis is particularly likely to become the basis in which the entanglement with the environment will take place. For example, a particle in a superposition state $|\psi\rangle = (|x_1\rangle + |x_2\rangle)/\sqrt{2}$ will entangle itself with the state of the environment ($|E\rangle$) in the following manner:

$$|\psi\rangle \otimes |E\rangle \to \frac{1}{\sqrt{2}} \left(|x_1\rangle \otimes |E_{x1}\rangle + |x_2\rangle \otimes |E_{x2}\rangle\right), \tag{A.11}$$

where $|E_{x1}\rangle$ and $|E_{x2}\rangle$ are two "incompatible" (i.e. orthogonal) states of the environment "having seen" the particle in positions x_1 and x_2, respectively. Because the environment cannot be measured, the state of the particle becomes a non-pure ensemble

$$\hat{\rho}_{\text{after}} = \frac{1}{2} \left(|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2|\right), \tag{A.12}$$

i.e. localized *either* at x_1 *or* at x_2, but showing no coherence of these two states. For this reason, macroscopic coherent superpositions (i.e. the "Schrödinger cat" states) do not occur in nature.

Appendix B

Solutions to Chapter 1 problems

Solution to Exercise 1.1. if we substitute Eqs. (1.5) and (1.6) into Maxwell equation (1.4) we find

$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \dot{\vec{E}} - \mu_0 \dot{\vec{P}}. \tag{B.1}$$

On the other hand, taking the curl of both sides Eq. (1.3) we obtain

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \dot{\vec{B}} \tag{B.2}$$

The left-hand side can be expanded according to the relation (1.10). We use Maxwell equation $\vec{\nabla} \cdot \vec{D} = 0$ to conclude that the first term in the right-hand side of Eq. (1.10) vanishes[1]. Accordingly, we have

$$-(\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\vec{\nabla} \times \dot{\vec{B}}. \tag{B.5}$$

Taking the time derivatives of both sides of Eq. (B.1) and equalizing its left-hand side with the right-hand side of Eq. (B.5), we find

$$-\vec{\nabla}^2 \vec{E} = -\epsilon_0 \mu_0 \ddot{\vec{E}} - \mu_0 \ddot{\vec{P}}. \tag{B.6}$$

Now substituting $c = 1/\sqrt{\epsilon_0 \mu_0}$ for the speed of light in vacuum leads us to Eq. (1.9).

Solution to Exercise 1.2. Let $\vec{E}(\vec{r}, t) = \vec{E}_0 \exp(i\vec{k} \cdot \vec{r} - i\omega t) + c.c$ be the solution to the wave equation. Then

$$\nabla^2 E = -k^2 [\vec{E}_0 \exp(i\vec{k} \cdot \vec{r} - i\omega t) + c.c] \tag{B.7}$$

and

$$\ddot{\vec{E}} = -\omega^2 [\vec{E}_0 \exp(i\vec{k} \cdot \vec{r} - i\omega t) + c.c] \tag{B.8}$$

In the absence of non-linearity, $\vec{P}_{NL} = 0$

$$\ddot{\vec{P}} = \ddot{\vec{P}}_L = -\epsilon_0 \chi \omega^2 [\vec{E}_0 \exp(i\vec{k} \cdot \vec{r} - i\omega t) + c.c]. \tag{B.9}$$

Substituting these results into in the electromagnetic wave equation (1.9), we have

$$-k^2 = -\frac{\omega^2}{c^2}(1 + \chi). \tag{B.10}$$

[1] This observation is trivial in a linear medium because then $\vec{D} = \epsilon_0(1+\chi)\vec{E}$. Nonlinear correction to the polarization is typically small and can be neglected. A meticulous reader can however obtain $\vec{\nabla}\vec{E}$ rigorously in the nonlinear case as follows. We use Eq. (1.6) to write

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\vec{\nabla} \cdot \vec{D} - \vec{\nabla} \cdot \vec{P}) = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \vec{P}. \tag{B.3}$$

Expressing the linear component of $\vec{P}_L = \epsilon_0 \chi \vec{E}$ and the nonlinear component as $\vec{P}_{NL} = \epsilon_0 \chi^{(2)} \vec{E}^2 + \epsilon_0 \chi^{(3)} \vec{E}^3 + \ldots$, we obtain from Eq. (B.3),

$$\vec{\nabla} \cdot \vec{E}(1 + \frac{\chi}{\epsilon} + 2\frac{\chi^{(2)}}{\epsilon_0}\vec{E} + 3\frac{\chi^{(3)}}{\epsilon_0}\vec{E}^2 + \ldots) = 0. \tag{B.4}$$

We see that the given solution (1.11) is in agreement with the electromagnetic wave equation as long as the wavevector k satisfies equation (B.10).

Solution to Exercise 1.3.

a) Assuming a plane wave solution (1.11) we get:

$$\begin{aligned}
\vec{P}_L(t) &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)\vec{E}(t-\tau)\mathrm{d}\tau \\
&= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)(\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t+i\omega\tau}+\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t-i\omega\tau})\mathrm{d}\tau \\
&= \frac{\epsilon_0}{2\pi}\left(\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{i\omega\tau}\mathrm{d}\tau+\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{-i\omega\tau}\mathrm{d}\tau\right) \\
&= \epsilon_0\left(\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}\chi(\omega)+\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}\chi(-\omega)\right) \\
&= \epsilon_0\chi(\omega)\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}+\epsilon_0\chi(-\omega)\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t} \\
&= \epsilon_0\chi(\omega)\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}+\epsilon_0\chi^*(\omega)\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}
\end{aligned}$$

In the last line above we used the fact that

$$\chi(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{-i\omega\tau}\mathrm{d}\tau = \chi^*(\omega) \tag{B.11}$$

since $\tilde{\chi}(\tau)$ is an observable and must be real.

Substituting the above expression for the polarization into the wave equation (1.9), which we rewrite as

$$\vec{\nabla}^2\vec{E}-\frac{1}{c^2}\ddot{\vec{E}}-\frac{1}{\epsilon_0c^2}\ddot{\vec{P}},$$

gives

$$\begin{aligned}
&\vec{\nabla}^2\vec{E}-\frac{1}{c^2}\ddot{\vec{E}}-\frac{1}{\epsilon_0c^2}\ddot{\vec{P}} \\
&= -\vec{k}^2\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}+-\vec{k}^{*2}\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}+\frac{\omega^2}{c^2}\vec{E}+\frac{\omega^2}{\epsilon_0c^2}\vec{P} \\
&= -\vec{k}^2\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}+-\vec{k}^{*2}\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t} \\
&\quad+\frac{\omega^2}{c^2}\left((1+\chi(\omega))\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t}+(1+\chi^*(\omega))\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}\right) \\
&= \left(-\vec{k}^2+\frac{\omega^2(1+\chi(\omega))}{c^2}\right)\vec{E}_0e^{i\vec{k}.\vec{r}-i\omega t} \\
&\quad\left(-\vec{k}^{*2}+\frac{\omega^2(1+\chi^*(\omega))}{c^2}\right)\vec{E}_0^*e^{-i\vec{k}^*.\vec{r}+i\omega t}
\end{aligned}$$

which vanishes for

$$\vec{k}^2 = \frac{\omega^2}{c^2}(1+\chi(\omega))$$

in agreement with Eq. (1.12).

b) Consider the special case where $\tilde{\chi}(\tau) = 2\pi\chi\delta(\tau)$ with $\chi = \text{const}$.
From Eq. (1.14) we get for $\chi(\omega)$:

$$\begin{aligned}
\chi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{i\omega\tau}\mathrm{d}\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\chi\delta(\tau)e^{i\omega\tau}\mathrm{d}\tau \\
&= \chi\int_{-\infty}^{\infty}\delta(\tau)e^{i\omega\tau}\mathrm{d}\tau \\
&= \chi
\end{aligned}$$

and from Eq. (1.13) we get for $\vec{P}_L(t)$:

$$\begin{aligned}\vec{P}_L(t) &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)\vec{E}(t-\tau)\mathrm{d}\tau\\ &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}2\pi\chi\delta(\tau)\vec{E}(t-\tau)\mathrm{d}\tau\\ &= \epsilon_0\chi\int_{-\infty}^{\infty}\delta(\tau)\vec{E}(t-\tau)\mathrm{d}\tau\\ &= \epsilon_0\chi\vec{E}(t)\end{aligned}$$

which is what we assumed in Eq. (1.8).

Solution to Exercise 1.4. If the susceptibility is a complex number $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$, so are the index of refraction and the wavevector, so we can write $n = n' + in''$ and $k = k' + ik''$. Inserting k into the expression for the field yields

$$E(z,t) = (E_0 e^{ik'z-i\omega t} + E_0^* e^{-ik'z+i\omega t})e^{-k''z}.$$

The intensity is proportional to the square of the amplitude of the field. The complex exponents in the above expression do not affect the amplitude, but the imaginary part of the wavevector translates into a real exponent, so we have

$$I \propto |E_0|^2 e^{-2k''z}.$$

Thus, the intensity is given by

$$I = I_0 e^{-\alpha z}$$

where $\alpha = 2k'' = 2\frac{\omega}{c}\mathrm{Im}\sqrt{1+\chi}$. For small susceptibilities, we approximate $\sqrt{1+\chi} \approx 1 + \chi/2$, thereby obtaining Eq. (1.17).

Solution to Exercise 1.5. To fulfill the condition of slowly varying envelopes, the change of $\mathcal{E}$ or $\mathcal{P}$ after one period T or one wavelength λ should be much smaller than the actual value. We begin with showing that the upper (spatial) part of Eq. (1.19) holds for slowly varying $\mathcal{E}$ with constant time t.

$$\begin{aligned}\Delta\vec{\mathcal{E}}(\Delta z = \lambda) &\ll \vec{\mathcal{E}}(z=0)\\ \vec{\mathcal{E}}(z=\lambda) - \vec{\mathcal{E}}(z=0) &\ll \vec{\mathcal{E}}(z=0)\end{aligned}$$

As λ is small, we will use Taylor expansion around $z = 0$ up to the first derivative.

$$\begin{aligned}\vec{\mathcal{E}}(z=0) + \partial_z\vec{\mathcal{E}}(z=0)(\lambda) - \vec{\mathcal{E}}(z=0) &\ll \vec{\mathcal{E}}(z=0)\\ \partial_z\vec{\mathcal{E}}(z=0)\lambda &\ll \vec{\mathcal{E}}(z=0)\\ \partial_z\vec{\mathcal{E}}(z=0) &\ll \frac{1}{\lambda}\vec{\mathcal{E}}(z=0)\end{aligned}$$

Since the right side is much smaller than the left, this also holds if we multiply with 2π on the right side, which gives us the wave vector k:

$$\partial_z\vec{\mathcal{E}}(z=0) \ll k\cdot\vec{\mathcal{E}}(z=0)$$

The argument for the lower (temporal) part of Eq. (1.19) is similar to the above, and so is the proof of that equation for $\vec{\mathcal{P}}$.

Solution to Exercise 1.6. In order to substitute Eqs. (1.18) into the wave equation, we need to calculate the second derivative of the electric field and polarization. For the spatial dependence we have

$$\vec{\nabla}\vec{E} = ik\vec{\mathcal{E}}e^{ikz-i\omega t} + \vec{\nabla}\vec{\mathcal{E}}e^{ikz-i\omega t} + c.c.$$

and hence

$$\vec{\nabla}^2\vec{E} = (-k^2\vec{\mathcal{E}} + 2ik\vec{\nabla}\vec{\mathcal{E}} + \vec{\nabla}^2\vec{\mathcal{E}}e^{ikz-i\omega t}) + c.c. \tag{B.12}$$

Similarly, for the time dependence we obtain

$$\ddot{\vec{E}} = (-\omega^2\vec{\mathcal{E}} - 2i\omega\dot{\vec{\mathcal{E}}} + \ddot{\vec{\mathcal{E}}})e^{ikz-i\omega t} + c.c. \tag{B.13}$$

$$\ddot{\vec{P}} = (-\omega^2\vec{\mathcal{P}} - 2i\omega\dot{\vec{\mathcal{P}}} + \ddot{\vec{\mathcal{P}}})e^{ikz-i\omega t} + c.c. \tag{B.14}$$

In these expressions, each subsequent term in the parentheses is much smaller than the previous one due to the slow varying envelope approximation (1.19).

Let us now specialize to part (a) of the problem. For the reason that will become clear in a moment, we keep two first terms in Eqs. (B.12) and (B.13). Inserting them into Eq. (1.9), we find

$$-k^2\vec{\mathcal{E}} + 2ik\vec{\nabla}\vec{\mathcal{E}} - \frac{1}{c^2}(-\omega^2\vec{\mathcal{E}} - 2i\omega\dot{\vec{\mathcal{E}}}) = -\frac{\omega^2}{c^2\epsilon_0}\vec{\mathcal{P}}$$

Since $k = \frac{\omega}{c}$, the first and third terms in the left-hand side of the above equation cancel (this is why we had to keep two terms). After multiplying both sides by $\frac{1}{2ik}$, we get:

$$\left(\partial_z + \frac{1}{c}\partial_t\right)\vec{\mathcal{E}}(t,z) = i\frac{\omega}{2\epsilon_0 c}\vec{\mathcal{P}}(t,z). \tag{B.15}$$

For (b) we have to take a closer look at the two parts of the polarization,

$$\vec{\mathcal{P}} = \vec{\mathcal{P}}_L + \vec{\mathcal{P}}_{NL} \tag{B.16}$$

with[2]

$$\vec{\mathcal{P}}_L = \chi\epsilon_0\vec{\mathcal{E}} \tag{B.17}$$

We rewrite Eq. (B.14) accordingly, keeping the first two terms in the linear part of the polarization, but only the first term in its nonlinear part:

$$\ddot{\vec{P}} = (\epsilon_0\chi\ddot{\vec{\mathcal{E}}} - 2i\omega\epsilon_0\chi\dot{\vec{\mathcal{E}}} - \epsilon_0\chi\omega^2\vec{\mathcal{E}})e^{ikz-i\omega t} - \omega^2\vec{\mathcal{P}}_{NL}e^{ikz-i\omega t} + c.c. \tag{B.18}$$

Now we insert this again into the wave equation:

$$-k^2\vec{\mathcal{E}} + 2ik\vec{\nabla}\vec{\mathcal{E}} - \frac{1}{c^2}(-\omega^2\vec{\mathcal{E}} - 2i\omega\dot{\vec{\mathcal{E}}}) = \frac{1}{c^2\epsilon_0}(-\epsilon_0\chi\omega^2\vec{\mathcal{E}} - 2i\omega\epsilon_0\chi\dot{\vec{\mathcal{E}}} - \omega^2\vec{\mathcal{P}}_{NL}) \tag{B.19}$$

If we use the wave vector $k = \frac{\omega}{c}\sqrt{1+\chi}$, the first and third terms of the left side cancel with the first term of the right side. Using again the expression for the wave vector, we simplify the above as follows:

$$2ik\vec{\nabla}\vec{\mathcal{E}} + 2i\frac{k^2}{\omega}\dot{\vec{\mathcal{E}}} = -\frac{\omega^2}{c^2\epsilon_0}\vec{\mathcal{P}}_{NL}, \tag{B.20}$$

which is the same as Eq. (1.21).

Solution to Exercise 1.7. Substituting the electric field from Eq. (1.18a) into Eq. (1.13) we express the linear part of the polarization as follows.

$$\vec{P}_L(t) = \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)\vec{\mathcal{E}}(z,t-\tau)e^{ikz-i\omega t+i\omega\tau}\,d\tau \tag{B.21}$$

Taylor expanding $\vec{\mathcal{E}}(t-\tau)$ up to the first derivative, we obtain:

$$\vec{\mathcal{E}}(t-\tau) = \vec{\mathcal{E}}(t) - \tau\dot{\vec{\mathcal{E}}}(t) \tag{B.22}$$

[2] This holds only if we have no dispersion.

Accordingly,

$$\begin{aligned}\vec{P}_L(t) &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(\tau)e^{i\omega\tau}\vec{\mathcal{E}}(t)e^{ikz-i\omega t}\,d\tau \\ &- \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty}\tau\tilde{\chi}(\tau)e^{i\omega\tau}\dot{\vec{\mathcal{E}}}(t)e^{ikz-i\omega t}\,d\tau + c.c.\end{aligned} \tag{B.23}$$

Taking the derivative of both sides of the Fourier transformation (1.15), we obtain

$$-i\frac{d\chi(\omega)}{d\omega} = \frac{1}{2\pi}\int\tau\tilde{\chi}(\tau)e^{i\omega\tau}d\tau \tag{B.24}$$

Equation (B.23) can now be rewritten as,

$$\vec{P}_L = \underbrace{\epsilon_0\left(\chi(\omega)\vec{\mathcal{E}} + i\frac{d\chi(\omega)}{d\omega}\frac{d\vec{\mathcal{E}}}{d\omega}\right)}_{\vec{\mathcal{P}}_L}e^{ikz-i\omega t} \tag{B.25}$$

Let us now find the second time derivative of this polarization so we can plug it into the wave equation. For the first derivative, we have

$$\dot{\vec{P}}_L = \epsilon_0\chi(\omega)(-i\omega\vec{\mathcal{E}} + \dot{\vec{\mathcal{E}}})e^{ikz-i\omega t} + i\epsilon_0\frac{d\chi(\omega)}{d\omega}(-i\omega\dot{\vec{\mathcal{E}}} + \ddot{\vec{\mathcal{E}}})e^{ikz-i\omega t}$$

We can safely ignore the last term in the above expression according to the slow-varying envelope assumption (1.19). Differentiating again and applying the same approximation, we get

$$\ddot{\vec{P}}_L = -\epsilon_0\chi(\omega)(2i\omega\dot{\vec{\mathcal{E}}} + \omega^2\vec{\mathcal{E}})e^{ikz-i\omega t} - i\omega^2\epsilon_0\frac{d\chi(\omega)}{d\omega}\dot{\vec{\mathcal{E}}}e^{ikz-i\omega t} \tag{B.26}$$

For the nonlinear part of the polarization, we keep only the most significant term:

$$\ddot{\vec{P}}_{NL} = -\omega^2\vec{\mathcal{P}}_{NL}e^{ikz-i\omega t} \tag{B.27}$$

Substituting $\ddot{\vec{P}} = \ddot{\vec{P}}_L + \ddot{\vec{P}}_{NL}$ along with Eqs. (B.12) and (B.13) into the electromagnetic wave equation and some rearranging yields the desired result

$$\left[\partial_z + \frac{1}{v_{gr}}\partial_t\right]\vec{\mathcal{E}} = i\frac{\omega^2}{2\epsilon_0 kc^2}\vec{\mathcal{P}}_{NL} \tag{B.28}$$

with

$$\frac{1}{v_{gr}} = \frac{dk}{d\omega} = \frac{k}{\omega} + \frac{\omega^2}{2kc^2}\frac{d\chi}{d\omega} \tag{B.29}$$

Solution to Exercise 1.7. See Eq. (B.11) above.

Solution to Exercise 1.10.

a) Because there are no poles inside the contour, the integral vanishes in accordance with the residue theorem.

b) We can parametrize the half circle integral by assuming $\omega' = Re^{i\phi}$, with $R \to \infty$, $\phi \in [0,\pi]$. Accordingly, $\mathrm{d}\omega' = iRe^{i\phi}\mathrm{d}\phi$. Since ω is just a constant in the denominator we neglect it for large R, which leaves the following integral:

$$\lim_{R\to\infty}\int_0^{\pi}\frac{\chi(\omega')}{Re^{i\phi}}iRe^{i\phi}\ \mathrm{d}\phi' = i\lim_{R\to\infty}\int_0^{\pi}\chi(\omega')\ \mathrm{d}\phi' \tag{B.30}$$

In order to prove that the above integral vanishes it suffices to show that $\chi(\omega') \to 0$ for $|\omega'| \to \infty$. To do that we use relationship (1.15) between $\chi(\omega')$ and its Fourier transform $\tilde{\chi}(\tau)$.

$$\chi(\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\chi}(\tau)e^{i\omega'\tau}\mathrm{d}\tau = -\frac{i}{2\pi\omega'}\int_{-\infty}^{\infty} \dot{\tilde{\chi}}(\tau)e^{i\omega'\tau}\mathrm{d}\tau. \tag{B.31}$$

Now we have to take some physics into account. Because the time-dependent susceptibility has the physical meaning of the medium's dipole moment response to an instantaneous excitation (see Sec. 1.1), it is bounded in time: the response eventually ends some time after the excitation has occurred. Furthermore, we do not expect any instant "jumps" in the time-dependent dipole moment, and hence $\dot{\tilde{\chi}}(\tau)$ is also bounded. Accordingly, the integral above is also bounded by a quantity that is independent of ω'. The factor of ω'^{-1} in the right-hand side of Eq. (B.31) therefore ensures that $\chi(\omega')$ tends to zero when $|\omega'|$ tends to infinity.

c) We have to evaluate the integral over an infinitesimal semi-circle BC in the clock-wise direction. The integration involves a simple pole on the path of integration. Then the integral in the counter-clockwise direction along an arc centered at ω of angle π equals

$$\int_{BC} \frac{\chi(\omega')}{\omega' - \omega}d\omega' = -i\pi\chi(\omega).$$

d) Summarizing parts (a)–(c) we have

$$\int_{AB} \frac{\chi(\omega')}{\omega' - \omega}d\omega' + \int_{CD} \frac{\chi(\omega')}{\omega' - \omega}d\omega' = \pi i\chi(\omega)$$

The left-hand side of the above equation is the principal value of the integral over the real axis. Accordingly,

$$\chi(\omega) = \frac{1}{i\pi}\mathcal{VP}\int_{-\infty}^{+\infty} \frac{\chi(\omega')}{\omega' - \omega}d\omega'$$

Solution to Exercise 1.11. We can Eq. (1.26) into the real and imaginary parts $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$ as follows:

$$\chi'(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega}\mathrm{d}\omega'; \tag{B.32}$$

$$\chi''(\omega) = = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega}\mathrm{d}\omega'. \tag{B.33}$$

We can split the integral for the real part into the positive and negative frequency parts.

$$\chi'(\omega) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega}\mathrm{d}\omega' + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\chi''(\omega')}{\omega' - \omega}\mathrm{d}\omega'. \tag{B.34}$$

Our goal is to eliminate the negative frequencies. To that end, we use Eq. (1.24), which we rewrite as $\chi'(-\omega) = \chi'(\omega)$ and $\chi''(-\omega) = -\chi''(\omega)$. Changing variables from ω' to $-\omega'$:

$$\begin{aligned}
\chi'(\omega) &= \frac{1}{\pi}\int_{0}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega}\mathrm{d}\omega' + \frac{1}{\pi}\int_{\infty}^{0} \frac{\chi''(-\omega')}{(-\omega') - \omega}\mathrm{d}(-\omega') \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega}\mathrm{d}\omega' + \frac{1}{\pi}\int_{0}^{\infty} \frac{-\chi''(\omega')}{-\omega' - \omega}\mathrm{d}\omega' \\
&= \frac{1}{\pi}\int_{0}^{\infty} \chi''(\omega')\left(\frac{1}{\omega' - \omega} - \frac{1}{-\omega' - \omega}\right)\mathrm{d}\omega' \\
&= \frac{1}{\pi}\int_{0}^{\infty} \chi''(\omega')\left(\frac{2\omega'}{\omega'^2 - \omega^2}\right)\mathrm{d}\omega' \\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{\omega'\chi''(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega'.
\end{aligned}$$

Similarly, for the imaginary part:

$$
\begin{aligned}
\chi''(\omega) &= -\frac{1}{\pi}\int_0^\infty \frac{\chi'(\omega')}{\omega'-\omega}\mathrm{d}\omega' - \frac{1}{\pi}\int_\infty^0 \frac{\chi'(-\omega')}{(-\omega')-\omega}\mathrm{d}(-\omega') \\
&= -\frac{1}{\pi}\int_0^\infty \frac{\chi'(\omega')}{\omega'-\omega}\mathrm{d}\omega' + \frac{1}{\pi}\int_0^\infty \frac{\chi'(\omega')}{\omega'+\omega}\mathrm{d}\omega' \\
&= -\frac{1}{\pi}\int_0^\infty \chi'(\omega')\left(\frac{1}{\omega'-\omega} - \frac{1}{\omega'+\omega}\right)\mathrm{d}\omega' \\
&= -\frac{1}{\pi}\int_0^\infty \chi'(\omega')\left(\frac{2\omega}{\omega'^2-\omega^2}\right)\mathrm{d}\omega' \\
&= -\frac{2\omega}{\pi}\int_0^\infty \frac{\chi'(\omega')}{\omega'^2-\omega^2}\mathrm{d}\omega'.
\end{aligned}
$$

Solution to Exercise 1.12. We start with Newton's law and sum up all forces acting on the electron:

$$m\ddot{x} = eE - \kappa x - m\Gamma\dot{x} \tag{B.35}$$

- The first term on the right-hand side is the Coulomb force,
- The second term is due to Hooke's law; the spring constant κ can be estimated according to $\omega_0 = \sqrt{\frac{\kappa}{m}}$
- The last term corresponds to the velocity dependent damping.

Let's now insert the usual oscillatory expression for x and the electric field E.

$$E = E_0 e^{ikz-i\omega t} + c.c. \tag{B.36}$$

$$x = x_0 e^{ikz-i\omega t} + c.c. \tag{B.37}$$

The exponential factor being common, it can be canceled from both sides and we obtain an equation for x_0

$$-m\omega^2 x_0 = eE_0 - m\omega_0^2 x_0 + i\omega m\Gamma x_0 \tag{B.38}$$

Solving this for x_0 leads to

$$x_0 = \frac{eE_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma} \tag{B.39}$$

To calculate $\chi(\omega)$ we need to compute the polarization P, which is the dipole moment per unit volume and can thus be written as $P = Nex$. On the other hand, we know that $P = \epsilon_0\chi(\omega)E$, i.e.:

$$Ne\left(x_0 e^{ikz-i\omega t} + c.c.\right) = \epsilon_0\chi(\omega)\left(E_0 e^{ikz-i\omega t} + c.c.\right),$$

from which we find

$$\chi(\omega) = \frac{Nex_0}{\epsilon_0 E_0} = \frac{e^2 N}{\epsilon_0 m}\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma}.$$

For $\omega \sim \omega_0$ and $\omega - \omega_0 = \Delta$, we can simplify the above expression using

$$\omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) = 2\omega_0\Delta,$$

so

$$\chi = -\frac{e^2 N}{\epsilon_0 m\omega_0}\frac{1}{2\Delta + i\Gamma}$$

which gives us the frequency dependent susceptibility in the classical theory of dispersion.

Solution to Exercise 1.13.

a) Because of the short electric field pulse, there is a change in momentum of the electron which is equal to the impulse Δp caused by the pulse.

$$\Delta p = \int eEdt = e\int A\delta(t)dt = Ae$$

Now initially, before the impulse, the electron was at rest. Hence, $\Delta p = mv_0$ where v_0 is the velocity of the electron immediately after the delta-function pulse. Therefore,

$$v_0 = Ae/m$$

b) We write the equation of motion of the electron,

$$m\ddot{x} + m\Gamma\dot{x} + m\omega_0^2 x = 0 \tag{B.40}$$

(since the electric field i.e. the driving force is zero right after the delta-function pulse), with ω_0 being the resonant frequency. This is just the differential equation for a damped simple harmonic oscillator.

If we look for a solution of the form $x = e^{\lambda t}$, Eq. (B.40) reduces to

$$\lambda^2 + \omega_0^2 + \Gamma\lambda = 0,$$

which we solve as

$$\lambda = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2}.$$

For $\omega_0 >> \Gamma$, we approximate $\sqrt{\Gamma^2 - 4\omega_0^2} \approx 2i\omega_0$ and hence

$$\lambda = -\Gamma/2 \pm i\omega_0.$$

Thus the solution is

$$x(t) = Ce^{(i\omega_0-\Gamma/2)t} + Be^{(-i\omega_0-\Gamma/2)t}.$$

Using the initial conditions, $x(0) = 0$ and $\dot{x}(0) = v_0$ we get,

$$C = -B = \frac{v_0}{2i\omega_0}.$$

Therefore the motion of the electrons after the pulse is described by

$$\begin{aligned} x(t) &= \frac{v_0}{2i\omega_0}\left(e^{(i\omega_0-\Gamma/2)t} - e^{(-i\omega_0-\Gamma/2)t}\right) \\ &= \frac{Ae}{m\omega_0}\sin(\omega_0 t)e^{-\frac{\Gamma t}{2}}. \end{aligned}$$

c) The time-dependent polarization is given by

$$P(t) = Nex(t) = \frac{NAe^2}{m\omega_0}\sin(\omega_0 t)e^{-\frac{\Gamma t}{2}}.$$

Comparing this expression with Eq. (1.14), we find

$$\chi(t) = \frac{2\pi Ne^2}{m\omega_0\epsilon_0}\sin(\omega_0 t)e^{-\frac{\Gamma t}{2}} = \frac{2\pi Ne^2}{2im\omega_0\epsilon_0}\left(e^{(i\omega_0-\Gamma/2)t} - e^{(-i\omega_0-\Gamma/2)t}\right).$$

In order to find the frequency-dependent susceptibility, we perform the Fourier transform:

$$\begin{aligned}
\chi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\chi}(t)e^{i\omega t}dt \\
&= \frac{1}{2\pi}\int_{0}^{\infty}\tilde{\chi}(t)e^{i\omega t}dt \\
&= \frac{Ne^2}{2i\epsilon_0 m\omega_0}\int_0^{\infty}\left(e^{-i\omega_0 t+i\omega t-\frac{\Gamma}{2}t}-e^{i\omega_0 t+i\omega t-\frac{\Gamma}{2}t}\right)dt \\
&= \frac{Ne^2}{2i\epsilon_0 m\omega_0}\left(\frac{1}{i\omega_0-i\omega+\frac{\Gamma}{2}}-\frac{1}{-i\omega_0-i\omega+\frac{\Gamma}{2}}\right) \\
&= \frac{Ne^2}{\epsilon_0 m\omega_0}\left(\frac{4\omega_0}{(-2\omega_0+2\omega+i\Gamma)(2\omega_0+2\omega+i\Gamma)}\right).
\end{aligned}$$

In calculating the integral, we used the fact that $\int_0^{-\infty} e^{-\alpha x} = 1/\alpha$ even is α is complex as long as $\text{Re}\alpha > 0$.

Let $\Delta = \omega - \omega_0$ and assume $\Delta << \omega_0$. Then the above result simplifies as follows:

$$\begin{aligned}
\chi(\omega) &\approx \frac{Ne^2}{\epsilon_0 m\omega_0}\left(\frac{4\omega_0}{(2\Delta+i\Gamma)(4\omega_0+i\Gamma)}\right) \\
&\approx \frac{Ne^2}{\epsilon_0 m\omega_0}\frac{1}{(2\Delta+i\Gamma)}.
\end{aligned}$$

Solution to Exercise 1.14.

a) From Eq. (1.17), we have

$$\alpha(\omega) = \frac{\omega}{c}\chi''(\omega)$$

where χ'' is the imaginary part of the susceptiblity of the material. From Eq. (1.29), we can calculate the imaginary part of susceptiblity to be

$$\chi''(\omega) \approx \frac{e^2 N}{m\epsilon_0\omega_0}\frac{\Gamma}{4\Delta^2+\Gamma^2}.$$

On resonance we have $\Delta = \omega - \omega_0 = 0$. Accordingly,

$$\alpha(\omega) \approx \frac{e^2 N}{mc\epsilon_0\Gamma} \tag{B.41}$$

Substituting all the values into the above equation we get

$$\alpha = 2.81 * 10^3\text{m}^{-1}.$$

b) The width (FWHM) of the Doppler broadened line equals

$$\Gamma_{\text{Doppler}} = \sqrt{\frac{8kT\ln 2}{m}}\frac{2\pi}{\lambda}$$

(see, e.g., *Wikipedia* for the derivation of this). At room temperature,

$$\Gamma_{\text{Doppler}} = 3.15 * 10^9\ \text{s}^{-1}.$$

We should not thoughtlessly substitute this result into Eq. (B.41), because that equation has been derived for oscillators that are not moving, and the situation we are dealing with now is different. Let us however consider the following handwaving argument. A wave of a particular frequency within the Doppler broadened line interacts only with those atoms whose velocity is such that they are in resonance with the wave. This means that their Doppler shifted frequency must differ from the frequency of the wave by no more than the natural linewidth. The probability of this to be the case for a particular atom is given by $\Gamma/\Gamma_{\rm Doppler}$. Accordingly, the absorption index is given by

$$\alpha = \frac{e^2 N}{mc\epsilon_0 \Gamma_{\rm Doppler}} = 33.5m^{-1}.$$

Notice that if we yielded to our initial temptation to substitute $\Gamma_{\rm Doppler}$ instead of Γ into Eq. (B.41), we would in fact have obtained the correct answer.

Solution to Exercise 1.17.

From Eq. (1.23) we have

$$\frac{1}{v_{gr}} \approx \frac{1}{c} + \frac{\omega}{2c}\frac{\mathrm{d}\chi'}{d\omega}.$$

In this case, the real part of the susceptibility is given by

$$\chi'(\omega) = -\frac{\alpha_0 c}{\omega_0}\frac{2(\omega-\omega_0)\Gamma}{4(\omega-\omega_0)^2+\Gamma^2}. \tag{B.42}$$

The derivative of this expression at resonance ($\omega = \omega_0$) is

$$\begin{aligned}\left.\frac{\mathrm{d}\chi'}{d\omega}\right|_{\omega=\omega_0} &= -\frac{\alpha_0 c}{\omega_0}2\Gamma\left[\frac{\Gamma^2-4(\omega-\omega_0)^2}{(4(\omega-\omega_0)^2+\Gamma^2)^2}\right]\Bigg|_{\omega=\omega_0}\\ &= -\frac{2\alpha_0 c}{\omega_0\Gamma}.\end{aligned}$$

Therefore, the group velocity is

$$\frac{1}{v_{gr}} \approx \frac{1}{c} - \frac{\alpha_0}{\Gamma}.$$

The amount that the pulse advances in time is

$$\frac{L}{v_{gr}} - \frac{L}{c} \approx -L\alpha_0\tau_p,$$

where we use $\tau_p \approx 1/\Gamma$. In order for this to be equal to $-M\tau_p$, we require $\alpha_0 L \approx M$.

Solution to Exercise 1.18.

b) The real part of the susceptibility is precisely the negative of that under the classical theory of dispersion (See Ex. 1.17). Thus, the derivative at resonance ($\omega = \omega_0$) is

$$\left.\frac{\mathrm{d}\chi'}{d\omega}\right|_{\omega=\omega_0} = \frac{2\alpha_0 c}{\omega_0\Gamma}$$

and the the group velocity is

$$\frac{1}{v_{gr}} \approx \frac{1}{c} + \frac{\alpha_0}{\Gamma}.$$

c) The time delay compared to propagation in vacuum is

$$\frac{L}{v_{gr}} - \frac{L}{c} \approx (\alpha_0 L)\tau_p,$$

where we use $\tau_p \approx 1/\Gamma$.

Appendix C

Solutions to Chapter 2 problems

Solution to Exercise 2.1. We restrict the derivation to the second-order term of the nonlinear polarization (higher-order terms are derived similarly). We start by expressing $\tilde{\chi}^{(2)}$ and $\vec{E}$ in Eq. (2.3) in terms of their Fourier images

$$\tilde{\chi}^{(2)}(\tau_1, \tau_2) = \iiint\limits_{-\infty}^{+\infty} \tilde{\chi}^{(2)}(\omega_1, \omega_2) e^{-i\omega_1 \tau_1} e^{-i\omega_2 \tau_2} \mathrm{d}\omega_1 \mathrm{d}\omega_2; \tag{C.1}$$

$$E(t) = \int\limits_{-\infty}^{+\infty} E_F(\omega) \mathrm{d}\omega \tag{C.2}$$

and performing the Fourier transform on both sides of Eq. (2.3).

$$P_F(\omega) = \frac{\epsilon_0}{(2\pi)^3} \int\limits_{-\infty}^{+\infty} \cdots \int \tilde{\chi}^{(2)}(\omega_1, \omega_2) e^{-i\omega_1 \tau_1} e^{-i\omega_2 \tau_2}$$
$$\times E(\omega_1') e^{-i\omega_1'(t-\tau_1)} E(\omega_2') e^{-i\omega_2'(t-\tau_2)} e^{i\omega t} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_1' \mathrm{d}\omega_2' \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}t.$$

In addition to double integrals in Eqs. (2.3) and (C.1) we have two instances of a single integral (C.2), plus an integral associated with the overall Fourier transformation. This leads to a total of seven integrals. We rearrange the above result as follows:

$$P_F(\omega) = \frac{\epsilon_0}{(2\pi)^3} \int\limits_{-\infty}^{+\infty} \cdots \int \tilde{\chi}^{(2)}(\omega_1, \omega_2) E(\omega_1') e^{-i\omega_1' t} E(\omega_2') e^{-i\omega_2' t}$$
$$\cdot e^{-i(\omega_1 - \omega_1')\tau_1} e^{-i(\omega_2 - \omega_2')\tau_2} e^{i\omega t} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_1' \mathrm{d}\omega_2' \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}t$$

Now we apply the following identity

$$\int\limits_{-\infty}^{+\infty} e^{i\omega t} \mathrm{d}t = 2\pi \delta(\omega) \tag{C.3}$$

to the integrals over τ_1 and τ_2:

$$P_F(\omega) = \frac{\epsilon_0}{(2\pi)} \int\limits_{-\infty}^{+\infty} \cdots \int \tilde{\chi}^{(2)}(\omega_1, \omega_2) E(\omega_1') e^{-i\omega_1' t} E(\omega_2') e^{-i\omega_2' t}$$
$$\cdot \delta(\omega_1' - \omega_1) \delta(\omega_2' - \omega_2) e^{i\omega t} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_1' \mathrm{d}\omega_2' \mathrm{d}t$$

Now we have only five integrals. As the next step, we perform integration over ω_1' and ω_2' to eliminate the delta functions.

$$P_F(\omega) = \frac{\epsilon_0}{(2\pi)} \iiint_{-\infty}^{+\infty} \tilde{\chi}^{(2)}(\omega_1,\omega_2)E(\omega_1)e^{-i\omega_1 t}E(\omega_2)e^{-i\omega_2 t} \cdot e^{i\omega t}\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}t.$$

Finally, we use identity (C.3) again to perform the integration over t:

$$P_F(\omega) = \epsilon_0 \iint_{-\infty}^{+\infty} \tilde{\chi}^{(2)}(\omega_1,\omega_2)E(\omega_1)E(\omega_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{C.4}$$

Solution to Exercise 2.2. Suppose there exists a medium for which Eq. (2.5) does not hold, i.e., for example, $\chi^{(2)}_{ijk}(\omega_1,\omega_2) \neq \chi^{(2)}_{ikj}(\omega_2,\omega_1)$. But then another susceptibility, given by

$$\chi'^{(2)}_{ijk}(\omega_1,\omega_2) = \chi^{(2)}_{ikj}(\omega_2,\omega_1), \tag{C.5}$$

also correctly describes second-order nonlinear properties of the material. To see this, we write the second-order term in Eq. (2.4) in the tensor form:

$$P^{(2)}_i(\omega) = \iint_{-\infty}^{+\infty} \chi^{(2)}_{ijk}(\omega_1,\omega_2)E_j(\omega_1)E_k(\omega_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{C.6}$$

In the integral above, ω_1 and ω_2 are dummy variables, and can be arbitrarily exchanged:

$$P^{(2)}_i(\omega) = \iint_{-\infty}^{+\infty} \chi^{(2)}_{ijk}(\omega_2,\omega_1)E_j(\omega_2)E_k(\omega_1)\delta(\omega-\omega_1-\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{C.7}$$

Dummy indices j and k can also be exchanged:

$$\begin{aligned} P^{(2)}_i(\omega) &= \iint_{-\infty}^{+\infty} \chi^{(2)}_{ijk}(\omega_2,\omega_1)E_k(\omega_2)E_j(\omega_1)\delta(\omega-\omega_1-\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2 \\ &= \iint_{-\infty}^{+\infty} \chi^{(2)}_{ijk}(\omega_2,\omega_1)E_j(\omega_1)E_k(\omega_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2 \end{aligned} \tag{C.8}$$

(in the last line, we used commutativity of multiplication). Comparing Eqs. C.6 and C.8, we see that susceptibility (C.9) indeed correctly describes the medium's nonlinear properties. But then so does the following:

$$\chi''^{(2)}_{ijk}(\omega_1,\omega_2) = \frac{1}{2}\left[\chi^{(2)}_{ikj}(\omega_2,\omega_1) + \chi''^{(2)}_{ijk}(\omega_1,\omega_2)\right]. \tag{C.9}$$

This susceptibility has symmetric properties according to Eq. (2.5a).

The proof for the time-domain susceptibility (2.5b) is analogous.

Solution to Exercise 2.3. First we find E in the frequency domain:

$$\begin{aligned} E_F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + c.c.\right)e^{i\omega t}dt \\ &= E_1\delta(\omega-\omega_1) + E_2\delta(\omega-\omega_2) + E_1^*\delta(\omega+\omega_1) + E_2^*\delta(\omega+\omega_2) \end{aligned}$$

The second order nonlinear term of $P(\omega)$ is obtained from Eq. (2.4):

$$P_F^{(2)}(\omega) = \epsilon_0 \iint_{-\infty}^{\infty} \chi^{(2)}(\omega_1', \omega_2') \underbrace{\left[E_1^2 \delta(\omega_1' - \omega_1)\delta(\omega_2' - \omega_1) + \ldots\right]}_{16 \text{ terms}} \delta(\omega - \omega_1' - \omega_2') d\omega_1' d\omega_2'$$

Integration over ω_1' and ω_2' and assuming $\chi^{(2)}(\omega_1, \omega_2) = \chi^{(2)}(\omega_2, \omega_1)$ yields:

$$\begin{aligned} P_F^{(2)}(\omega) = \epsilon_0 \Big[& \chi^{(2)}(\omega_1, \omega_1) E_1^2 \delta(\omega - \omega_1 - \omega_1) + 2\chi^{(2)}(\omega_1, \omega_2) E_1 E_2 \delta(\omega - \omega_1 - \omega_2) \\ & + \chi^{(2)}(\omega_1, -\omega_1) E_1 E_1^* \delta(\omega - \omega_1 + \omega_1) + 2\chi^{(2)}(\omega_1, -\omega_2) E_1 E_2^* \delta(\omega - \omega_1 + \omega_2) \\ & + \chi^{(2)}(\omega_2, \omega_2) E_2^2 \delta(\omega - \omega_2 - \omega_2) + \chi^{(2)}(\omega_2, -\omega_2) E_2 E_2^* \delta(\omega - \omega_2 + \omega_2) + c.c. \Big] \end{aligned}$$

Taking the inverse Fourier transform gives $P^{(2)}(t)$:

$$\begin{aligned} P^{(2)}(t) &= \int_{-\infty}^{\infty} P^{(2)}(\omega) e^{-i\omega t} d\omega \\ &= \epsilon_0 \Big[\chi^{(2)}(\omega_1, \omega_1) E_1^2 e^{-2i\omega_1 t} + 2\chi^{(2)}(\omega_1, \omega_2) E_1 E_2 e^{-i(\omega_1 + \omega_2)t} \\ &\quad + \chi^{(2)}(\omega_1, -\omega_1) E_1 E_1^* + 2\chi^{(2)}(\omega_1, -\omega_2) E_1 E_2^* e^{-i(\omega_1 - \omega_2)t} \\ &\quad + \chi^{(2)}(\omega_2, \omega_2) E_2^2 e^{-2i\omega_2 t} + \chi^{(2)}(\omega_2, -\omega_2) E_2 E_2^* + c.c. \Big]. \end{aligned}$$

Solution to Exercise 2.4. The dimensionlessness of the first-order susceptibility can be obtained, for example, by comparing Eqs. (1.8) and (1.9) or simply recalling that the refractive index of a medium, $n = \sqrt{1 + \chi^{(1)}}$, is dimensionless. This is true for the time-independent susceptibility, or the frequency-domain susceptibility $\chi^{(1)}(\omega)$. On the other hand, $\chi^{(1)}(t)$ has a dimension of s^{-1} in accordance with Eq. (1.15).

The non-linear polarization is given by Eq. (2.4). The dimension of the first term is just the dimension of the electric field.

$$[E_F(\omega)] = \frac{\mathrm{Vs}}{\mathrm{m}} \tag{C.10}$$

The dimension of the second term is

$$[E_F(\omega)]^2 [d\omega]^2 [\delta(\omega)] = \left(\frac{\mathrm{Vs}}{\mathrm{m}}\right)^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{s} = \left(\frac{\mathrm{V}}{m}\right)^2 \mathrm{s} \tag{C.11}$$

The dimension of the third term is

$$[E_F(\omega)]^3 [d\omega]^3 [\delta(\omega)] = \left(\frac{\mathrm{Vs}}{\mathrm{m}}\right)^3 \cdot \mathrm{s}^{-3} \cdot \mathrm{s} = \left(\frac{\mathrm{V}}{\mathrm{m}}\right)^3 \mathrm{s}. \tag{C.12}$$

The dimension of $\chi^{(2)}(\omega)$ is obtained by dividing Eq. (C.10) by Eq. (C.11).

$$[\chi^{(2)}(\omega)] = \frac{m}{V} \tag{C.13}$$

The dimension of $\chi^{(3)}(\omega)$ is obtained by dividing Eq. (C.10) by Eq. (C.12).

$$[\chi^{(3)}(\omega)] = \left(\frac{m}{V}\right)^2 \tag{C.14}$$

Solution to Exercise 2.6. We limit the analysis to the second-order nonlinearity; the third-order case is analyzed similarly. The contribution to the net force on the electron comes from the potential energy given by (2.9), damping force, $-\gamma\dot{x}$, and external electric field, $\vec{E}$. The equation of motion for the electron is

$$\ddot{x} = \frac{eE}{m}x - \gamma\dot{x} - \omega_0^2 - ax^2. \tag{C.15}$$

Supposing the electric field has two components as in Problem 2.3, we first solve for $x(t)$ by neglecting the nonlinear term. The first-order solution has the form

$$x = x_1 e^{-i\omega_1 t} + x_2 e^{-i\omega_2 t} + c.c., \tag{C.16}$$

which we can substitute into eq. (C.15), neglecting nonlinearity. Equating only the terms containing $e^{-i\omega_1 t}$, we have

$$\begin{aligned} -\omega_1^2 x_1 &= -\omega_0^2 x_1 + i\gamma\omega_1 x_1 + \tfrac{e}{m}E_1 \\ \Rightarrow x_1 &= \frac{\frac{e}{m}E_1}{D(\omega_1)}, \end{aligned}$$

where $D(\omega_1) = \omega_0^2 - \omega_1^2 - i\gamma\omega_1$. Similarly, equating only the terms containing $e^{-i\omega_2 t}$, we obtain[1]

$$x_2 = \frac{\frac{e}{m}E_2}{D(\omega_2)}.$$

Substituting this solution into the nonlinear term ax^2 in Eq. (C.15), we obtain terms oscillating at the second harmonic of both input fields, their sum and difference frequencies, as well as the zero frequency. These terms serve as "driving forces" that lead to the oscillation of the electron's position $x(t)$ with similar frequencies. Let us find the amplitude of the sum-frequency oscillation

$$x_{\rm SFG}(t) = x_{12} e^{-i(\omega_1+\omega_2)t} + c.c.$$

that is driven by the sum frequency term in ax^2,

$$2ax_1 e^{-i\omega_1 t} x_2 e^{-i\omega_2 t} = \frac{2aE_1E_2\left(\frac{e}{m}\right)^2}{D(\omega_1)D(\omega_2)} e^{-i\omega_1 t - i\omega_2 t},$$

Substituting the above into Eq. (C.15), we obtain

$$\begin{aligned} -(\omega_1+\omega_2)^2 x_{12} &= i\gamma(\omega_1+\omega_2)x_{12} - \omega_0^2 x_{12} - ax_1x_2 \\ \Rightarrow x_{12} &= \frac{-2aE_1E_2e^2}{m^2 D(\omega_1+\omega_2)D(\omega_1)D(\omega_2)}. \end{aligned}$$

The sum-frequency component of the polarization then is given by

$$P_{\rm SFG} = Nex_{\rm SFG} = -\frac{Nae^3}{m^2}\frac{-2E_1E_2}{D(\omega_1+\omega_2)D(\omega_1)D(\omega_2)} e^{-i(\omega_1+\omega_2)t} + c.c.$$

Comparing this result with Eq. (2.7), we obtain

$$\chi^{(2)} = -\frac{Nae^3}{\epsilon_0 m^2}\frac{1}{D(\omega_1+\omega_2)D(\omega_1)D(\omega_2)}.$$

Solution to Exercise 2.7. To estimate the order of the nonlinear susceptibilities we employ the expressions (1.29), (2.10) and (2.11) obtained using the classical theory of dispersion. As we are

[1]Here we have simply repeated our calculation for the classical theory of dispersion, see Sec. 1.4.

interested in the estimate far away from the resonance we neglect ω_0^2 and $i\omega\gamma$ in $D(\omega)$, so $D(\omega) \sim \omega^2$. So we can the suscettibilities as follows:

$$\chi^{(1)} \sim \frac{e^2 N}{\epsilon_0 m} \frac{1}{\omega^2} \tag{C.17}$$

$$\chi^{(2)} \sim \frac{e^3 N a}{\epsilon_0 m^2} \frac{1}{\omega^6} \tag{C.18}$$

$$\chi^{(3)} \sim \frac{e^4 N b}{\epsilon_0 m^3} \frac{1}{\omega^8} \tag{C.19}$$

We now substitute the following values.

- Unit charge $e \sim 10^{-19}$ C;
- Electric constant $\epsilon_0 \sim 10^{-11}$ C V^{-1} m^{-1};
- Optical frequency $\omega \sim 10^{15}$ s^{-1};
- Mass of an electron $m \sim 10^{-30}$ kg;
- Lattice constant $d \sim 10$Å$= 10^{-9}$ m;
- Number density $N \sim d^{-3} \sim 10^{27}$ m^{-3}.

In order to determine the constants a and b, we use Eq. (2.13) to write for a

$$m\omega_0^2 d \sim mad^2; \tag{C.20}$$

$$a \sim \frac{\omega_0^2}{d} \sim 10^{39} \text{ m}^{-1} \text{ s}^{-2}. \tag{C.21}$$

Similarly we get for b

$$m\omega_0^2 d \sim mbd^3; \tag{C.22}$$

$$b \sim \frac{\omega_0^2}{d^2} \sim 10^{48} \text{ m}^{-2} \text{ s}^{-2}. \tag{C.23}$$

Now we can insert all our determined values and obtain:

$$\chi^{(1)} \sim 1 \tag{C.24}$$

$$\chi^{(2)} \sim 10^{-10} \frac{\text{m}}{\text{V}} \tag{C.25}$$

$$\chi^{(3)} \sim 10^{-20} \frac{\text{m}^2}{\text{V}^2} \tag{C.26}$$

Solution to Exercise 2.9.

$$\begin{aligned}
&\chi^{(2)}_{xxx} \\
&\chi^{(2)}_{yyy} \\
&\chi^{(2)}_{zzz} \\
\chi^{(2)}_{xxy} = \chi^{(2)}_{xyx} = {}&\chi^{(2)}_{yxx} \\
\chi^{(2)}_{yyx} = \chi^{(2)}_{yxy} = {}&\chi^{(2)}_{xyy} \\
\chi^{(2)}_{yyz} = \chi^{(2)}_{yzy} = {}&\chi^{(2)}_{zyy} \\
\chi^{(2)}_{zzy} = \chi^{(2)}_{zyz} = {}&\chi^{(2)}_{yzz} \\
\chi^{(2)}_{xxz} = \chi^{(2)}_{xzx} = {}&\chi^{(2)}_{zxx} \\
\chi^{(2)}_{zzx} = \chi^{(2)}_{zxz} = {}&\chi^{(2)}_{xzz} \\
\chi^{(2)}_{xyz} = \chi^{(2)}_{xzy} = \chi^{(2)}_{yxz} = \chi^{(2)}_{yzx} = \chi^{(2)}_{zxy} = {}&\chi^{(2)}_{zyx}
\end{aligned}$$

Solution to Exercise 2.10.

a) Tensors link vectors which transform according to certain rules when the reference frame is rotated. Thus, the tensor elements have well-defined transformation properties as well. Consider, for example, rotation by angle π around the z axis. In this rotation, the x and y components of the electric field and polarization vectors will change their sign. However, these components are connected by the following equation:

$$P_i = \epsilon_0 \chi^{(2)}_{ijk} E_j E_k \tag{C.27}$$

This equation must remain valid after the rotation. This means that each component of the susceptibility tensor must change its sign according to the number of times symbols x and y occur in its index. For example, the index of $\chi^{(2)}_{xyz}$ contains one x symbol and one y symbol, so it will not change its sign. On the other hand, $\chi^{(2)}_{xzz}$ contains only one x symbol, so it will change its sign.

b) As discussed, rotation by angle π around the z axis transforms $\chi'^{(2)}_{xxy} = -\chi^{(2)}_{xxy}$. On the other hand, rotational symmetry of the material tells us that we must have $\chi'^{(2)}_{xxy} = \chi^{(2)}_{xxy}$. On comparing the two equations, we get $\chi^{(2)}_{xxy} = 0$. Using a similar analysis we can see that all the components of $\chi^{(2)}$ with repeating indices must vanish.

Solution to Exercise 2.11.

a) Mirror inversion about each of the primary Cartesian planes changes the sign according to the number of symbols that do not refer to that plane. For example, mirror inversion about the yz plane inverts inverts the sign of xxx, xyy, xzz and xyz components of the susceptibility, as well as any permutations of the above.

b) For each component of the second-order susceptibility there exists a primary Cartesian plane such that reflection about that plane will change the component's sign. But mirror symmetry tells us that the value of susceptibility should not change. This implies that the mentioned components are all identically equal to zero.

Solution to Exercise 2.12. As discussed in Problem 2.11, complete mirror symmetry implies that any terms with an odd number of indices in any of the x,y or z directions must vanish. This leaves only elements with the following forms: $\chi^{(3)}_{iijj}$, $\chi^{(3)}_{ijji}$, $\chi^{(3)}_{ijij}$ and $\chi^{(3)}_{iiii}$.

We can also show that each of these elements does not depend on any choice of $i, j \in \{x, y, z\}$ as long as $i \neq j$ by using isotropy. For example:

$$\chi^{(3)}_{xxyy} \xrightarrow{-90^\circ \text{ about x}} \chi^{(3)}_{xxzz} \xrightarrow{90^\circ \text{ about z}} \chi^{(3)}_{yyzz}$$

We now prove that $\chi^{(3)}_{xxxx} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}$. Suppose only one fundamental wave is present in the medium, and the reference frame is such that the electric field in that wave is along the x axis: $\vec{E} \parallel \hat{x}$. Then the only component of the susceptibility tensor that plays a role in the nonlinear process is $\chi^{(3)}_{xxxx}$, so we have $\vec{P} \parallel \hat{x}$ and $P = P_x = \epsilon_0 \chi^{(3)}_{xxxx} E^3$. Now suppose the reference frame is rotated about the z axis by 45°. The electric field and polarization vectors in the new reference frame are

$$\vec{P}_{new} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} P;$$

$$\vec{E}_{new} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} E.$$

The rotation of the reference frame does not change the susceptibility tensor, so we have

$$\begin{aligned}
P_{new,x} &= \epsilon_0 \left[\chi^{(3)}_{xxxx} E^3_{new,x} + \chi^{(3)}_{xyxy} E_{new,x} E^2_{new,y} + \chi^{(3)}_{xyyx} E_{new,x} E^2_{new,y} + \chi^{(3)}_{xxyy} E_{new,x} E^2_{new,y}\right] \\
&= \epsilon_0 \left[\chi^{(3)}_{xxxx} \left(\frac{E}{\sqrt{2}}\right)^3 + \chi^{(3)}_{xyxy} \left(\frac{E}{\sqrt{2}}\right)^3 + \chi^{(3)}_{xyyx} \left(\frac{E}{\sqrt{2}}\right)^3 + \chi^{(3)}_{xxyy} \left(\frac{E}{\sqrt{2}}\right)^3\right] \\
&= \frac{\epsilon_0}{2\sqrt{2}} \left[\chi^{(3)}_{xxxx} E^3 + \chi^{(3)}_{xyxy} E^3 + \chi^{(3)}_{xyyx} E^3 + \chi^{(3)}_{xxyy} E^3\right]
\end{aligned}$$

But we also have:

$$P_{new,x} = \frac{P}{\sqrt{2}} = \frac{\epsilon_0}{\sqrt{2}} \chi^{(3)}_{xxxx} E^3$$

Equating the two results gives:

$$\begin{aligned}
\frac{1}{\sqrt{2}} \chi^{(3)}_{xxxx} &= \frac{1}{2\sqrt{2}} \left[\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}\right] \\
\chi^{(3)}_{xxxx} &= \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}
\end{aligned}$$

Solution to Exercise 2.15. Consider the wave equation for sum-frequency generation with $\omega_3 = \omega_1 + \omega_2$. Assume a slowly-varying envelope and continuous-wave regime. The wave equation (1.22) takes the form:

$$\begin{aligned}
\left[\partial_z + \frac{1}{v_{gr}} \partial_t\right] \mathcal{E}_3 &= \frac{i\omega^2}{2\epsilon_0 k c^2} \mathcal{P}_3 \\
&= \frac{i\omega}{2\epsilon_0 n c} \mathcal{P}_3
\end{aligned}$$

We adapt Eq. (2.6) to slowly-varying envelopes

$$\begin{aligned}
E_1(z,t) &= \mathcal{E}_1(z) e^{ik_1 z - i\omega_1 t} + c.c. \\
E_2(z,t) &= \mathcal{E}_2(z) e^{ik_2 z - i\omega_2 t} + c.c.
\end{aligned}$$

and write the sum-frequency term in Eq. (2.7) as

$$P^{(2)}_3(t) = \epsilon_0 \left[4 d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 e^{-i(\omega_1+\omega_2)t + i(k_1+k_2)z} + c.c.\right] \tag{C.28}$$

On the other hand, the sum-frequency polarization is expressed in terms of its slowly-varying envelope according to

$$P^{(2)}_3(z,t) = \mathcal{P}^{(2)}_3(z) e^{ik_3 z - i\omega_3 t} + c.c. \tag{C.29}$$

Note that this polarization oscillates with respect to its envelope at frequency ω_3 and propagates with wavevector k_3. Comparing Eqs. (C.28) and (C.29) yields:

$$\begin{aligned}
\mathcal{P}^{(2)}_3 &= \epsilon_0 \left[4 d_{\text{eff}} \left(\mathcal{E}_1 e^{-i\omega_1 t + ik_1 z}\right) \left(\mathcal{E}_2 e^{-i\omega_1 t + ik_2 z}\right) e^{-ik_3 z + i\omega_3 t}\right] \\
&= \epsilon_0 \left[4 d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 e^{i(k_z + k_2 - k_3)z}\right]
\end{aligned}$$

since $\omega_3 = \omega_1 + \omega_2$. However, in general $k_3 \neq k_1 + k_2$ since the material may have a different index of refraction for different frequencies and polarizations. We can define $\Delta k = k_1 + k_2 - k_3$ and we get:

$$\mathcal{P}^{(2)}_3 = \epsilon_0 \left[4 d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 e^{i\Delta k z}\right]$$

Substituting this into the wave equation gives:

$$\begin{aligned}
\left[\partial_z + \frac{1}{v_{gr,3}} \partial_t\right] \mathcal{E}_3 &= \frac{i\omega_3}{2\epsilon_0 n_3 c} \mathcal{P}_{NL,3} \\
\partial_z \mathcal{E}_3 &= \frac{i\omega_3}{2\epsilon_0 n_3 c} \left(\epsilon_0 \left[4 d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 e^{i\Delta k z}\right]\right) \\
\partial_z \mathcal{E}_3 &= \frac{2 i d_{\text{eff}} \omega_3}{n_3 c} \mathcal{E}_1 \mathcal{E}_2 e^{i\Delta k z}
\end{aligned}$$

where we used $\partial_t\mathcal{E} = 0$ since we are in the continuous wave regime. Following a similar derivation for the difference frequency generation between ω_3 and either ω_1 or ω_2 we also get:

$$\partial_z\mathcal{E}_1 = \frac{2id_{\text{eff}}\omega_1}{n_1c}\mathcal{E}_3\mathcal{E}_2^*e^{-i\Delta kz} \tag{C.30}$$

$$\partial_z\mathcal{E}_2 = \frac{2id_{\text{eff}}\omega_2}{n_2c}\mathcal{E}_3\mathcal{E}_1^*e^{-i\Delta kz} \tag{C.31}$$

Note that the complex conjugate comes from the formula for DFG in Eq. (2.7). The k components also have an opposite sign. For example in Eq. (C.30), corresponding to the difference frequency of waves 3 and 2, the E_3 factor contributes an e^{ik_3z} component, the E_2^* factor contributes an e^{-ik_2z} component, and there is a e^{-ik_1z} component from the $\mathcal{P}_1 \to P_1$ substitution.

Solution to Exercise 2.17.

a) Assuming that both waves 1 and 2 do not get depleted, so their intensity remains constant, and integrating Eq. (2.19a) with respect to the propagation distance,

$$\begin{aligned}\mathcal{E}_3 &= \frac{2i\omega_3d_{\text{eff}}}{\epsilon_0n_3c}\mathcal{E}_1\mathcal{E}_2\int_0^z e^{i\Delta kz'}dz' \\ &= \frac{2i\omega_3d_{\text{eff}}}{\epsilon_0n_3c}\mathcal{E}_1\mathcal{E}_2\left(\frac{e^{i\Delta kL}-1}{i\Delta k}\right) \\ &= \frac{2i\omega_3d_{\text{eff}}}{\epsilon_0n_3c}\mathcal{E}_1\mathcal{E}_2e^{i\Delta kL/2}\sin\left(\frac{\Delta kL}{2}\right)\end{aligned} \tag{C.32}$$

Expressing this result in terms of the intensities,

$$I_j = 2n_j\epsilon_0c\mathcal{E}_j\mathcal{E}_j^* \tag{C.33}$$

(where $j = 1,2,3$), we obtain Eqs. (2.22) and (2.23).

b) As I_2 does not get depleted follows $\partial_z\mathcal{E}_2 = 0$. Hence we have to solve the coupled differential equations (2.19a) and (2.19c) with the condition $\Delta k = 0$. Redefining $\mathcal{E}_2' = i\mathcal{E}_2$, the coupled equations change to:

$$\partial_z\mathcal{E}_1 = -\frac{2d_{\text{eff}}\omega_1}{n_1c}\mathcal{E}_3\mathcal{E}_2 \tag{C.34}$$

$$\partial_z\mathcal{E}_3 = \frac{2d_{\text{eff}}\omega_3}{n_3c}\mathcal{E}_1\mathcal{E}_2 \tag{C.35}$$

To decouple the equations, we differentiate Eq. (C.35) once more:

$$\partial_z^2\mathcal{E}_3 = \frac{2d_{\text{eff}}\omega_3}{n_3c}(\mathcal{E}_2\partial_z\mathcal{E}_1 + \mathcal{E}_1\partial_z\mathcal{E}_2) \tag{C.36}$$

Again $\partial_z\mathcal{E}_2 = 0$. We can insert $\partial_z\mathcal{E}_1$ from Eq. (C.35). This leads to the following equation:

$$\partial_z^2\mathcal{E}_3 + \frac{4d^2\omega_3\omega_1}{n_1n_3c^2}(\mathcal{E}_2')^2\mathcal{E}_3 = 0 \tag{C.37}$$

Without loss of generality, we can assume the constant amplitude $(\mathcal{E}_2')$ of field 2 to be real. Expressing this quantity from the relation $I_2 = 2n_2c\epsilon_0(\mathcal{E}_2')^2$, we obtain

$$\partial_z^2\mathcal{E}_3 + \frac{4d^2\omega_3\omega_1I_2}{2n_1n_2n_3c^3\epsilon_0}\mathcal{E}_3 = 0 \tag{C.38}$$

Introducing the nonlinear length according to Eq. (2.23), the above equation becomes:

$$\partial_z^2 \mathcal{E}_3 + \frac{1}{L_{NL}^2} \mathcal{E}_3 = 0 \tag{C.39}$$

This equation has the general solution:

$$\mathcal{E}_3(z) = A \sin\left(\frac{z}{L_{NL}}\right) + B \cos\left(\frac{z}{L_{NL}}\right) \tag{C.40}$$

From the boundary condition $\mathcal{E}_3(0) = 0$ we have $B = 0$ to fulfill the boundary condition. To calculate A we insert Eq. (C.40) into Eq. (C.35) and evaluate the latter at $z = 0$.

$$\begin{aligned} \frac{A}{L_{NL}} \cos\left(\frac{0}{L_{NL}}\right) &= \frac{2d\omega_3}{n_3 c} \mathcal{E}_{10} \mathcal{E}_2; \\ A &= \frac{2d\omega_3 L_{NL}}{n_3 c} \mathcal{E}_{10} \mathcal{E}_2, \end{aligned} \tag{C.41}$$

where $\mathcal{E}_{10}$ is the initial amplitude of wave 1. This leads to the final expression for $\mathcal{E}_3(z)$

$$\mathcal{E}_3(z) = \frac{2d\omega_3 L_{NL}}{n_3 c} \mathcal{E}_{10} \mathcal{E}_2 \sin\left(\frac{z}{L_{NL}}\right) \tag{C.42}$$

Expressing the field amplitudes in terms of intensities, we can simplify the above result to

$$I_3(z) = I_{10} \frac{\omega_3}{\omega_1} \sin^2\left(\frac{z}{L_{NL}}\right) \tag{C.43}$$

Solution to Exercise 2.16.

a) From the Manley-Rowe relations, we have that

$$\begin{aligned} \partial_z I_1 &= -\frac{\omega_1}{\omega_3} \partial_z I_3 \text{ and} \\ \partial_z I_2 &= -\frac{\omega_2}{\omega_3} \partial_z I_3. \\ \Rightarrow \partial_z I_1 + \partial_z I_2 &= -\frac{\omega_1 + \omega_2}{\omega_3} \partial_z I_3 = -\partial_z I_3, \end{aligned} \tag{C.44}$$

using the fact that $\omega_1 + \omega_2 = \omega_3$.

b) The intensity in field mode i is

$$\begin{aligned} I_i &= 2n_i \epsilon_0 c |\mathcal{E}_i|^2 \\ &= 2n_i \epsilon_0 c \mathcal{E}_i^* \mathcal{E}_i. \end{aligned} \tag{C.45}$$

The position derivative of this expression is

$$\partial_z I_1 = 2n_1 \epsilon_0 c (\mathcal{E}_1^* \partial_z \mathcal{E}_1 + \mathcal{E}_1^* \partial_z \mathcal{E}_1). \tag{C.46}$$

Using the coupled-wave equations for sum-frequency generation, we find that

$$\begin{aligned} \partial_z I_1 &= 2n_1 \epsilon_0 c \left(\frac{2id_{\text{eff}}\omega_1}{n_1 c} \mathcal{E}_3^* \mathcal{E}_2 e^{+i\Delta kz} \mathcal{E}_1 + \mathcal{E}_1^* \frac{2id_{\text{eff}}\omega_1}{n_1 c} \mathcal{E}_3 \mathcal{E}_2^* e^{-i\Delta kz} \right) \\ &= 4\epsilon_0 i d_{\text{eff}} \omega_1 \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3^* e^{i\Delta kz} + c.c. \\ \Rightarrow \frac{\partial_z I_1}{\omega_1} &= 4\epsilon_0 i d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3^* e^{i\Delta kz} + c.c.. \end{aligned} \tag{C.47}$$

Similarly, we have

$$\frac{\partial_z I_2}{\omega_2} = 4\epsilon_0 i d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3^* e^{i\Delta kz} + c.c. \tag{C.48}$$

and

$$\frac{\partial_z I_3}{\omega_2} = -4\epsilon_0 i d_{\text{eff}} \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3^* e^{i\Delta kz} + c.c.. \tag{C.49}$$

Thus we have shown that

$$\frac{\partial_z I_1}{\omega_1} = \frac{\partial_z I_2}{\omega_2} = -\frac{\partial_z I_3}{\omega_3}. \tag{C.50}$$

Solution to Exercise 2.18.

a) We begin with the coupled wave equations (2.19) for SFG. As the ingoing wave is not depleting $\partial_z \mathcal{E}_1 = 0$ and we only have to look onto the equation for $\mathcal{E}_2$:

$$\partial_z \mathcal{E}_2(z) = \frac{i d_{\text{eff}} \times 2\omega}{n_2 c} \mathcal{E}_1^2 e^{i\Delta kz} \tag{C.51}$$

This equation obtains from (2.19b), but there is no factor of 2 in front of the right-hand side because it is only there for SFG, but not for SHG [see Eq. (2.7)].

To get $\mathcal{E}_2(z)$ we simply integrate Eq. (C.51) from 0 to z, recall the expression for the intensity $\mathcal{E}_1^2 = I_1/2n_1\epsilon_0 c$ and get

$$\mathcal{E}_2(z) = \frac{2i\omega d_{\text{eff}}}{n_2 c} \frac{I_1}{2n_1\epsilon_0 c} \frac{1}{i\Delta k} \left(e^{i\Delta kz} - 1\right) \tag{C.52}$$

Now we rearrange the equation to get a more familiar looking result:

$$\mathcal{E}_2(z) = \frac{i\omega d_{\text{eff}} I_1}{n_1 n_2 \epsilon_0 c^2} \frac{2e^{\frac{i\Delta kz}{2}}}{\Delta k} \frac{e^{\frac{i\Delta kz}{2}} - e^{-\frac{i\Delta kz}{2}}}{2i} \tag{C.53}$$

The last term can be identified as $\sin\left(\frac{\Delta kz}{2}\right)$. Now we can calculate the intensity I_2

$$I_2(z) = 2n_2\epsilon_0 c |\mathcal{E}_2|^2 = 2n_2\epsilon_0 c \left(\frac{\omega d_{\text{eff}} I_1}{n_1 n_2 \epsilon_0 c^2}\right)^2 \frac{4}{(\Delta k)^2} \sin^2\left(\frac{\Delta kz}{2}\right). \tag{C.54}$$

This can be rewritten as

$$I_2(z) = I_1 \frac{1}{L_{NL}^2} z^2 \frac{\sin^2\left(\frac{\Delta kz}{2}\right)}{\left(\frac{\Delta kz}{2}\right)^2} = I_1 \frac{z^2}{L_{NL}^2} \operatorname{sinc}^2\left(\frac{\Delta kz}{2}\right), \tag{C.55}$$

where

$$L_{NL} = \sqrt{\frac{\epsilon_0 c^3 n_1^2 n_2}{2\omega^2 I_1 d_{\text{eff}}^2}}. \tag{C.56}$$

b) Now in addition to Eq. (C.51) we also have to consider the competing process of DFG between the generated second harmonic wave and the fundamental:

$$\partial_z \mathcal{E}_1 = \frac{2i\omega d_{\text{eff}}}{n_1 c} \mathcal{E}_1^* \mathcal{E}_2 e^{-i\Delta kz}$$

In this case, the factor of 2 in the right-hand side is retained because this is a DFG process.

Now consider the case where $\Delta k = 0$. We can also simplify our coupled wave equations slightly by assuming $\mathcal{E}_1$ is real and taking $\mathcal{E}_2 \to i\mathcal{E}_2$. The resulting coupled waves equations are:

$$\partial_z(i\mathcal{E}_2) = \frac{2i\omega d_{\text{eff}}}{n_2 c} \mathcal{E}_1^2 \Longrightarrow \partial_z \mathcal{E}_2 = \frac{2\omega d_{\text{eff}}}{n_2 c} \mathcal{E}_1^2 \tag{C.57}$$

$$\partial_z \mathcal{E}_1 = \frac{2i\omega d_{\text{eff}}}{n_1 c} \mathcal{E}_1(i\mathcal{E}_2) \Longrightarrow \partial_z \mathcal{E}_1 = \frac{-2\omega d_{\text{eff}}}{n_1 c} \mathcal{E}_1 \mathcal{E}_2 \tag{C.58}$$

We can solve this by dividing Eq. (C.58) by $\mathcal{E}_1$ and taking the ∂_z derivative, then substituting into Eq. (C.57) to eliminate $\mathcal{E}_2$. This leads to a solution of the form:

$$\mathcal{E}_2 = \mathcal{E}_{10}\sqrt{\frac{n_1}{n_2}}\tanh\left(\frac{z}{L_{NL}}\right)$$
$$\mathcal{E}_1 = \mathcal{E}_{10}\frac{1}{\cosh\left(\frac{z}{L_{NL}}\right)}$$

where $\mathcal{E}_{10}$ is a constant that must be equal to $\mathcal{E}_1(z=0)$ from boundary conditions.

From here we can calculate the intensities I_1 and I_2 as a function of z:

$$I_1 = 2n_1\epsilon_0 c\mathcal{E}_1\mathcal{E}_1^* = 2n_1\epsilon_0 c|\mathcal{E}_{10}|^2\frac{1}{\cosh^2\frac{z}{L_{\text{NL}}}} = I_{10}\left(1-\tanh^2\frac{z}{L_{\text{NL}}}\right)$$

$$I_2 = 2n_2\epsilon_0 c\mathcal{E}_2\mathcal{E}_2^* = 2n_1\epsilon_0 c|\mathcal{E}_{10}|^2\tanh^2\frac{z}{L_{\text{NL}}} = I_{10}\tanh^2\frac{z}{L_{\text{NL}}},$$

where $I_{10} = I_1(z=0)$.

Solution to Exercise 2.19. The numerical data are as follows:

- power $P = 1$ W;
- diameter of the beam $d = 0.1$ mm $= 10^{-4}$ m;
- second order susceptibility $d_{eff} = 1$ pm/V $= 10^{-12}$ m/V;
- angular frequency $\omega \sim 10^{15}$ s^{-1};
- refractive indices $n_1 \sim n_2 \sim 1$;
- speed of light $c \sim 3\times 10^8$ m/s;
- electric constant $\epsilon_0 \sim 10^{-11}$ C^2/(N m^2);
- laser pulse repetition rate $f = 10^8$ Hz;
- laser pulse duration $\tau = 1$ ps $= 10^{-12}$ s.

As far as the fundamental wave's intensity is concerned, it depends on whether we are dealing with a CW or pulsed wave. For CW, the intensity is the laser power divided by the beam area, $I_1 \sim P/d^2 \sim 10^8$ W/m^2. In the pulsed case, the energy of a single pulse is given by $P/f = 10^{-8}$ J so the power within the pulse is $P' = P/f\tau = 10^4$ W and the intensity $I_1' \sim P'/d^2 \sim 10^{12}$ W/m^2. Substituting all these parameters into Eq. (2.31) we obtain $L_{NL} \sim 1$ m por the CW wave and $L_{NL} \sim 1$ cm for the pulsed wave.

Solution to Exercise 2.20.

a) The k-vector lies in the xz-plane, and is given by

$$\vec{k} = \begin{pmatrix} k\sin\theta \\ 0 \\ k\cos\theta \end{pmatrix},$$

where θ is the angle between $\vec{k}$ and the z-axis. The electric displacement field is given by $\vec{D} = \vec{D}_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}$, and taking its gradient yields $\nabla\cdot\vec{D} = i(\vec{D}_0\cdot\vec{k})e^{i\vec{k}\cdot\vec{r}-i\omega t}$. From Maxwell's equation (1.1), we require $\nabla\cdot\vec{D} = 0$, so $\vec{D}_0\cdot\vec{k}$ must vanish. Therefore,

$$\vec{D}_0 = \begin{pmatrix} D_0\cos\theta \\ 0 \\ -D_0\sin\theta \end{pmatrix},$$

since $\vec{D}$ must also lie in the xz-plane. Recalling that $\vec{D} = \epsilon_0(\mathbf{1} + \vec{\chi})\vec{E}$, the electric field can be determined by

$$\begin{aligned}
\vec{E} &= \tfrac{1}{\epsilon_0}(\mathbf{1}+\chi)^{-1}\vec{D} \\
&= \tfrac{1}{\epsilon_0}\begin{pmatrix} \frac{1}{1+\chi_{xx}} & 0 & 0 \\ 0 & \frac{1}{1+\chi_{yy}} & 0 \\ 0 & 0 & \frac{1}{1+\chi_{zz}} \end{pmatrix}\begin{pmatrix} D_0\cos\theta \\ 0 \\ -D_0\sin\theta \end{pmatrix} e^{i\vec{k}\cdot\vec{r}-i\omega t} \\
&= \tfrac{D_0}{\epsilon_0}\begin{pmatrix} \cos\theta/n_x^2 \\ 0 \\ -\sin\theta/n_z^2 \end{pmatrix} e^{i\vec{k}\cdot\vec{r}-i\omega t},
\end{aligned} \tag{C.59}$$

assuming that χ is diagonal and $n_i = \sqrt{1+\chi_{ii}}$.

We now address Maxwell's equation Eq. (1.3) Taking the curl of $\vec{E}$, the only element that survives is the y-component:

$$\partial_z E_x - \partial_x E_z = \frac{iD_0}{\epsilon_0}k\left(\frac{\cos^2\theta}{n_x^2} + \frac{\sin^2\theta}{n_z^2}\right)e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

which must be equal to $-\dot{B}_y$. Therefore, we have $\vec{B} = \vec{B}_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}$, with

$$\vec{B}_0 = \frac{D_0}{\epsilon_0}\frac{k}{\omega}\frac{1}{n_e^2(\theta)}\begin{pmatrix}0\\1\\0\end{pmatrix}, \tag{C.60}$$

where we define $\frac{1}{n_e(\theta)} = \sqrt{\frac{\cos^2\theta}{n_x^2} + \frac{\sin^2\theta}{n_z^2}}$. This satisfies Maxwell's equation (1.2): $\nabla\cdot\vec{B} = 0$. Finally, in order to verify (1.4), we let $\vec{H} = \frac{1}{\mu_0}\vec{B}$, and determine $\nabla\times\vec{H}$:

$$\begin{aligned}
&\partial_y H_z - \partial_z H_y = 0 - i\frac{D_0}{\epsilon_0\mu_0}\frac{k}{\omega}\frac{k\cos\theta}{n_e^2(\theta)}e^{i\vec{k}\cdot\vec{r}-i\omega t} \\
&\partial_z H_x - \partial_x H_z = 0 \\
&\partial_x H_y - \partial_y H_x = i\frac{D_0}{\epsilon_0\mu_0}\frac{k}{\omega}\frac{k\sin\theta}{n_e^2(\theta)}e^{i\vec{k}\cdot\vec{r}-i\omega t} - 0
\end{aligned}$$

and $\dot{\vec{E}}$:

$$\dot{\vec{D}} = -i\omega D_0\begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}e^{i\vec{k}\cdot\vec{r}-i\omega t}.$$

Maxwell's equation $\nabla\times\vec{H} = \dot{\vec{D}}$ is satisfied as long as

$$\frac{k^2}{\omega^2} = \frac{n_e^2(\theta)}{c^2}.$$

Therefore, the wave described above is a valid solution to Maxwell's equations. The index of refraction is defined by the ratio of the speed of light and the phase velocity of the waveform, $n = c/v_{ph}$, where the phase velocity is $v_{ph} = \omega/k$. Thus,

$$v_{ph} = \frac{\omega}{k} = \frac{c}{n_e(\theta)} \qquad \text{and} \qquad n = \frac{c}{v_{ph}} = n_e(\theta),$$

as desired.

b) Combining Eqs. (C.59) and (C.60), we find the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}^*$:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \propto \frac{D_0 B_0}{\mu_0\epsilon_0}\begin{pmatrix} -\sin\theta/n_z^2 \\ 0 \\ -\cos\theta/n_y^2 \end{pmatrix},$$

which is not parallel to $\vec{k}$ but oriented at angle

$$\tan\theta' = \frac{-\sin\theta/n_z^2}{-\cos\theta/n_y^2} = \frac{n_y^2}{n_z^2}\tan\theta$$

with respect to the z axis.

Solution to Exercise 2.22. We consider a wave propagating along x-z plane with the k vector oriented at angle θ with respect to thte z axis. We have two cases:

- When the electric field vector of wave is also polarized in the same plane (i.e. xz plane) the refractive index n_e of the extraordinary wave is given by Eq. (2.37)
- When the electric field vector is polarized perpendicular to the xz plane (i.e. in y direction) , the refractive index of the ordinary wave is $n_o = n_y$.

The wave will experience no birefringence when the refractive indices associated with the two cases discussed above are equal:

$$\begin{aligned} n_e(\theta) &= n_o \\ \frac{1}{n_y^2} &= \frac{1}{n_x^2} - \sin^2\theta\left(\frac{1}{n_x^2} - \frac{1}{n_z^2}\right) \\ \sin^2\theta &= \frac{1/n_x^2 - 1/n_y^2}{1/n_x^2 - 1/n_z^2} \end{aligned} \tag{C.61}$$

Solution to Exercise 2.23. Following the same logic as in the Solution to Exercise 2.22, we write

$$\begin{aligned} n_e(2\omega,\theta) &= n_O(\omega) \\ \frac{1}{n_o^2(\omega)} &= \frac{1}{n_O^2(2\omega)} - \sin^2\theta_{ph}\left(\frac{1}{n_O^2(2\omega)} - \frac{1}{n_E^2(2\omega)}\right) \\ \sin^2\theta_{ph} &= \frac{1/n_O^2(2\omega) - 1/n_o^2(\omega)}{1/n_O^2(2\omega) - 1/n_E^2(2\omega)} \end{aligned} \tag{C.62}$$

Substituting the numerical values for the BBO crystal we find

$$\begin{aligned} \sin\theta_{ph} &= 0.389, \\ \theta_{ph} &= 22.9^\circ \end{aligned}$$

Solution to Exercise 2.24. We assume that the phase mismatch varies linearly with the wavelength. Then the phase matching bandwidth $\Delta\lambda_{\mathrm{FWHM}}$, i.e. the allowed range of wavelengths within which the value of Δk is within the range prescribed by Eq. (2.40), can be related to the latter as:

$$\Delta(\Delta k)_{\mathrm{FWHM}} = \left|\frac{d\Delta k}{d\lambda}\right|\Delta\lambda_{\mathrm{FWHM}} \tag{C.63}$$

Since our goal is to relate the phase matching bandwidth to the group velocity mismatch, it is convenient to rewrite the above derivative in terms of the frequency, $\omega = 2\pi c/\lambda$,

$$\frac{d\Delta k}{d\lambda} = -\frac{2\pi c}{\lambda_0^2}\frac{d\Delta k}{d\omega}. \tag{C.64}$$

Using

$$\frac{dk_1}{d\omega} = \frac{dk_2}{d(\omega)} = \frac{1}{v_{\text{gr}}(\omega)} \tag{C.65}$$

$$\frac{dk_2}{d\omega} = 2\frac{dk_2}{d(2\omega)} = \frac{2}{v_{\text{gr}}(2\omega)} \tag{C.66}$$

$$\tag{C.67}$$

we obtain

$$\frac{d\Delta k}{d\omega} = 2\frac{d\Delta k_1}{d\omega} - \frac{d\Delta k_2}{d\omega} = 2\text{GVM} \tag{C.68}$$

and hence

$$\frac{d\Delta k}{d\lambda} = -\frac{4\pi c}{\lambda_0^2}\text{GVM} \tag{C.69}$$

Since, according to Eq. (2.40), we must have $-1.39 \le \Delta kL/2 \le 1.39$, the allowed range of Δk is

$$\Delta(\Delta k)_{\text{FWHM}} = \frac{5.56}{L}. \tag{C.70}$$

Hence

$$\begin{aligned}\Delta\lambda &= \frac{\lambda_0^2}{Lc(\text{GVM})}\frac{5.56}{4\pi} \\ &= 0.44\frac{\lambda^2}{Lc}\frac{1}{\text{GVM}}\end{aligned} \tag{C.71}$$

which is the required expression.

Solution to Exercise 2.25. The range of wavelengths over which the phase matching condition holds for a constant propagation angle and given length of crystal L, is given by Eq. (C.70). Assuming that the phase matching is present in a small range of angles about the phase matching angle, we can write the acceptance angle $\Delta\theta$ as

$$\Delta\theta = \frac{\Delta(\Delta k)}{\mathrm{d}\Delta k/\mathrm{d}\theta} = \frac{5.56}{L}\frac{1}{\mathrm{d}\Delta k/\mathrm{d}\theta} \tag{C.72}$$

where $\Delta k = k_2 - k_1$, with the wave vector $\vec{k_1}$ representing the ordinary fundamental wave and $\vec{k_2}$ the extraordinary second harmonic wave.

Since

$$\Rightarrow \Delta k = \frac{n_e(2\omega)2\omega}{c} - \frac{n_O(\omega)\omega}{c}, \tag{C.73}$$

we find

$$\frac{d\Delta k}{d\theta} = \frac{2\omega}{c}\left(\frac{dn_e(2\omega)}{d\theta} - \frac{1}{2}\frac{dn_O(\omega)}{d\theta}\right) \tag{C.74}$$

We know the refractive index for the extraordinary wave is given by Eq. (2.37). Differentiating with respect to θ, we obtain

$$\begin{aligned}\frac{dn_e}{d\theta} &= -\frac{1}{2}n_e^3\left(\frac{-2\sin\theta\cos\theta}{n_E^2(2\omega)} + \frac{2\sin\theta\cos\theta}{n_O^2(2\omega)}\right) \\ \frac{dn_e}{d\theta} &= \frac{1}{2}n_e^3\sin 2\theta\left(\frac{1}{n_E^2(2\omega)} - \frac{1}{n_O^2(2\omega)}\right).\end{aligned} \tag{C.75}$$

On the other hand, $\mathrm{d}n_O(\omega)/\mathrm{d}\theta = 0$ because the refractive index of an ordinary wave in a uniaxial crystal is the same along all directions. Thus

$$\Rightarrow \frac{d\Delta k}{d\theta} = \frac{2\pi}{\lambda}\sin 2\theta n_e^3\left(\frac{1}{n_E^2(2\omega)} - \frac{1}{n_O^2(2\omega)}\right). \tag{C.76}$$

Substituting this result into Eq. (C.72), we find

$$\Delta\theta = 0.88\frac{\lambda}{L}\left[\sin 2\theta_{\rm ph} n_e^3(\theta_{\rm ph})\left(\frac{1}{n_E^2(2\omega)} - \frac{1}{n_O^2(2\omega)}\right)\right]^{-1} \tag{C.77}$$

where $\theta_{\rm ph}$ is the phase matching angle.

Solution to Exercise 2.30. In the case of a regular crystal with no phase mismatch and no depletion, the coupled-wave equation (2.19c) for SFG ($\omega_1 + \omega_2 \to \omega_3$) gives the following:

$$\partial_z \mathcal{E}_3 = iAe^{i\Delta kz} \tag{C.78}$$

where

$$A = \frac{2id_{\rm eff}\omega_3}{n_3 c}\mathcal{E}_1\mathcal{E}_2$$

is a constant and $\Delta k = 0$, which gives us

$$\mathcal{E}_3(z) = iAz. \tag{C.79}$$

On the other hand, in case of periodic poling where the orientation of the $\chi^{(2)}$ tensor is periodically inverted with period $\Lambda = \frac{2\pi}{\Delta k}$, A is no longer a constant and changes sign every half-period. We want to consider the case where $\Delta k \neq 0$. The coupled wave equations are then integrated piece-wise according to the duty cycle:

$$\begin{aligned}
\mathcal{E}_z &= i\int_0^z A(z)e^{i\Delta kz}dz \\
&= i\left(\int_0^{\pi/\Delta k} |A|e^{i\Delta kz}dz + \int_{\pi/\Delta k}^{2\pi/\Delta k} -|A|e^{i\Delta kz}dz + \dots\right) \\
&= \frac{A}{\Delta k}\left(e^{i\pi} - 1\right) + \frac{A}{\Delta k}\left(e^{i2\pi} - e^{i\pi}\right) + \dots \\
&\approx -\frac{2A}{\Delta k} \times (\text{number of half periods between 0 and } z) \\
&= -\frac{2A}{\Delta k}\left(\frac{z}{\pi/\Delta k}\right) \\
&= -\frac{2A}{\pi}z.
\end{aligned}$$

This behavior (Fig. 2.13) is similar to that of a non-poled material with $\Delta k = 0$ and

$$\begin{aligned}
|A'_{\rm eff}| &= \frac{2}{\pi}|A_{\rm eff}| \\
\Rightarrow |d'_{\rm eff}| &= \frac{2}{\pi}|d_{\rm eff}|.
\end{aligned}$$

Solution to Exercise 2.31. We are generating a second harmonic wave in a PPKTP crystal. The wavelength of the generating wave is $1.064\mu m$ and the wavelength of the second harmonic following $0.532\mu m$. This crystal has the following nonlinear coefficients:

- $d_{31} = d_{zxx} = 1.4\,\text{pm/V}$
- $d_{32} = d_{zyy} = 2.65\,\text{pm/V}$
- $d_{33} = d_{zzz} = 10.7\,\text{pm/V}$

For a) we choose d_{zzz} as effective nonlinearity, because it has the highest value and therefore the highest change of polarization, if an electric field is applied. Further on it satisfies the condition of Type I. The poling period Λ is given by Eq. (2.44). Taking into account that $\omega_2 = 2\omega_1$, we have

$$\Delta k = 2k(\omega_1) - k(\omega_2) = \frac{2\omega_1}{c} n_z(\omega_1) - \frac{2\omega_1}{c} n_z(\omega_2).$$

So we can use $n_z(\omega_1) = 1.830$ and $n_z(\omega_2) = 1.889$ to determine the poling period

$$\Lambda = \frac{2\pi c}{2\omega_1} \frac{1}{|n_z(\omega_1) - n_z(\omega_2)|} = \frac{\lambda(2\omega_1)}{0.059} \approx 9.02\ \mu\text{m}$$

We take the absolute value of the difference in the refraction indices because Λ must be positive. Let us consider now case b). As the waves' polarizations are now perpendicular, it is not possible to choose d_{zzz}, but we use the second largest nonlinearity d_{zyy}. This corresponds to the first harmonic being polarized along the z-axis and the second harmonic along the y-axis. It would also be possible to choose d_{zxx}, but its value is lower than d_{zyy}. The refraction indices for this scenario are $n_z(\omega_1) = 1.830$ and $n_y(\omega_2) = 1.789$. Following the same calculation as for Type I, we obtain the poling period:

$$\Lambda = \frac{2\pi c}{2\omega_1} \frac{1}{|n_z(\omega_1) - n_y(\omega_2)|} = \frac{\lambda(2\omega_1)}{0.041} \approx 12.98\ \mu\text{m}.$$

Appendix D

Solutions to Chapter 3 problems

Solution to Exercise 3.1.

a) Starting from the definition of the correlation function $\Gamma(t,\tau)$ we have for the stationary processes:

$$\begin{aligned}
\Gamma(\tau) &= \langle E^-(t)E^+(t+\tau)\rangle \\
&= \langle E^-(t'-\tau)E^+(t')\rangle && \text{(substituting } t' = t+\tau) \\
&= \langle E^-(t-\tau)E^+(t)\rangle && \text{(Replacing dummy variable } t' \text{ by } t) \\
&= \langle E^+(t-\tau)E^-(t)\rangle^* && \text{(because } E^-(t) = (E^+(t))^*) \\
&= \Gamma(-\tau)^*.
\end{aligned}$$

Hence, according to the definition (3.3) of the degree of coherence, and because $\Gamma(0)$ is a real number, we have $g^{(1)}(\tau) = [g^{(1)}(-\tau)]*$.

b) The expected value $\langle E^-(t)E^+(t+\tau)\rangle$ can be considered as an average of many samples $E_i(t)$ of the process. can then be interpreted as an inner product and the Cauchy-Schwartz inequality can be applied:

$$\begin{aligned}
|\Gamma(\tau)|^2 &= \frac{1}{N^2}\Bigg|\underbrace{\sum_{i=1}^{N} E_i^-(t_i)E_i^+(t+\tau)}_{\text{inner product}}\Bigg|^2 \\
&\le \frac{1}{N^2}\sum_{i=1}^{N}|E_i^-(t)|^2\sum_{j=1}^{N}|E_j^+(t+\tau)|^2 && \text{(From Cauchy Schwartz inequality)} \\
&= \left[\frac{1}{N}\sum_{i=1}^{N}E_i^-(t)E_i^+(t)\right]\left[\frac{1}{N}\sum_{j=1}^{N}E_j^-(t')E_j^+(t')\right] && \text{(set } (t+\tau) = t') \\
&= |\langle E^-(t)E^+(t)\rangle|^2 && \text{(use stationarity)} \\
&= |\Gamma(0)|^2 .
\end{aligned}$$

That is, $|\Gamma(\tau)| \le |\Gamma(0)|$ for all τ. From the definition of $g^{(1)}(\tau)$, we have $\left|g^{(1)}(\tau)\right| \le 1$ as desired.

Solution to Exercise 3.3. Let the positive-frequency part of the interferometer input field be $E^+(t)$. The first beam splitter divides the input energy into two equal parts; accordingly, the field in each of the interferometer arms after the first beam splitter is given by $E^+(t)/\sqrt{2}$. And as the Fiber Mach-Zender interferometer introduces a temporal path-length difference τ, the electric field in the two arms of the interferometer before the second beam splitter are given by $E^+(t+\tau)/\sqrt{2}$ and $E^+(t)/\sqrt{2}$. The effect of the second beam splitter is then

$$E_f^+(t) = \frac{E^+(t+\tau)e^{i\phi} + E^+(t)}{2} \tag{D.1}$$

The phase factor is added to account for the interference fringes. Writing the above equation requires an assumption that a small change of the path length difference is equivalent to multiplying by a phase factor. This assumption is valid when the coherence time is much greater than the inverse optical frequency.

We now proceed to calculating the intensity. Because we are observing the fringes over a time interval that is much longer than the coherence time, we must take a statistical average:

$$\begin{aligned} I(t) &\propto \langle E_f^+(t)E_f^-(t)\rangle = \frac{1}{4}\langle (E^-(t+\tau)e^{-i\phi} + E^-(t))(E^+(t+\tau)e^{i\phi} + E^+(t))\rangle \qquad \text{(D.2)} \\ &= \frac{1}{4}(\langle E^-(t+\tau)E^+(t+\tau)\rangle + \langle E^-(t+\tau)E^+(t)\rangle e^{-i\phi} + \langle E^-(t)E^+(t+\tau)\rangle e^{i\phi} + \langle E^-(t)E^+(t)\rangle) \\ &= \frac{1}{4}(\Gamma(0) + \Gamma^*(\tau)e^{-i\phi} + \Gamma(\tau)e^{i\phi} + \Gamma(0)). \end{aligned}$$

We can write $\Gamma(\tau) = |\Gamma(\tau)|e^{i\alpha}$ where $\alpha = \mathrm{Arg}\Gamma(\tau)$. Then

$$I(t) \propto \frac{1}{2}(\Gamma(0) + |\Gamma(\tau)|\cos(\alpha+\phi)).$$

Hence

$$I_{\min}(t) = \frac{1}{2}(\Gamma(0) - |\Gamma(\tau)|); \qquad \text{(D.3)}$$

$$I_{\max}(t) = \frac{1}{2}(\Gamma(0) + |\Gamma(\tau)|), \qquad \text{(D.4)}$$

so the visibility of the interference pattern,

$$\mathcal{V}(\tau) = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = |g^{(1)}(\tau)|. \qquad \text{(D.5)}$$

Solution to Exercise 3.4. We recall that

$$E_F(\omega) = \frac{1}{2\pi}E^+(t)e^{i\omega t}\mathrm{d}t.$$

Accordingly,

$$\langle E_F^*(\omega)E_F(\omega')\rangle = \frac{1}{4\pi^2}\left\langle \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E^-(t)e^{-i\omega t}E^+(t')e^{i\omega t'}\mathrm{d}t\mathrm{d}t'\right\rangle \qquad \text{(D.6)}$$

Now, let us substitute the variable t' with $\tau = t' - t$. Since $\mathrm{d}t\mathrm{d}t' = \mathrm{d}t\mathrm{d}t'$ (the determinant of the Jacobian, in this case, is 1), the above equation now becomes:

$$\begin{aligned} \langle E_F^*(\omega)E_F(\omega')\rangle &= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle E^-(t)E^+(t+\tau)\rangle e^{-i(\omega-\omega')t}e^{i\omega\tau}\mathrm{d}t\mathrm{d}\tau \qquad \text{(D.7)} \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma(\tau)e^{-i(\omega-\omega')t}e^{i\omega\tau}\mathrm{d}t\mathrm{d}\tau. \end{aligned}$$

We now take advantage of the fact that in the integral above, the only quantity that depends on t is the complex exponential.

$$\langle E_F^*(\omega)E_F(\omega')\rangle = \frac{1}{2\pi}\delta(\omega-\omega')\int_{-\infty}^{+\infty}\Gamma(\tau)e^{i\omega'\tau}(d)\tau = \Gamma_F(\omega)\delta(\omega-\omega'),$$

as required.

Solution to Exercise 3.7. We assume that the spectrum of the light generated by the bulb is perfectly white: $\Gamma_F(\omega) = c$. This assumption is valid if the bandwidth δ of the filter is much less than an octave (see Fig. 3.2). The intensity spectrum after the filter is then

$$\Gamma_{1F}(\omega) = \Gamma_F(\omega)T(\omega) = ce^{-(\omega-\omega_0)^2/2\delta^2}.$$

The correlation function of the filtered field is then

$$\Gamma_1(\tau) = \left| \int_{-\infty}^{\infty} \Gamma_{1F}(\omega) e^{-i\omega t} \mathrm{d}\omega \right| = c_1 e^{-\frac{\tau^2\delta^2}{2}},$$

where c_1 is a constant. To calculate the interference visibility we use Eq. (3.5):

$$\mathcal{V}(\tau) = |g^{(1)}(\tau)| = \frac{|\Gamma(\tau)|}{\Gamma(0}) = e^{-\frac{\tau^2\delta^2}{2}}.$$

To calculate the coherence time τ_c, we use Eq. (3.4):

$$\tau_c = \sqrt{\frac{\int_{-\infty}^{\infty} \tau^2 e^{-\tau^2\delta^2} \mathrm{d}\tau}{\int_{-\infty}^{\infty} e^{-\tau^2\delta^2} \mathrm{d}\tau}} = \sqrt{\frac{\sqrt{\pi}/2\delta^3}{\sqrt{\pi}/\delta}} = \frac{1}{\sqrt{2}\delta}.$$

Solution to Exercise 3.8. We can derive a few useful properties of $g^{(2)}(\tau)$ from its definition:

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}$$

a) Given that $I(t) = 2n\epsilon_0 c|E|^2$ is always positive, then all of the terms in $g^{(2)}(\tau)$ are also positive. Hence:

$$g^{(2)}(\tau) \geq 0$$

b) Given that the process is stationary, t is an arbitrary variable in the definition of $g^{(2)}(\tau)$ and we can perform a change of variables using $t' = t + \tau$ to get:

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle} \\ &= \frac{\langle I(t'-\tau)I(t')\rangle}{\langle I(t'-\tau)\rangle\langle I(t')\rangle} \\ &= g^{(2)}(-\tau) \end{aligned}$$

c) Consider $I(t) = \langle I(t)\rangle + \delta I(t)$ where $\langle \delta I(t)\rangle = 0$. Then:

$$\begin{aligned} g^{(2)}(0) &= \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2} \\ &= \frac{\langle I(t)\rangle^2 + 2\langle\langle I(t)\rangle\delta I(t)\rangle + \langle(\delta I(t))^2\rangle}{\langle I(t)\rangle^2} \\ &= \frac{\langle I(t)\rangle^2 + \langle(\delta I(t))^2\rangle}{\langle I(t)\rangle^2} \quad \text{(since } \langle I(t)\rangle \text{ is a constant and } \langle \delta I(t)\rangle = 0\text{)} \\ &\geq \frac{\langle I(t)\rangle^2}{\langle I(t)\rangle^2} \\ &= 1 \\ \Longrightarrow\ & g^{(2)}(0) \geq 1 \end{aligned}$$

d) We can show $\langle I(t)I(t+\tau)\rangle \leq \langle I^2(t)\rangle$ using the Cauchy-Schwarz inequality:

$$\begin{aligned} \langle I(t)I(t+\tau)\rangle &= \frac{1}{N}\sum_i^N I_i(t)I_i(t+\tau) \\ &\leq \frac{1}{N}\sqrt{\sum_i^N |I_i(t)|^2 \sum_j^N |I_j(t+\tau)|^2} \\ &= \frac{1}{N}\sqrt{\sum_i^N |I_i(t)|^2 \sum_j^N |I_j(t)|^2} \\ &= \langle I^2(t)\rangle \end{aligned}$$

Then the ratio of $g^{(2)}(0)$ to $g^{(2)}(\tau)$ is given by:

$$\begin{aligned}
R &= \frac{g^{(2)}(0)}{g^{(2)}(\tau)} \\
&= \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2}\frac{\langle I(t)\rangle\langle I(t+\tau)\rangle}{\langle I(t)I(t+\tau)\rangle} \\
&= \frac{\langle I^2(t)\rangle}{\langle I(t)\rangle^2}\frac{\langle I(t)\rangle\langle I(t)\rangle}{\langle I(t)I(t+\tau)\rangle} \\
&= \frac{\langle I^2(t)\rangle}{\langle I(t)I(t+\tau)\rangle} \\
&\geq 1
\end{aligned}$$

which gives us

$$g^{(2)}(0) \leq g^{(2)}(\tau).$$

Solution to Exercise 3.11. Consider the representation of the electric field $E^{+}(t)$ from a filtered thermal light source in a complex plane. Each point in this plane corresponds to a random sample of the electric field obtained from multiple partially coherent sources. Both the real and imaginary parts of the field vector obey normal distribution,

$$\mathrm{pr}[\mathrm{Re}E^{+}] = \frac{1}{E_0\sqrt{\pi}}\exp\left(-\frac{(\mathrm{Re}E^{+})^2}{E_0^2}\right),$$

$$\mathrm{pr}[\mathrm{Im}E^{+}] = \frac{1}{E_0\sqrt{\pi}}\exp\left(-\frac{(\mathrm{Im}E^{+})^2}{E_0^2}\right).$$

This can be written in a two-dimensional form

$$\mathrm{pr}[\mathrm{Re}E^{+},\mathrm{Im}E^{+}] = \frac{1}{\pi E_0^2}\exp\left(-\frac{r^2}{E_0^2}\right), \tag{D.8}$$

where $r = (\mathrm{Re}E^{+})^2 + (\mathrm{Im}E^{+})^2$. Let us convert this into a single distribution for the intensity

$$I = 2n\epsilon_0 c|E^{+}|^2 = 2n\epsilon_0 c[r^2].$$

Inside a ring of width dr and radius r in the complex plane, the number of field samples would be proportional to the probability distribution (D.8) times the area dA of that ring.

$$dN(r) \propto \mathrm{pr}[E]dA = K\mathrm{pr}[E]d(\pi r^2),$$

where K is the proportionality constant. On the other hand, the differential of the intensity equals

$$\mathrm{d}I = 2n\epsilon_0 c\mathrm{d}(r^2).$$

Accordingly, the probability distribution of the intensity of the electric field is given by

$$\begin{aligned}
\mathrm{pr}(I) &= \frac{dN(r)}{dI} \\
&= \frac{1}{2n\epsilon_0 c)}\frac{dN}{dr^2} \\
&= \frac{K\pi}{2n\epsilon_0 c}e^{-(\frac{r}{E_0})^2} \\
&= K' e^{-(\frac{I(t)}{I_0})^2},
\end{aligned}$$

where K' is a constant and $I_0 = 2n\epsilon_0 c E_0^2 =$.

Normalizing probability $\mathrm{pr}(I)$ over the range of 0 to ∞, i.e. requiring that

$$\int_0^\infty pr(I) = 1,$$

we find $K' = 1/I_0$.

The relation between $\Gamma(0)$ and I_0 is determined by calculating the average on both sides of Eq. (3.8)

$$\begin{aligned}\langle I(t)\rangle &= 2nc\epsilon_0 \langle E^-(t)E^+(t)\rangle \\ &= 2nc\epsilon_0 \Gamma(0).\end{aligned}$$

On the other hand, according to Eq. (3.15) (which we just proved), $\langle I(t)\rangle = I_0$ and hence $I_0 = 2nc\epsilon_0\Gamma(0)$.

Solution to Exercise 3.12. For Gaussian random variables, from the Gaussian moment theorem, we have

$$\left\langle E_1^- E_2^- E_3^+ E_4^+ \right\rangle = \langle E_1^- E_2^- \rangle \langle E_3^+ E_4^+ \rangle + \langle E_1^- E_3^+ \rangle \langle E_2^- E_4^+ \rangle + \langle E_1^- E_4^+ \rangle \langle E_1^- E_4^+ \rangle . \tag{D.9}$$

Set $E_1^- = E^-(t)$, $E_2^- = E^-(t+\tau)$, $E_3^+ = E^+(t)$ and $E_4^+ = E^+(t+\tau)$, each of which are complex stochastic random variables whose probability distributions are Gaussian and centered at the origin. Then the first term in Eq. (D.9) must vanish because the phase of $E(t)$, and hence of the products $E_1^- E_2^-$ and $E_3^+ E_4^+$ is completely random. Thus (**??**)educes to

$$\begin{aligned}&\left\langle E^-(t)E^-(t+\tau)E^+(t)E^+(t+\tau)\right\rangle \\ &\quad = \underbrace{\langle E^-(t)E^+(t)\rangle \langle E^-(t+\tau)E^+(t+\tau)\rangle}_{\Gamma(0)^2} + \underbrace{\langle E^-(t)E^+(t+\tau)\rangle}_{\Gamma(\tau)} \underbrace{\langle E^+(t)E^-(t+\tau)\rangle}_{\Gamma(\tau)^*} \\ &\quad = \Gamma(0)^2 + |\Gamma(\tau)|^2 .\end{aligned} \tag{D.10}$$

From the definition of $g^{(2)}$,

$$\begin{aligned}g^{(2)}(\tau) &= \frac{\langle E^-(t)E^-(t+\tau)E^+(t)E^+(t+\tau)\rangle}{\langle E^-(t)E^-(t+\tau)\rangle \langle E^+(t)E^+(t+\tau)\rangle} \\ &= \frac{\Gamma(0)^2 + |\Gamma(\tau)|^2}{\Gamma(0)^2} \\ &= 1 + \left|g^{(1)}\right|^2.\end{aligned} \tag{D.11}$$

Appendix E

Solutions to Chapter 4 problems

Solution to Exercise 4.1.

a) We substitute Eqs. (4.1) into Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}} \tag{E.1}$$

keeping in mind that, for a plane wave, taking the curl is equivalent to multiplying by ik and the time derivative leads to factor $-i\omega$. The relationship between $u_{0\vec{k},s}$ and $w_{0\vec{k},s}$ therefore looks as follows

$$iku_{0\vec{k},s} = i\omega w_{0\vec{k},s}$$

or

$$w_{0\vec{k},s} = \frac{u_{0\vec{k},s}}{c}. \tag{E.2}$$

b) We rewrite Eqs. (4.1) as follows:

$$E_{\vec{k},s} = |u_{0\vec{k},s}|\left(e^{i\varphi}e^{i\vec{k}\cdot\vec{r}-i\omega_k t} + e^{-i\varphi}e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}\right)$$
$$E_{\vec{k},s} = 2|u_{0\vec{k},s}|\cos(\vec{k}\cdot\vec{r} - \omega_k t + \varphi)$$

with φ defined as $u_{0\vec{k},s} = |u_{0\vec{k},s}|e^{i\varphi_1}$. Similarly we have

$$B_{\vec{k},s} = 2|w_{0\vec{k},s}|\cos(\vec{k}\cdot\vec{r} - \omega_k t + \varphi) = 2\frac{|u_{0\vec{k},s}|}{c}\cos(\vec{k}\cdot\vec{r} - \omega_k t + \varphi).$$

To get the Hamiltonian $H_{\vec{k},s}$, we integrate the electromagnetic energy density over the quantization volume V.

$$H_{\vec{k},s} = \frac{1}{2}\int\left(\epsilon_0 E^2_{\vec{k},s} + \frac{B^2_{\vec{k},s}}{\mu_0}\right)\mathrm{d}V = \frac{1}{2}\int\left(\epsilon_0\left\langle E^2_{\vec{k},s}\right\rangle + c^2\epsilon_0\left\langle B^2_{\vec{k},s}\right\rangle\right)\mathrm{d}V, \tag{E.3}$$

where the average is taken over the quantization volume. Since $\left\langle\cos(x)^2\right\rangle = \frac{1}{2}$ for x varying over any interval of length 2π, we get

$$H_{\vec{k},s} = \frac{1}{2}\int\left(2\epsilon_0|u_{0\vec{k},s}|^2 + 2\epsilon_0|u_{0\vec{k},s}|^2\right)\mathrm{d}V = 2\epsilon_0|u_{0\vec{k},s}|^2 V \tag{E.4}$$

Solution to Exercise 4.3.

a) On adding the two equations (4.6), we find

$$u(t) = \sqrt{\frac{\hbar\omega_{\vec{k}}}{4\epsilon_0 V}}(X + iP) \tag{E.5}$$

$$u(t)^* = \sqrt{\frac{\hbar\omega_{\vec{k}}}{4\epsilon_0 V}}(X - iP) \tag{E.6}$$

Next, we substitute these equations into 4.3 to obtain

$$H = \frac{\hbar\omega}{2}(X^2 + P^2). \tag{E.7}$$

b) We use the relations

$$\begin{aligned} \dot{u}(t) &= -i\omega u(t); && \text{(E.8)} \\ \dot{u^*}(t) &= i\omega u(t). && \text{(E.9)} \end{aligned}$$

to find

$$\begin{aligned} \dot{P} &= \sqrt{\frac{2\epsilon_0 V}{\hbar\omega_{\vec{k}}}}\frac{-i\omega u(t) - i\omega u(t)^*}{\sqrt{2}i} \\ &= -\omega X \\ &= -\frac{\partial H}{\partial X} \end{aligned} \tag{E.10}$$

Similarly for the case of $\dot{X}$, we obtain

$$\dot{X} = \omega P = \frac{\partial H}{\partial P} \tag{E.11}$$

which are the required equations.

Solution to Exercise 4.4. For part (a), we have

$$\begin{aligned} \left\langle X\right|\hat{P}\left|\psi\right\rangle &= \langle X|\left(\int_{-\infty}^{+\infty} P\,|P\rangle\langle P|\,\mathrm{d}P\right)|\psi\rangle \quad \left(\text{because } \hat{P} = \int_{-\infty}^{+\infty} P\,|P\rangle\langle P|\,\mathrm{d}P\right) \\ &= \int_{-\infty}^{+\infty} P\,\langle X|\,P\rangle\,\langle P|\,\psi\rangle\,\mathrm{d}P \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} P e^{iPX}\,\langle P|\,\psi\rangle\,\mathrm{d}P \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}X} e^{iPX}\,\langle P|\,\psi\rangle\,\mathrm{d}P \\ &= -i\frac{\mathrm{d}}{\mathrm{d}X}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{iPX}\,\langle P|\,\psi\rangle\,\mathrm{d}P \\ &= -i\frac{\mathrm{d}}{\mathrm{d}X}\int_{-\infty}^{+\infty} \langle X|\,P\rangle\,\langle P|\,\psi\rangle\,\mathrm{d}P \\ &= -i\frac{\mathrm{d}}{\mathrm{d}X}\,\langle X|\,\psi\rangle \quad \left(\text{because } \int_{-\infty}^{+\infty} |P\rangle\langle P|\,\mathrm{d}P\right) = \hat{\mathbf{1}} \\ &= -i\frac{\mathrm{d}}{\mathrm{d}X}\psi(X) \end{aligned}$$

The proof for part(b) is analogous.

Solution to Exercise 4.5.

a) Because the position and momentum operators are Hermitian, $\hat{X}^\dagger = \hat{X}$ and $(i\hat{P})^\dagger = -i\hat{P}$. Therefore, $\hat{a}^\dagger = (\hat{X} + i\hat{P})^\dagger/\sqrt{2} = (\hat{X} - i\hat{P})/\sqrt{2}$.

b) From part (a), $\hat{a} \neq \hat{a}^\dagger$.

c)
$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2}[\hat{X} + i\hat{P}, \hat{X} - i\hat{P}] = \frac{1}{2}\big([\hat{X}, \hat{X}] - i[\hat{X}, \hat{P}] + i[\hat{P}, \hat{X}] + [\hat{P}, \hat{P}]\big) = 1.$$

d) The position and momentum operators are expressed through $\hat{a}$ and $\hat{a}^\dagger$ by solving Eqs. (4.12) and (4.13).

e)
$$\begin{aligned}
\hat{H} &= \frac{1}{2}\hbar\omega(\hat{X}^2 + \hat{P}^2) \\
&= \frac{1}{4}\hbar\omega\left[(\hat{a} + \hat{a}^\dagger)^2 + \frac{1}{i^2}(\hat{a} - \hat{a}^\dagger)^2\right] \\
&= \frac{1}{4}\hbar\omega\left[(\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) + \frac{1}{i^2}(\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\right] \\
&= \frac{1}{4}\hbar\omega[2\hat{a}\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a}] \\
&\overset{(4.14)}{=} \frac{1}{4}\hbar\omega[2\hat{a}^\dagger\hat{a} + 2 + 2\hat{a}^\dagger\hat{a}] \\
&= \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right].
\end{aligned}$$

f) For any operators $\hat{b}, \hat{c}, \hat{d}$ the following general relation holds: $[\hat{a}, \hat{b}\hat{c}] = [\hat{a}, \hat{b}]\hat{c} + \hat{b}[\hat{a}, \hat{c}]$. Accordingly,
$$[\hat{a}, \hat{a}^\dagger\hat{a}]\hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}, \hat{a}^\dagger]\hat{a} = \hat{a};$$
$$[\hat{a}^\dagger, \hat{a}^\dagger\hat{a}]\hat{a}^\dagger[\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = -\hat{a}^\dagger.$$

Solution to Exercise 4.6.

a) In order to verify if the state $\hat{a}\,|n\rangle$ is an eigenstate of the photon number operator $\hat{a}^\dagger\hat{a}$, let us subject this state to the action of this operator and employ the result (4.17), rewritten in the form $\hat{a}^\dagger\hat{a}\hat{a} = \hat{a}\hat{a}^\dagger\hat{a} - \hat{a}$:
$$\hat{a}^\dagger\hat{a}\hat{a}\,|n\rangle = (\hat{a}\hat{a}^\dagger\hat{a} - \hat{a})\,|n\rangle = (\hat{a}n - \hat{a})\,|n\rangle = (n-1)\hat{a}\,|n\rangle\,,$$
as was required.

b) Similarly, from Eq. (4.17) we find $\hat{a}^\dagger\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger$ and thus
$$\hat{a}^\dagger\hat{a}\hat{a}^\dagger\,|n\rangle = (\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger)\,|n\rangle = (\hat{a}^\dagger n + \hat{a}^\dagger)\,|n\rangle = (n+1)\hat{a}^\dagger\,|n\rangle\,.$$

Solution to Exercise 4.7.

a) Let $|\psi\rangle = \hat{a}\,|n\rangle$. From the previous exercise, we know that $|\psi\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n-1$, i.e. $|\psi\rangle = A\,|n-1\rangle$, where A is some constant. We need to find A. To this end, we notice that $\langle\psi| = \langle n|\,\hat{a}^\dagger$ and calculate
$$\langle\psi|\,\psi\rangle = \left\langle n\right|\,\hat{a}^\dagger\hat{a}\left|\,n\right\rangle = n.$$

But on the other hand,
$$\langle\psi|\,\psi\rangle = |A|^2\,\langle n-1|\,n-1\rangle = |A|^2,$$
where in the last equality we have used the fact that the eigenstates of the number operator are normalized. From the last two equations, we find $|A| = \sqrt{n}$.

The phase of A is arbitrary. By convention, it is chosen equal to zero, so A is real and positive: $A = \sqrt{n}$.

b) Similarly, if $|\phi\rangle = \hat{a}^\dagger |n\rangle = B|n+1\rangle$, then, on one hand,

$$\langle\phi|\,\phi\rangle = \left\langle n\right|\hat{a}\hat{a}^\dagger\left|\,n\right\rangle = \left\langle n\right|\hat{a}^\dagger\hat{a}+1\left|\,n\right\rangle = n+1,$$

and on the other hand

$$\langle\phi|\,\phi\rangle = |B|^2\langle n+1|\,n+1\rangle = |B|^2.$$

Therefore (invoking a similar convention), $B = \sqrt{n+1}$.

Solution to Exercise 4.8. The vacuum state obeys the equation $\hat{a}|0\rangle = 0$, or

$$(\hat{X}+i\hat{P})|0\rangle = 0. \tag{E.12}$$

In order to find the wavefunction in the position basis, we use Eq. (4.11) to write the momentum operator in this basis. Equation (E.12) then becomes

$$\left(X+\frac{\mathrm{d}}{\mathrm{d}X}\right)\psi(X) = 0.$$

This is a first order ordinary differential equation whose solution is

$$\psi(x) = Ae^{-X^2/2},$$

where A is the normalization constant, calculated in the usual manner:

$$\langle\psi|\,\psi\rangle = \int_{-\infty}^{+\infty}|\psi(X)|^2\mathrm{d}x = |A|^2\int_{-\infty}^{+\infty}e^{-X^2}\mathrm{d}x = |A|^2\sqrt{\pi}.$$

Requiring the norm of $|\psi\rangle$ to equal 1, we find $A = \pi^{-1/4}$.

The wavefunction in the momentum basis is calculated similarly.

Solution to Exercise 4.9.

a) The single-photon Fock state is obtained from the vacuum state by applying a single creation operator. Using Eq. (4.11), we express the creation operator in the position basis as

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X}-i\hat{P}) \leftrightarrow \frac{1}{\sqrt{2}}\left(X-\frac{\mathrm{d}}{\mathrm{d}X}\right)$$

and thus the wavefunction of the state $|1\rangle = \hat{a}^\dagger|0\rangle$ is

$$\psi_1(X) = \frac{1}{\sqrt{2}\pi^{1/4}}\left(X-\frac{\mathrm{d}}{\mathrm{d}X}\right)e^{-X^2/2} = \frac{\sqrt{2}}{\pi^{1/4}}Xe^{-X^2/2}.$$

The two-photon Fock state is obtained by applying the creation operator to the single-photon state:

$$|2\rangle \overset{(4.19)}{=} \frac{\hat{a}^\dagger}{\sqrt{2}}|1\rangle\,.$$

In the position basis,

$$\psi_2(X) = \frac{1}{2}\left(X-\frac{\mathrm{d}}{\mathrm{d}X}\right)\psi_1(x) = \frac{1}{\sqrt{2}\pi^{1/4}}\left(X-\frac{\mathrm{d}}{\mathrm{d}X}\right)Xe^{-X^2/2} = \frac{1}{\sqrt{2}\pi^{1/4}}(2X^2-1)e^{-X^2/2}.$$

b) We now show by induction that Eq. (4.22) describes the wavefunction of the Fock state $|n\rangle$. First, from the definition of the Hermite polynomial,

$$H_n(X) = (-1)^n e^{X^2}\frac{\mathrm{d}^n}{\mathrm{d}X^n}e^{-X^2},$$

we find that $H_0(X) = 1$ and thus the wavefunction of the vacuum state obtained from Eq. (4.22) is $\psi_0(X) = \pi^{-1/4}e^{-X^2/2}$, which is consistent with Eq. (4.21). Second, assuming that if Eq. (4.22) is valid for a specific Fock state $|n\rangle$, we need to prove it to be also valid for the next Fock state $|n+1\rangle = \hat{a}^\dagger |n\rangle / \sqrt{n+1}$. We apply the creation operator in the position basis:

$$\begin{aligned}
|n+1\rangle &= \frac{\hat{a}^\dagger}{\sqrt{n+1}}|n\rangle \\
&\leftrightarrow \frac{X - \mathrm{d}/\mathrm{d}X}{\sqrt{2}\sqrt{n+1}} \frac{H_n(X)}{\pi^{1/4}\sqrt{2^n n!}} e^{-X^2/2} \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} \left[X H_n(X) e^{-X^2/2} - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} e^{-X^2/2} - H_n(X) \frac{\mathrm{d}e^{-X^2/2}}{\mathrm{d}X} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} \left[\left(2X H_n(X) - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} \right) e^{-X^2/2} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} H_{n+1}(X) e^{-X^2/2},
\end{aligned}$$

which, according to Eq. (4.21), is the wavefunction of the state $|n+1\rangle$. In the final step in the above transformation, we have applied the known recursion relation for the Hermite polynomials,

$$H_{n+1}(X) = 2X\, H_n(X) - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} H_n(X).$$

Solution to Exercise 4.10. For an arbitrary Fock state $|n\rangle$, we have

$$\begin{aligned}
\langle n | \hat{X} | n \rangle &= \frac{1}{\sqrt{2}} \langle n | (\hat{a} + \hat{a}^\dagger) | n \rangle \\
&= \frac{1}{\sqrt{2}} \langle n | \left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle \right) \\
&= 0.
\end{aligned} \tag{E.13}$$

Similarly,

$$\langle n | \hat{P} | n \rangle = 0.$$

For the uncertainties, we have

$$\begin{aligned}
\langle \Delta X^2 \rangle &= \langle n | \hat{X}^2 | n \rangle \\
&= \frac{1}{2} \langle n | (\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) | n \rangle \\
&= \frac{1}{2} \langle n | \left[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{n+1}^2\,|n\rangle + \sqrt{n}^2\,|n\rangle + \sqrt{(n+1(n+2)}\,|n+2\rangle \right] \\
&= \frac{1}{2}(2n+1).
\end{aligned} \tag{E.14}$$

The same answer holds for the momentum uncertainty:

$$\langle \Delta P^2 \rangle = \frac{1}{2}(2n+1).$$

Solution to Exercise 4.11. Both $|0\rangle$ and $|1\rangle$ are energy eigenstates with the eigenvalues being, respectively, $\hbar\omega/2$ and $3\hbar\omega/2$. The evolution of the superposition of these states is then given by

$$|\psi(t)\rangle = e^{-i\omega t/2}\,|0\rangle + e^{-3i\omega t/2}\,|1\rangle\,.$$

The expectation value of the position observable is then given by

$$\langle X\rangle \overset{(4.15)}{=} \frac{1}{\sqrt{2}}\langle\psi(t)|\left(\hat{a}+\hat{a}^\dagger\right)|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(\langle 0| + e^{i\omega t}\langle 1|\Big)\left(\hat{a}+\hat{a}^\dagger\right)\Big(|0\rangle + e^{-i\omega t}|1\rangle\Big).$$

The only nonvanishing matrix elements in the expression above are $\langle 0|\,\hat{a}\,|1\rangle = \left\langle 1\right|\hat{a}^\dagger\left|0\right\rangle = 1$. We then conclude that

$$\langle X\rangle = \frac{1}{\sqrt{2}}\left(e^{i\omega t}+e^{-i\omega t}\right) = \sqrt{2}\cos\omega t.$$

Similarly, for the momentum observable we find

$$\langle X\rangle = \frac{1}{\sqrt{2}i}\Big(\langle 0| + e^{i\omega t}\langle 1|\Big)\left(\hat{a}-\hat{a}^\dagger\right)\Big(|0\rangle + e^{-i\omega t}|1\rangle\Big) = \frac{1}{\sqrt{2}i}\left(-e^{i\omega t}+e^{-i\omega t}\right) = -\sqrt{2}\sin\omega t.$$

The trajectory in the phase space is a clockwise circle with the center in the origin and a radius of $\sqrt{2}$.

Solution to Exercise 4.12. Let us assume some decomposition of the coherent state into the number basis,

$$|\alpha\rangle = \sum_{n=0}^{\infty}\alpha_n|n\rangle, \tag{E.15}$$

and apply the definition (4.23) of the coherent state to this decomposition. For the left-hand side of Eq. (4.23), we have in accordance with Eq. (4.18),

$$\begin{aligned}\hat{a}|\alpha\rangle &= \sum_{n=0}^{\infty}\alpha_n\hat{a}|n\rangle\\ &= \sum_{n=1}^{\infty}\alpha_n\sqrt{n}|n-1\rangle\\ &\overset{n'=n-1}{=} \sum_{n'=0}^{\infty}\alpha_{n'+1}\sqrt{n'+1}|n'\rangle.\end{aligned} \tag{E.16}$$

At the same time, the right-hand side of (4.23) can be written as

$$\alpha|\alpha\rangle = \sum_{n'=0}^{\infty}\alpha\alpha_{n'}|n'\rangle. \tag{E.17}$$

Equalizing both sides, we find a recursive relation

$$\alpha_{n'+1} = \frac{\alpha\alpha_{n'}}{\sqrt{n'+1}}, \tag{E.18}$$

so

$$\begin{aligned}\alpha_1 &= \alpha\alpha_0;\\ \alpha_2 &= \frac{\alpha\alpha_1}{\sqrt{2}} = \frac{\alpha^2\alpha_0}{\sqrt{2}};\\ \alpha_3 &= \frac{\alpha\alpha_2}{\sqrt{3}} = \frac{\alpha^3\alpha_0}{\sqrt{6}};\\ &\ldots,\end{aligned} \tag{E.19}$$

or in general

$$\alpha_n = \frac{\alpha^n\alpha_0}{\sqrt{n!}}. \tag{E.20}$$

It remains to find such a value of α_0 that state (E.15) is normalized to one. We find

$$\langle\alpha|\,\alpha\rangle = \sum_{n=0}^{\infty}|\alpha_n|^2 = |\alpha_0|^2\sum_{n=0}^{\infty}\frac{(|\alpha|^2)^n}{n!}. \tag{E.21}$$

If we look carefully at the sum in the above expression, we will find it to be the Taylor decomposition of $e^{|\alpha|^2}$, so we have $\langle\alpha|\,\alpha\rangle = |\alpha_0|^2 e^{|\alpha|^2}$. Setting $\langle\alpha|\,\alpha\rangle = 1$, we find

$$|\alpha_0|^2 = e^{-|\alpha|^2} \tag{E.22}$$

or

$$\alpha_0 = e^{i\varphi_\alpha} e^{-|\alpha|^2/2}. \tag{E.23}$$

The quantum phase factor $e^{i\varphi_\alpha}$ is a matter of convention, but it must be consistent with the convention chosen for the phase of the coherent state wavefunction (**??**). This phase cannot be determined based on these elementary considerations. However, a rigorous derivation shown in the beginning of this solution demonstrates that $\varphi_\alpha = 0$ for all α.

Combining Eqs. (E.20) and (E.23) we obtain

$$\alpha_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}. \tag{E.24}$$

Solution to Exercise 4.13. For the mean number of excitation quanta, we write:

$$\begin{aligned}
\langle n\rangle &= \sum_{n=0}^{\infty} n\,\mathrm{pr}_n \\
&= e^{-|\alpha|^2}\sum_{n=0}^{\infty} n\frac{|\alpha|^{2n}}{n!} \\
&= e^{-|\alpha|^2}\sum_{n=1}^{\infty} \frac{|\alpha|^{2n}}{(n-1)!} \\
&\overset{n'=n-1}{=} |\alpha|^2 e^{-|\alpha|^2}\sum_{n'=0}^{\infty} \frac{|\alpha|^{2n'}}{n'!}.
\end{aligned} \tag{E.25}$$

As we know from calculus, the sum equals $e^{|\alpha|^2}$. Accordingly, $\langle n\rangle = |\alpha|^2$.

For the mean square number of excitation quanta, the calculation is similar, but somewhat more complicated:

$$\begin{aligned}
\left\langle n^2\right\rangle &= e^{-|\alpha|^2}\sum_{n=0}^{\infty} n^2\frac{|\alpha|^{2n}}{n!} \\
&= e^{-|\alpha|^2}\sum_{n=1}^{\infty} n\frac{|\alpha|^{2n}}{(n-1)!} \\
&\overset{n'=n-1}{=} |\alpha|^2 e^{-|\alpha|^2}\sum_{n'=0}^{\infty} (n'+1)\frac{|\alpha|^{2n'}}{n'!} \\
&= |\alpha|^2 e^{-|\alpha|^2}\left[\sum_{n'=0}^{\infty} n'\frac{|\alpha|^{2n'}}{n'!} + \sum_{n'=0}^{\infty}\frac{|\alpha|^{2n'}}{n'!}\right] \\
&= |\alpha|^2\left[\sum_{n'=0}^{\infty} n'\mathrm{pr}_{n'} + \sum_{n'=0}^{\infty}\mathrm{pr}_{n'}\right] \\
&\overset{(E.25)}{=} |\alpha|^2\left[|\alpha|^2+1\right].
\end{aligned} \tag{E.26}$$

Hence the variance of n is $\left\langle\Delta n^2\right\rangle = \left\langle n^2\right\rangle - \langle n\rangle^2 = |\alpha|^2$.

Solution to Exercise 4.21.

a) Let us derive this result with the Wigner formula:

$$\iint_{-\infty}^{+\infty} W_{\hat\rho}(X,P)\mathrm{d}X\mathrm{d}P = \frac{1}{2\pi}\iiint_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|\hat\rho\left|X+\frac{Q}{2}\right\rangle \mathrm{d}Q\mathrm{d}X\mathrm{d}P \tag{E.27}$$

We start with performing the integration over P, which gives a delta function, as the only part of the integrand that depends on P is the exponential

$$\begin{aligned}\iint_{-\infty}^{+\infty} W_{\hat\rho}(X,P)\mathrm{d}X\mathrm{d}P &= \frac{1}{2\pi}\iint_{-\infty}^{+\infty} 2\pi\delta(Q)\left\langle X-\frac{Q}{2}\right|\hat\rho\left|X+\frac{Q}{2}\right\rangle \mathrm{d}Q\mathrm{d}X \\ &= \int_{-\infty}^{+\infty}\langle X|\hat\rho|X\rangle\,\mathrm{d}X \\ &= \mathrm{Tr}(\hat\rho)=1 \end{aligned} \tag{E.28}$$

This result is not surprising if we examine the physical definition (4.29) of the Wigner function, which we rewrite for the case $\theta=0$:

$$\int_{-\infty}^{+\infty} W_{\hat\rho}(X,P)\mathrm{d}P = \mathrm{pr}(X), \tag{E.29}$$

where $\mathrm{pr}(X)$ is the marginal distribution for X. Furthermore,

$$\int_{-\infty}^{+\infty}\mathrm{pr}(X)\mathrm{d}X = 1 \tag{E.30}$$

because $\mathrm{pr}(X)$ is a probability distribution. Accordingly, the integral of the Wigner function over the phase space must equal 1.

b) Substituting $\hat\rho=\alpha\hat\rho_1+\beta\hat\rho_2$ in the Wigner formula, we obtain

$$\begin{aligned}W_{\hat\rho}(X,P) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|\hat\rho\left|X+\frac{Q}{2}\right\rangle dQ \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|(\alpha\hat\rho_1+\beta\hat\rho_2)\left|X+\frac{Q}{2}\right\rangle dQ \\ &= \alpha\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|\hat\rho_1\left|X+\frac{Q}{2}\right\rangle dQ \\ &\qquad + \quad \beta\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X-\frac{Q}{2}\right|\hat\rho_2\left|X+\frac{Q}{2}\right\rangle dQ \\ &= \alpha W_{\hat\rho_1}(X,P)+\beta W_{\hat\rho_2}(X,P)\end{aligned}$$

Again, this result is consistent with the definition of the Wigner function. From Eqs. (4.29) and (4.30) we have

$$\int W_{\hat\rho}(X,P)dP = \mathrm{pr}(X) = \mathrm{Tr}(\hat\rho\hat X) \tag{E.31}$$

Substituting again $\hat\rho=\alpha\hat\rho_1+\beta\hat\rho_2$ into the right-hand side of the above equation we get

$$\begin{aligned}\mathrm{Tr}(\hat\rho\hat X_\theta) &= \mathrm{Tr}[(\alpha\hat\rho_1+\beta\hat\rho_2)\hat X_\theta] \\ &= \alpha\mathrm{Tr}(\hat\rho_1\hat X_\theta)+\beta\mathrm{Tr}(\hat\rho_2\hat X_\theta) && \text{(linearity of trace)} \\ &= \alpha\int W_{\hat\rho_1}(X,P)dP+\beta\int W_{\hat\rho_2}(X,P)dP && \text{(from (E.31))} \\ &= \int W_{\hat\rho}(X,P)dP.\end{aligned}$$

c) We can show that the Wigner function uniquely defines a state by taking the Fourier transform of both sides of the Wigner formula:

$$\langle X-\frac{Q}{2}|\hat\rho|X+\frac{Q}{2}\rangle = \int_{-\infty}^{\infty} e^{-iPQ}W_{\hat\rho}(X,P)\mathrm{d}P$$

We can then make the change of variables $X = \frac{X_1+X_2}{2}$ and $Q = X_2 - X_1$:

$$\langle X_1|\hat{\rho}|X_2\rangle = \int_{-\infty}^{\infty} e^{-iP(X_2-X_1)} W_{\hat{\rho}}\left(\frac{X_1+X_2}{2}, P\right) \mathrm{d}P$$

Hence each element of the density matrix $\hat{\rho} = \sum_{X_1,X_2} \rho_{X_1,X_2}|X_1\rangle\langle X_2|$ is uniquely given by the integral in the right hand side, which depends solely on the Wigner function $W_{\hat{\rho}}\left(\frac{X_1+X_2}{2}, P\right)$.

d) We can show that the Wigner function is real by taking the complex conjugate of the definition and then making the change of variables $Q' = -Q$:

$$\begin{aligned}
W^*_{\hat{\rho}}(X,P) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iPQ}\langle X - \frac{Q}{2}|\hat{\rho}|X + \frac{Q}{2}\rangle^* dQ \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iPQ}\langle X + \frac{Q}{2}|\hat{\rho}|X - \frac{Q}{2}\rangle dQ \qquad \text{(because the density operator is Hermitian)} \\
&= \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{iPQ'}\langle X - \frac{Q'}{2}|\hat{\rho}|X + \frac{Q'}{2}\rangle d(-Q') \qquad \text{(change of integration variable)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iPQ}\langle X - \frac{Q'}{2}|\hat{\rho}|X + \frac{Q'}{2}\rangle dQ' \\
&= W_{\hat{\rho}}(X,P)
\end{aligned}$$

Therefore, the Wigner function must be real.

Solution to Exercise 4.22. We write the Wigner formula for both operators

$$W_{\hat{A}}(X,P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\langle X - \frac{Q}{2}|\hat{A}|X + \frac{Q}{2}\rangle dQ; \tag{E.32}$$

$$W_{\hat{B}}(X,P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ'}\langle X - \frac{Q'}{2}|\hat{B}|X + \frac{Q'}{2}\rangle dQ', \tag{E.33}$$

and integrate the product of the two Wigner functions over the phase space:

$$\begin{aligned}
&\iint_{-\infty}^{+\infty} W_{\hat{A}}(X,P) W_{\hat{B}}(X,P)\mathrm{d}X\mathrm{d}P \\
&= \frac{1}{(2\pi)^2}\iiiint e^{iPQ}\langle X - \frac{Q}{2}|\hat{A}|X + \frac{Q}{2}\rangle e^{iPQ'}\langle X - \frac{Q'}{2}|\hat{B}|X + \frac{Q'}{2}\rangle \mathrm{d}Q\mathrm{d}Q'\mathrm{d}X\mathrm{d}P \\
&= \frac{1}{(2\pi)^2}\iiiint e^{iP(Q+Q')}\langle X - \frac{Q}{2}|\hat{A}|X + \frac{Q}{2}\rangle\langle X - \frac{Q'}{2}|\hat{B}|X + \frac{Q'}{2}\rangle \mathrm{d}Q\mathrm{d}Q'\mathrm{d}X\mathrm{d}P \\
&= \frac{1}{(2\pi)}\iiint \langle X - \frac{Q}{2}|\hat{A}|X + \frac{Q}{2}\rangle\langle X - \frac{Q'}{2}|\hat{B}|X + \frac{Q'}{2}\rangle\delta(Q+Q')\mathrm{d}Q\mathrm{d}Q'\mathrm{d}X \\
&= \frac{1}{(2\pi)}\iint \langle X - \frac{Q}{2}|\hat{A}|X + \frac{Q}{2}\rangle\langle X + \frac{Q}{2}|\hat{B}|X - \frac{Q}{2}\rangle \mathrm{d}Q\mathrm{d}X
\end{aligned} \tag{E.34}$$

(where the integration limits $\pm\infty$ have been omitted for clarity. Now let us redefine the integration variables: $X_1 = X - \frac{Q}{2}$ $X_2 = X + \frac{Q}{2}$. The Jacobian is $\begin{vmatrix} 1 & \frac{-1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} = 1$. Hence

$$\begin{aligned} \iint W_{\hat A}(X,P)W_{\hat B}(X,P)\mathrm{d}X\mathrm{d}P &= \frac{1}{(2\pi)}\iint \langle X_1|\hat A|X_2\rangle\langle X_2|\hat B|X_1\rangle \mathrm{d}X_1\mathrm{d}X_2 \\ &= \frac{1}{(2\pi)}\int \langle X_1|\hat A\Big[\int |X_2\rangle\langle X_2|\mathrm{d}X_2\Big]\hat B|X_1\rangle\mathrm{d}X_1. \end{aligned} \tag{E.35}$$

The integral in parentheses equals the identity operator. Therefore the above expression becomes

$$\iint W_{\hat A}(X,P)W_{\hat B}(X,P)\mathrm{d}X\mathrm{d}P = \frac{1}{(2\pi)}\int \langle X_1|\hat A\hat B|X_1\rangle\mathrm{d}X_1 = \frac{1}{(2\pi)}\mathrm{Tr}(\hat A\hat B).$$

Solution to Exercise 4.24.

a) substitute $X = P = 0$ in the Wigner formula (4.31). We obtain

$$\begin{aligned} W_{\hat\rho}(0,0) &= \frac{1}{2\pi}\int_{\infty}^{-\infty}\langle -\frac{Q}{2}|\hat\rho|\frac{Q}{2}\rangle dQ \\ &= \frac{1}{\pi}\int_{\infty}^{-\infty}\langle x|\hat\rho|-x\rangle dx && \text{substitute x = (Q/2)} \\ &= \frac{1}{\pi}\int_{\infty}^{-\infty}\langle x|\hat\rho\hat\Pi|x\rangle dx && \text{since } \hat\Pi|-x\rangle = |x\rangle \\ &= \frac{1}{\pi}\mathrm{Tr}(\hat\rho\hat\Pi) \end{aligned}$$

Solution to Exercise 4.25.

a) As we found in Problem 4.20,

$$W_{|\alpha\rangle\langle\alpha|}(X,P) = W_{|0\rangle\langle 0|}(X - X_\alpha, P - P_\alpha) \tag{E.36}$$

with $\alpha = (X_\alpha + iP_\alpha)/\sqrt{2}$. Comparing this with the definition (4.40) of the Q function, we obtain Eq. (4.41).

b) Using Eq. (4.34) we get

$$\begin{aligned} Q_{\hat\rho}(X,P) &= \frac{1}{2\pi}\mathrm{Tr}(\hat\rho|\alpha\rangle\langle\alpha|) \\ &= \frac{1}{2\pi}\langle\alpha|\hat\rho|\alpha\rangle \end{aligned}$$

c) This follows from the previous result because the density operator is positive.

d) To prove that the Q-function is normalized we use the convolution theorem:

$$\mathcal{F}(Q_{\hat\rho}(X,P)) = (2\pi)^2\mathcal{F}(W_{\hat\rho}(X,P))\mathcal{F}(W_{|0\rangle\langle 0|}(X,P)) \tag{E.37}$$

where

$$Q_{\hat\rho,F}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} Q_{\hat\rho}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P; \tag{E.38}$$

$$W_{\hat\rho}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} W_{\hat\rho}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P; \tag{E.39}$$

and

$$W_{|0\rangle\langle 0|}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} W_{|0\rangle\langle 0|}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P. \tag{E.40}$$

The Wigner functions themselves are normalized, i.e.

$$W_{\hat\rho,F}(0,0) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} W_{\hat\rho}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P = \frac{1}{(2\pi)^2}; \tag{E.41}$$

$$W_{|0\rangle\langle 0|,F}(0,0) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} W_{|0\rangle\langle 0|}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P = \frac{1}{(2\pi)^2}; \tag{E.42}$$

$$\tag{E.43}$$

Using Eq. (E.37), we find that

$$(2\pi)^2 Q_{|0\rangle\langle 0|,F}(0,0) = \iint\limits_{-\infty}^{\infty} Q_{|0\rangle\langle 0|}(X,P)e^{-iX\xi_X - iP\xi_P}\mathrm{d}X\mathrm{d}P = 1. \tag{E.44}$$

Solution to Exercise 4.27.

a) Convolving Eq. (4.46) with the vacuum state Wigner function, we obtain

$$\iint_{-\infty}^{+\infty} \delta(X'-X_\alpha, P'-P_\alpha)W_{|0\rangle\langle 0|}(X-X', P-P')\,\mathrm{d}X'\mathrm{d}P' = W_{|0\rangle\langle 0|}(X-X_\alpha, P-P_\alpha), \tag{E.45}$$

which equals the coherent state Wigner function according to Eq. (E.36).

b) We solve this part by taking the Fouier transform of both sides of Eq. (4.45) and recalling that the Fourier transform of a convolution is a product. The wavefunction of a squeezed vacuum state with squeezing parameter r is $\psi_r(x) = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{r}}e^{-x^2/2r^2}$, so its Wigner function is

$$\begin{aligned} W_r(X,P) &= \frac{1}{2\pi\sqrt{\pi}}\frac{1}{r}\int\limits_{-\infty}^{\infty} e^{iPQ}e^{-\frac{(X-Q/2)^2}{2r^2}}e^{-\frac{(X+Q/2)^2}{2r^2}}\mathrm{d}Q \\ &= \frac{1}{2\pi\sqrt{\pi}}\frac{1}{r}e^{-\frac{X^2}{r^2}}\underbrace{\int\limits_{-\infty}^{\infty} e^{iPQ}e^{-\frac{Q^2}{4r^2}}\mathrm{d}Q}_{=2r\sqrt{\pi}e^{-P^2r^2}} \\ &= \frac{1}{\pi}e^{-\frac{X^2}{r^2}-P^2r^2}. \end{aligned}$$

Subsequently, its Fourier transform is

$$W_{r,F}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}e^{i(\xi_X X+\xi_P P)}W_r(X,P) = \frac{1}{(2\pi)^2}e^{-\frac{\xi_X^2 r^2}{4}-\frac{\xi_P^2}{4r^2}}. \tag{E.46}$$

whereas the Fourier transform of the vacuum state Wigner function is simply

$$W_{0,F}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}e^{-\frac{\xi_x^2+\xi_p^2}{4}}. \tag{E.47}$$

The ratio of these two expressions yields

$$P_{r,F}(\xi_X,\xi_P) = \frac{1}{(2\pi)^2}\frac{W_{r,F}(\xi_X,\xi_P)}{W_{0,F}(\xi_X,\xi_P)} = \frac{1}{(2\pi)^2}e^{\frac{\xi_x^2}{4}(1-r^2)+\frac{\xi_p^2}{4}\left(1-\frac{1}{r^2}\right)}.$$

This is an infinitely growing exponential, which cannot have a properly defined inverse transform.

Solution to Exercise 4.28. This is equivalent to showing that the two sides of eq. (4.47) have the same representation in the position basis. That is, we want to show that

$$\langle X_1 |\hat\rho| X_2\rangle = \iint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')\,\langle X_1|\alpha\rangle\,\langle\alpha|X_2\rangle\,\mathrm{d}X'\mathrm{d}P'. \tag{E.48}$$

Recalling that the wave function of a coherent state in the position basis (neglecting a phase factor) is given by $\langle X_1|\alpha\rangle = \pi^{-\frac{1}{4}}e^{-\frac{(X_1-X)^2}{2}}e^{iPX_1}$ for $\alpha = \frac{X+iP}{\sqrt{2}}$, the right-hand side of eq. (E.48) can be expressed as

$$\frac{1}{\sqrt{\pi}}\iint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')e^{-\frac{(X_1-X')^2}{2}}e^{-\frac{(X_2-X')^2}{2}}e^{iP(X_1-X_2)}\mathrm{d}X'\mathrm{d}P'. \tag{E.49}$$

We now want to show that this is equivalent to the left-hand side of eq. (E.48).

Taking the inverse Fourier Transform of the Wigner formula (4.31) yields

$$\left\langle X-\tfrac{Q}{2}\,|\hat\rho|\,X+\tfrac{Q}{2}\right\rangle = \int\limits_{-\infty}^{+\infty} e^{-iPQ}W_{\hat\rho}(X,P)\mathrm{d}P. \tag{E.50}$$

Let us now insert definition (4.45) of the Glauber-Sudarshan function into (E.50), noting that the Wigner function of the vacuum state is given by $\frac{1}{\pi}e^{-(X-X')^2}e^{-(P-P')^2}$. Thus we have

$$\begin{aligned}
&\left\langle X-\tfrac{Q}{2}\,|\hat\rho|\,X+\tfrac{Q}{2}\right\rangle = \\
&= \frac{1}{\pi}\iiint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')e^{-(X-X')^2}e^{-(P-P')^2}e^{-iPQ}\mathrm{d}X'\mathrm{d}P'\mathrm{d}P \\
&= \frac{1}{\pi}\iint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')e^{-(X-X')^2}\mathrm{d}X'\mathrm{d}P'\left(\int\limits_{-\infty}^{+\infty} e^{-(P-P')^2}e^{-iPQ}\mathrm{d}P\right).
\end{aligned} \tag{E.51}$$

We now evaluate the integral in parentheses in Eq. (E.51):

$$\begin{aligned}
\int\limits_{-\infty}^{+\infty} e^{-(P-P')^2}e^{-iPQ}\mathrm{d}P &= e^{-iP'Q}\int\limits_{-\infty}^{+\infty} e^{-(P-P')^2}e^{-i(P-P')Q}\mathrm{d}P \\
&= e^{-iP'Q}e^{\left(i\frac{Q}{2}\right)^2}\int\limits_{-\infty}^{+\infty} e^{-\left(P-P'+i\frac{Q}{2}\right)^2}\mathrm{d}P.
\end{aligned}$$

Performing the change of variables $\xi = P - P' + i\frac{Q}{2}$ in the integration, we obtain

$$\begin{aligned}
\int\limits_{-\infty}^{+\infty} e^{-(P-P')^2}e^{-iPQ}\mathrm{d}P &= e^{-iP'Q}e^{\left(i\frac{Q}{2}\right)^2}\int\limits_{-\infty}^{+\infty} e^{-\xi^2}\mathrm{d}\xi \\
&= \sqrt{\pi}e^{-iP'Q}e^{-\frac{Q^2}{4}}.
\end{aligned}$$

Substituting this into Eq. (E.51), we get

$$\left\langle X-\tfrac{Q}{2}\,|\hat\rho|\,X+\tfrac{Q}{2}\right\rangle = \frac{1}{\sqrt{\pi}}\iint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')e^{-(X-X')^2}e^{-\frac{Q^2}{4}}e^{-iP'Q}\mathrm{d}X'\mathrm{d}P'. \tag{E.52}$$

Finally, we make the change of variables $X_1 = X - \frac{Q}{2}$ and $X_2 = X + \frac{Q}{2}$, which results in $X = \frac{1}{2}(X_1 + X_2)$ and $Q = X_2 - X_1$. It follows that

$$\langle X_1 |\hat\rho| X_2\rangle = \frac{1}{\sqrt{\pi}}\iint\limits_{-\infty}^{+\infty} P_{\hat\rho}(X',P')e^{-(\frac{X_1+X_2}{2}-X')^2}e^{-\frac{(X_2-X_1)^2}{4}}e^{-iP'(X_2-X_1)}\mathrm{d}X'\mathrm{d}P'. \tag{E.53}$$

Analyzing the exponents in the integrand of the previous equation, we see that

$$\begin{aligned} e^{-(\frac{X_1+X_2}{2}-X')^2} e^{-\frac{(X_2-X_1)^2}{4}} &= e^{-\left(\frac{(X_1+X_2)^2+(X_1-X_2)^2}{4}-X'(X_1+X_2)+X'^2\right)} \\ &= e^{-\left(\frac{X_1^2}{2}-X_1X'+\frac{X'^2}{2}\right)} e^{-\left(\frac{X_2^2}{2}-X_2X'+\frac{X'^2}{2}\right)} \\ &= e^{-\frac{(X_1-X')^2}{2}} e^{-\frac{(X_2-X')^2}{2}}, \end{aligned}$$

so

$$\langle X_1 |\hat{\rho}| X_2 \rangle = \frac{1}{\sqrt{\pi}} \iint_{-\infty}^{+\infty} P_{\hat{\rho}}(X', P') e^{-\frac{(X_1-X')^2}{2}} e^{\frac{(X_2-X')^2}{2}} e^{iP'(X_1-X_2)} \mathrm{d}X'\mathrm{d}P'. \tag{E.54}$$

This is the same as Eq. (E.49), which concludes the proof.

Solution to Exercise 4.28. Coherent states, as well as the vacuum state (the coherent state with $\alpha = 0$) are classical: they are trivial cases of statistical mixtures of coherent states.

The Fock states are nonclassical The value of $g^2(0) = 0$ and the Mandel parameter is equal to -1, both of which point to the nonclassical nature of the state.

Superpositions and statistical mixtures of the single-photon and vacuum states are nonclassical because they have zero probability to contain exactly n photons for any $n > 2$.

The squeezed vacuum state is non-classical. This follows from the squeezing criterion. For the specific state in the problem, the $\langle \Delta P^2 \rangle = \langle \Delta X^2_{\theta=\pi/2} \rangle = \frac{1}{2r^2} = \frac{1}{8} < \frac{1}{2}$.

The Schrödinger cat state $|\alpha\rangle \pm |-\alpha\rangle$ is nonclassical. This can be seen, for example, from its Fock decomposition. Using Eq. (4.24), we find

$$|\alpha\rangle + |-\alpha\rangle = 2e^{-|\alpha|^2/2} \sum_{\text{even } n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{E.55}$$

$$|\alpha\rangle - |-\alpha\rangle = 2e^{-|\alpha|^2/2} \sum_{\text{odd } n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{E.56}$$

The contributions of odd and even photon number states to these sums vanish, so the criterion of Problem 4.32 applies.

The state with density matrix $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$ is a statistical mixture of coherent states and hence classical.

Solution to Exercise 4.37.

a)

$$\begin{aligned} \langle n \rangle = \mathrm{Tr}(\hat{\rho}\hat{n}) = (1-x)\sum_{n=0}^{\infty} nx^n &= x(1-x)\sum_{n=0}^{\infty} nx^{n-1} \\ = x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} x^n\right) \end{aligned}$$

where $x = e^{-\beta}$. The sum of the geometric progression in parentheses equals $1/(1-x)$. Hence

$$\begin{aligned} \langle n \rangle &= x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) \\ &= (1-x)\frac{x}{(1-x)^2} \\ &= \frac{x}{1-x} \\ &= \frac{1}{e^{\beta}-1} \end{aligned}$$

We have obtained Bose-Einstein statistics, which is not surprising given that photons are bosons.

b) Prior to calculating the second-order coherence function, let us find the expectation value of the squared photon number in the thermal state.

$$\begin{aligned}
\langle n^2\rangle &= (1-x)\sum_{n=0}^{\infty} n^2 x^n \\
&= (1-x)\sum_{n=0}^{\infty} x^2 n(n-1)x^{n-2} + xnx^{n-1} \\
&= (1-x)\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{1-x}\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right)\right] \\
&= (1-x)\left[\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}\right] \\
&= \frac{2x^2}{(1-x)^2} + \frac{x}{(1-x)}
\end{aligned}$$

For $\tau = 0$ (the same mode),

$$\begin{aligned}
g^{(2)}(0) &= \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2} \\
&= \frac{\langle \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rangle}{\langle \hat{a}^\dagger\hat{a}\rangle^2} \\
&= \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} = 2.
\end{aligned}$$

For $\tau >> \tau_c$, the operators at t and $t+\tau$ act on completely independent modes, so they commute with each other:

$$\begin{aligned}
g^{(2)}(\tau) &= \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2} \\
&= \frac{\langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2} \\
&= \frac{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2} \\
&= \frac{\langle n\rangle^2}{\langle n\rangle^2} = 1.
\end{aligned}$$

c) Let us first find the Q function for the thermal state:

$$\begin{aligned}
Q_{\hat{\rho}}(X,P) &= \frac{1}{2\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \qquad \text{where } \alpha = \frac{X+iP}{\sqrt{2}} \\
&= \frac{1}{2\pi}(1-e^{-\beta})\sum_{n=0}^{\infty} e^{-n\beta}\langle\alpha|n\rangle\langle n|\alpha\rangle \\
&= \frac{1}{2\pi}(1-e^{-\beta})\sum_{n=0}^{\infty} e^{-n\beta}e^{-|\alpha|^2}\frac{(\alpha\alpha^*)^2}{n!} \\
&= \frac{1}{2\pi}(1-e^{-\beta})e^{\frac{-X^2-P^2}{2}(1-e^{-\beta})}.
\end{aligned}$$

This is a Gaussian function with variance $\sigma_Q^2 = 1/(1-e^{-\beta})$. On the other hand, we know that the Q function is the convolution between Wigner functions $W_{\hat{\rho}}$ and $W_{|0\rangle\langle 0|}$. The latter is also a Gaussian function

$$W_{|0\rangle\langle 0|}(X,P) = \frac{1}{\pi}e^{-X^2-P^2}$$

with variance $\sigma^2_{|0\rangle\langle 0|} = 1/2$. Since both Q and $W_{|0\rangle\langle 0|}$ are Gaussian, the Wigner function must then be also Gaussian of the form:

$$W_{\hat{\rho}}(X,P) = \frac{1}{2\pi\sigma_W^2} e^{(-X^2-P^2)/2\sigma_W^2}$$

with variance

$$\begin{aligned}\sigma_W^2 &= \sigma_Q^2 - \sigma^2_{|0\rangle\langle 0|}\\ &= \frac{1}{1-e^{-\beta}} - \frac{1}{2}\\ &= \frac{1+e^{-\beta}}{2(1-e^{-\beta})}\\ &= \frac{1}{2\tanh^{-1}(\beta/2)}\end{aligned}$$

Hence

$$W_{\hat{\rho}}(X,P) = \frac{\tanh(\beta/2)}{\pi} e^{(-X^2-P^2)\tanh(\beta/2)}.$$

Solution to Exercise 4.38.

a) Let us calculate the expectation values for an operator $\hat{A}$ in the Schrödinger and Heisenberg pictures. First, the Schrödiner picture which tells us that:

$$|\psi(t)\rangle = e^{-i(\hat{H}/\hbar)t}|\psi_0\rangle \tag{E.57}$$

from which the expectation value is

$$\langle \hat{A}(t) = \left\langle \psi_0 \left| e^{i(\hat{H}/\hbar)t} \hat{A} e^{-i(\hat{H}/\hbar)t} \right| \psi_0 \right\rangle$$

which is the same as the expectation value of operator(4.54) evolving in accordance with the Heisenberg Picture.

b) Let us differentiate both sides of Eq. (4.54) with respect to time.

$$\begin{aligned}\partial_t \hat{A}(t) &= \partial_t \left(e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t} \right)\\ &= \partial_t \left(e^{i(\hat{H}/\hbar)t} \right) \hat{A}_0 e^{-i(\hat{H}/\hbar)t} + e^{i(\hat{H}/\hbar)t} \hat{A}_0 \partial_t \left(e^{-i(\hat{H}/\hbar)t} \right)\\ &= \frac{i}{\hbar} \left(\hat{H} e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t} - e^{i(\hat{H}/\hbar)t} \hat{A}_0 \hat{H} (e^{-i(\hat{H}/\hbar)t} \right)\\ &= \frac{i}{\hbar} \left(\hat{H} e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t} - e^{i(\hat{H}/\hbar)t} \hat{A}_0 (e^{-i(\hat{H}/\hbar)t} \hat{H} \right)\end{aligned}$$

where the last line follows from the comutativity of $\hat{H}$ and $e^{i\hat{H}/\hbar}$. Hence we have

$$\begin{aligned}\partial_t \hat{A}(t) &= \frac{i}{\hbar} \left(\hat{H}\hat{A}(t) - \hat{A}(t)\hat{H} \right)\\ &= \frac{i}{\hbar} \left[\hat{H}, \hat{A}(t) \right].\end{aligned}$$

Solution to Exercise 4.39. Let us calculate the time evolution of $\hat{x}$ under the harmonic oscillator Hamiltonian.

$$\begin{aligned}
\partial_t \hat{x} &= \frac{i}{\hbar}[\hat{H}, \hat{x}(t)] \\
&= \frac{i}{\hbar}[\hat{p}^2/2m, \hat{x}] + \frac{i}{\hbar}[k\hat{x}^2/2, \hat{x}] \\
&= \frac{i}{\hbar}\frac{1}{2m}[\hat{p}^2, \hat{x}] \\
&= \frac{i}{\hbar}\frac{1}{2m}[\hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p}] \\
&= \frac{\hat{p}}{m} \\
&= \frac{\partial H}{\partial \hat{p}}
\end{aligned}$$

because $[\hat{p}, \hat{x}] = -i\hbar$. The other canonical equation is proven similarly.

Solution to Exercise 4.40.

a) The Hamiltonian $\hat{H}$ such that the evolution $e^{-i\hat{H}t_0/\hbar}$ under $\hbar H$ for time t_0 is equal to $\hat{D}(X_0, P_0)$ must comply with

$$\hat{D}(X_0, P_0) = e^{iP_0\hat{X} - iX_0\hat{P}} = e^{-i\hat{H}t_0/\hbar}.$$

Hence

$$\hat{H} = \frac{\hbar X_0}{t_0}\hat{P} - \frac{\hbar P_0}{t_0}\hat{X}. \tag{E.58}$$

b) We have $[\hat{X}\ \hat{P}] = i$. Therefore, under the displacement Hamiltonian,

$$\begin{aligned}
\dot{X} &= \frac{i}{\hbar}[\hat{H}, \hat{X}] = \frac{X_0}{t_0}; \\
\dot{P} &= \frac{i}{\hbar}[\hat{H}\ \hat{P}] = \frac{P_0}{t_0},
\end{aligned}$$

and hence

$$\begin{aligned}
\hat{X}(t_0) &= \hat{X}(0) + X_0; \\
\hat{P}(t_0) &= \hat{P}(0) + P_0.
\end{aligned}$$

Solution to Exercise 4.40. The Hamiltonian $\hat{H}$ such that the evolution $e^{-i(\hat{H}/\hbar)t_0}$ under $\hat{H}$ for time t_0 is equal to the two mode squeezing operator $\hat{S}_2(\zeta)$ is

$$\hat{H} = \frac{i\hbar\zeta}{t_0}(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger) \tag{E.59}$$

Under this Hamiltonian, the differential equation for the evolution of $\hat{a}_1$ is:

$$\begin{aligned}
\dot{\hat{a}}_1 &= \frac{i}{\hbar}[\hat{H}, \hat{a}] \\
&= \frac{i}{\hbar}\left[\frac{i\hbar\zeta}{t_0}(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger), \hat{a}_1\right] \\
&= -\frac{\zeta}{t_0}\hat{a}_2^\dagger
\end{aligned}$$

(the last line follows from $[\hat{a}_1, \hat{a}_2] = [\hat{a}_1, \hat{a}_2^\dagger] = 0$). Doing the same for the $\hat{a}_2$, $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$, we obtain a set of coupled differential equations:

$$\dot{\hat{a}}_1 = -\frac{\zeta}{t_0}\hat{a}_2^\dagger \tag{E.60}$$

$$\dot{\hat{a}}_2 = -\frac{\zeta}{t_0}\hat{a}_1^\dagger \tag{E.61}$$

$$\dot{\hat{a}}_1^\dagger = -\frac{\zeta}{t_0}\hat{a}_2 \tag{E.62}$$

$$\dot{\hat{a}}_2^\dagger = -\frac{\zeta}{t_0}\hat{a}_1 \tag{E.63}$$

Differentiating Eq. (E.60) and substituting $\dot{\hat{a}}_2^\dagger$ from Eq. (E.63) yields

$$\ddot{\hat{a}}_1 = (\frac{\zeta}{t_0})^2 \hat{a}_1$$

which has general solution

$$\hat{a}_1 = A\cosh\left(\frac{\zeta}{t_0}t\right) + B\sinh\left(\frac{\zeta}{t_0}t\right)$$

with A and B being arbitrary operators. From the initial conditions, $\dot{\hat{a}}_1(t=0) = -\frac{\zeta}{t_0}\hat{a}_2^\dagger(0)$ [according to Eq. (E.60)], and hence $A = \hat{a}_1(0)$ and $B = -\hat{a}_2^\dagger(0)$. The evolution of the annihilation operator of mode 1 therefore is

$$\hat{a}_1(t) = \hat{a}_1(0)\cosh\left(\frac{\zeta}{t_0}t\right) - \hat{a}_2^\dagger(0)\sinh\left(\frac{\zeta}{t_0}t\right)$$

Rewriting the above equation for time $t = t_0$ yields Eq. (4.69). Equation (4.69) obtains in a similar fashion.

For the mode 1 quadrature observables, the evolution is as follows

$$\dot{\hat{X}}_1 = \frac{\dot{\hat{a}}_1 + \dot{\hat{a}}_1^\dagger}{\sqrt{2}} = -\frac{\zeta}{t_0}(\hat{a}_2^\dagger + \hat{a}_2) = -\frac{\zeta}{t_0}\hat{X}_2 \tag{E.64}$$

$$\dot{\hat{P}}_1 = \frac{\dot{\hat{a}}_1 - \dot{\hat{a}}_1^\dagger}{\sqrt{2}i} = -\frac{\zeta}{t_0}\frac{(\hat{a}_2^\dagger - \hat{a}_2)}{\sqrt{2}i} = \frac{\zeta}{t_0}\hat{P}_2 \tag{E.65}$$

and similarly

$$\dot{\hat{X}}_2 = -\frac{\zeta}{t_0}\hat{X}_1 \tag{E.66}$$

$$\dot{\hat{P}}_2 = \frac{\zeta}{t_0}\hat{P}_1 \tag{E.67}$$

From Eqs. (E.64) and (E.66), we get,

$$\dot{\hat{X}}_1 \pm \dot{\hat{X}}_2 = \mp\frac{\zeta}{t_0}(\hat{X}_1 \pm \hat{X}_2)$$

which upon Integration yields,

$$\hat{X}_1(t) \pm \hat{X}_2(t) = (\hat{X}_1(0) \pm \hat{X}_2(0))e^{\mp\frac{\zeta}{t_0}t}$$

Rewriting the above result for time $t = t_0$ yields Eq. (4.71). The derivation of is similar.

Solution to Exercise 4.75. The beam splitter operation can be written in the form

$$\vec{E}_0' = \begin{pmatrix} E_{01}' \\ E_{02}' \end{pmatrix} = \underline{B}\begin{pmatrix} E_{01} \\ E_{02} \end{pmatrix} = \underline{B}\vec{E}_0,$$

where B is the operation of the beam splitter and "vectors" refer to the pair of field amplitudes in the input and output beam splitter modes. The intensity of the input and output fields are given, respectively, by

$$I_0 = 2\epsilon_0 c(E_{01}^{(+)}E_{01}^{(-)} + E_{02}^{(+)}E_{02}^{(-)}) = 2\epsilon_0 c\vec{E}_0^\dagger \vec{E}_0,$$

and

$$I_0' = 2\epsilon_0 c\underline{B}^\dagger \vec{E}_0^\dagger \vec{E}_0 \underline{B}.$$

Since $\underline{B}$ must conserve energy, we must have that

$$\vec{E}^\dagger \underline{B}^\dagger \underline{B}\vec{E} = \vec{E}^\dagger \vec{E}$$

for any $\vec{E}$, and so $\underline{B}^\dagger \underline{B} = \mathbf{1}$, i.e. $\underline{B}$ is unitary.

Solution to Exercise 4.48.

a) Given that $r = t = 1/\sqrt{2}$, we can simplify $\underline{\mathrm{B}}$ to:

$$\underline{\mathrm{B}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{E.68}$$

Applying this to the annihilation and creation operators in the Heisenberg picture gives

$$\begin{aligned} \hat{X}_1' &= \frac{\hat{a}_1' + \hat{a}_1^{\dagger'}}{\sqrt{2}} \\ &= \frac{1}{2}(\hat{a}_1 - \hat{a}_2 + \hat{a}_1^\dagger - \hat{a}_2^\dagger) \\ &= \frac{1}{\sqrt{2}}(\hat{X}_1 - \hat{X}_2) \end{aligned}$$

and similarly:

$$\begin{aligned} \hat{X}_2' &= \frac{1}{\sqrt{2}}(\hat{X}_1 + \hat{X}_2) \\ \hat{P}_1' &= \frac{1}{\sqrt{2}}(\hat{P}_1 - \hat{P}_2) \\ \hat{P}_2' &= \frac{1}{\sqrt{2}}(\hat{P}_1 + \hat{P}_2) \end{aligned}$$

Now we consider the input state as a two-mode squeezed vacuum state with:

$$\begin{aligned} \hat{X}_1 \pm \hat{X}_2 &= (\hat{X}_{10} \pm \hat{X}_{20})e^{\mp\zeta}; \\ \hat{P}_1 \pm \hat{P}_2 &= (\hat{P}_{10} \pm \hat{P}_{20})e^{\pm\zeta}, \end{aligned}$$

where $\hat{X}_{i0}$ and $\hat{P}_{i0}$ refer to the position and momentum of the two vacuum states from which the two-mode squeezed state has been generated. These states can be treated, without loss of generality, as having been obtained from two further vacuum states that have been transformed by a beam splitter defined by Eq. (E.68):

$$\begin{aligned} X_{10} &= \frac{1}{\sqrt{2}}(\hat{X}_{100} - \hat{X}_{200}); \\ X_{20} &= \frac{1}{\sqrt{2}}(\hat{X}_{100} + \hat{X}_{200}); \\ P_{10} &= \frac{1}{\sqrt{2}}(\hat{P}_{100} - \hat{P}_{200}); \\ P_{20} &= \frac{1}{\sqrt{2}}(\hat{P}_{100} + \hat{P}_{200}), \end{aligned}$$

where subscript 00 refers to these latter states.

Combining the above expressions, we find the observables after the beam splitter as

$$\hat{X}_1' = \frac{1}{\sqrt{2}}(\hat{X}_1 - \hat{X}_2) = -\hat{X}_{200}e^{\zeta};$$
$$\hat{P}_1' = \frac{1}{\sqrt{2}}(\hat{P}_1 - \hat{P}_2) = -\hat{P}_{200}e^{-\zeta};$$
$$\hat{X}_2' = \frac{1}{\sqrt{2}}(\hat{X}_1 + \hat{X}_2) = \hat{X}_{100}e^{-\zeta};$$
$$\hat{P}_2' = \frac{1}{\sqrt{2}}(\hat{P}_1 + \hat{P}_2) = \hat{P}_{100}e^{\zeta}.$$

We see that the final states can be viewed as results of applying the single-mode squeezing operations (one being position-squeezing and the other momentum-squeezing) to the two initial vacuum states.

b) The beam splitter operation is reversible, so reversing the direction of the beams would mean that overlapping two oppositely-squeezed single mode states on a beam splitter would result in a two-mode squeezed state. The same math can be applied to find this result.

Solution to Exercise 4.49. Consider the action of Hamiltonian given by (4.81) on a two-mode state. Using Eq. (4.55) we get

$$\begin{aligned} \dot{\hat{a}}_1 &= \frac{i}{\hbar}[H, \hat{a}_1] \\ &= \Omega[\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger, \hat{a}_1] \\ &= -\Omega \hat{a}_2 \end{aligned} \tag{E.69}$$

and

$$\begin{aligned} \dot{\hat{a}}_2 &= \frac{i}{\hbar}[H, \hat{a}_2] \\ &= \Omega[\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger, \hat{a}_2] \\ &= \Omega \hat{a}_1. \end{aligned} \tag{E.70}$$

Differentiating both sides of the above equations with respect to time, we find

$$\ddot{\hat{a}}_1 = -\Omega^2 \hat{a}_1; \tag{E.71}$$
$$\ddot{\hat{a}}_2 = -\Omega^2 \hat{a}_2. \tag{E.72}$$

The solution to this system of differential equations can be written as

$$\hat{a}_1(\tau) = \hat{a}_1(0)\cos(\Omega\tau) - \hat{a}_2(0)\sin(\Omega\tau); \tag{E.73}$$
$$\hat{a}_2(\tau) = \hat{a}_1(0)\cos(\Omega\tau) + \hat{a}_2(0)\sin(\Omega\tau). \tag{E.74}$$

Defining

$$\begin{aligned} \cos(\Omega\tau) &= t \\ \sin(\Omega\tau) &= r, \end{aligned}$$

we find

$$\begin{pmatrix} \hat{a}_1(\tau) \\ \hat{a}_2(\tau) \end{pmatrix} = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix},$$

which is equivalent to transformation (4.79) with the beam splitter matrix (4.78).

Solution to Exercise 4.50.

a) Since any Fock state can be constructed from the vacuum state using Eq. (4.20), we have

$$\begin{aligned}\langle m_1, m_2| &= (|m_1, m_2\rangle)^\dagger \\ &= \left(\frac{(\hat a_1^\dagger)^{m_1}}{\sqrt{m_1!}}\frac{(\hat a_2^\dagger)^{m_2}}{\sqrt{m_2!}}|0,0\rangle\right)^\dagger \\ &= \left(\frac{(\hat a_1^\dagger)^{m_1}}{\sqrt{m_1!}}\frac{(\hat a_2^\dagger)^{m_2}}{\sqrt{m_2!}}\hat U|0,0\rangle\right)^\dagger \qquad \text{(beam splitter acting on vacuum gives vacuum)} \\ &= \frac{1}{\sqrt{m_1!m_2!}}\langle 0,0|\hat U^\dagger(\hat a_1)^{m_1}(\hat a_2)^{m_2}\end{aligned}$$

Substituting the above into the left-hand side of Eq. (4.82), we obtain the desired result.

b) We rewrite Eq. (4.82) as

$$\begin{aligned}\langle m_1, m_2|\hat U|n_1, n_2\rangle &= \tfrac{1}{\sqrt{m_1!m_2!}}\langle 0,0|\hat U^\dagger(\hat a_1)^{m_1}\hat U\hat U^\dagger(\hat a_2)^{m_2}\hat U|n_1,n_2\rangle; &\text{(E.75)}\\ &= \tfrac{1}{\sqrt{m_1!m_2!}}\langle 0,0|(\hat U^\dagger\hat a_1\hat U)^{m_1}(\hat U^\dagger\hat a_2\hat U)^{m_2}|n_1,n_2.\rangle &\text{(E.76)}\end{aligned}$$

Using

$$\begin{aligned}\hat a_1' &= \hat U^\dagger\hat a_1\hat U = t\hat a_1 - r\hat a_2; &\text{(E.77)}\\ \hat a_2' &= \hat U^\dagger\hat a_2\hat U = r\hat a_1 + t\hat a_2, &\text{(E.78)}\end{aligned}$$

we obtain

$$\begin{aligned}&\langle m_1, m_2|\,\hat U\,|n_1, n_2\rangle \\ &= \frac{1}{\sqrt{m_1!m_2!}}\langle 0,0|(t\hat a_1 - r\hat a_2)^{m_1}(r\hat a_1 + t\hat a_2)^{m_2}|n_1,n_2\rangle. \\ &= \frac{1}{\sqrt{m_1!m_2!}}\,\langle 0,0|\left(\sum_{k_1=0}^{m_1}\frac{(-1)^{m_1-k_1!}n_1!}{k_1!(m_1-k_1)!}(t\hat a_1)^{k_1}(r\hat a_2)^{m_1-k_1}\right) \\ &\qquad\qquad\times\left(\sum_{k_2=0}^{m_2}\frac{m_2!}{k_2!(m_2-k_2)!}(r\hat a_1)^{k_2}(t\hat a_2)^{m_2-k_2}\right)|n_1,n_2\rangle \\ &= \frac{1}{\sqrt{m_1!m_2!}}\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\frac{(-1)^{m_1-k_1!}n_1!}{k_1!(m_1-k_1)!}\frac{m_2!}{k_2!(m_2-k_2)!}t^{k_1+m_2-k_2}r^{m_1-k_1+k_2} \\ &\qquad\qquad\times\langle 0,0|\,\hat a_1^{k_1+k_2}\hat a_2^{m_1-k_1+m_2-k_2}\,|n_1,n_2\rangle \\ &= \frac{1}{\sqrt{m_1!m_2!}}\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\frac{(-1)^{m_1-k_1!}n_1!}{k_1!(m_1-k_1)!}\frac{m_2!}{k_2!(m_2-k_2)!}t^{k_1+m_2-k_2}r^{m_1-k_1+k_2} \\ &\qquad\qquad\times\delta_{k_1+k_2,n_1}\delta_{m_1-k_1+m_2-k_2,n_2}\sqrt{n_1!n_2!} \qquad\qquad \text{(E.79)}\end{aligned}$$

The condition $k_1 + k_2, n_1$ imposed by the first of the above Kronecker deltas implies that the second Kronecker delta can be substituted by $\delta_{m_1+m_2-n_1,m_2} = \delta_{n_1+n_2,m_1+m_2}$ (which is simply energy conservation). With this substitution, we obtain Eq. (4.83).

Solution to Exercise 4.51. For $n_2 = 0$ and $n_1 = n$, Eq. (4.83) takes the form

$$\begin{aligned}&\langle m_1, m_2|\hat U|n, 0\rangle = \\ &= \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\delta_{k_1+k_2,n}\delta_{n,m_1+m_2}\frac{(-1)^{m_1-k_1}\sqrt{n!m_1!m_2!}}{k_1!(m_1-k_1)!k_2!(m_2-k_2)!}t^{m_2+k_1-k_2}r^{m_1-k_1+k_2}\end{aligned}$$

Since the Kronecker deltas impose $k_1 + k_2 = m_1 + m_2 = n$, the only nonvanishing element in the sum is the one with $k_1 = m_1$ and $k_2 = m_2$. Accordingly,

$$\begin{aligned}\langle m_1, m_2|\hat{U}|n, 0\rangle &= \delta_{n,m_1+m_2}\frac{\sqrt{n!m_1!m_2!}}{m_1!m_2!}t^{m_1}r^{m_2}\\ &= \delta_{n,m_1+m_2}\sqrt{\begin{pmatrix}\hat{n}\\ \hat{m}_2\end{pmatrix}}t^{m_1}r^{m_2},\end{aligned}$$

which is equivalent to Eq. (4.84).

Solution to Exercise 4.51. Applying Eq. (4.83) to the $|1,1\rangle$ state, we can see that

$$\hat{U}\,|1,1\rangle = -r\tau\sqrt{2}\,|0,2\rangle - r^2\,|1,1\rangle + \tau^2\,|1,1\rangle + r\tau\sqrt{2}\,|2,0\rangle$$

For a $50:50$ beam-splitter we have $r = \frac{1}{\sqrt{2}}$ and $\tau = \frac{1}{\sqrt{2}}$ and thus

$$\hat{U}\,|1,1\rangle = \frac{1}{\sqrt{2}}(-\,|0,2\rangle + |2,0\rangle)$$

The amplitude of output $|1,1\rangle$ vanishes, so there can be no coincident photon detection after the beam splitter.

Solution to Exercise 4.54. To show that two coherent states after a beam splitter are still coherent states we look at $\hat{a}_1\hat{U}|\alpha\rangle\otimes|\beta\rangle$. First we multiply with $\hat{U}\hat{U}^\dagger = \hat{\mathbf{1}}$ from the left.

$$\hat{a}_1\hat{U}|\alpha\rangle\otimes|\beta\rangle = \hat{U}\hat{U}^\dagger\hat{a}_1\hat{U}|\alpha\rangle\otimes|\beta\rangle,$$

with $\hat{U}^\dagger\hat{a}_1\hat{U} = B_{11}\hat{a}_1 + B_{12}\hat{a}_2$ according to Eq. (4.79). Hence

$$\begin{aligned}\hat{a}_1\hat{U}|\alpha\rangle\otimes|\beta\rangle &= \hat{U}(B_{11}\hat{a}_1 + B_{12}\hat{a}_2)|\alpha\rangle\otimes|\beta\rangle\\ &= \hat{U}(B_{11}\alpha + B_{12}\beta)|\alpha\rangle\otimes|\beta\rangle\\ &= (B_{11}\alpha + B_{12}\beta)\hat{U}|\alpha\rangle\otimes|\beta\rangle\end{aligned}$$

In the last line, we exchanged $(B_{11}\alpha + B_{12}\beta)$ with $\hat{U}$ since the former is just a number. We see that $\hat{U}|\alpha\rangle\otimes|\beta\rangle$ is an eigenstate of $\hat{a}_1$ with the eigenvalue $B_{11}\alpha + B_{12}\beta$.

$$\hat{a}_1\hat{U}|\alpha\rangle\otimes|\beta\rangle = (t\alpha - r\beta)\hat{U}|\alpha\rangle\otimes|\beta\rangle$$

Similarly we get for $\hat{a}_2$

$$\hat{a}_2\hat{U}|\alpha\rangle\otimes|\beta\rangle = (B_{21}\alpha + B_{22}\beta)\hat{U}|\alpha\rangle\otimes|\beta\rangle$$

This means that the states after the beam splitter are eigenstates of the annihilation operators and therefore coherent states:

$$\hat{U}|\alpha\rangle\otimes|\beta\rangle = |B_{11}\alpha + B_{12}\beta\rangle\otimes|B_{21}\alpha + B_{22}\beta\rangle.$$

Solution to Exercise 4.58. We have to find the behaviour of the Wigner function when the P functions shrinks from $P_{\rm in}(X,P)$ to $P_{\rm out}(X,P) = \frac{1}{t^2}P_{\rm in}(\frac{X}{t},\frac{P}{t})$. We use the fact that the Wigner function is equal to the convolution of the P function and the vacuum state Wigner function:

$$W_{\hat{\rho},\rm out}(X,P) = P_{\hat{\rho},\rm out}(X,P) * W_{|0\rangle\langle 0|}(X,P) \tag{E.80}$$

$$= \tfrac{1}{t^2}P_{\hat{\rho},\rm in}(\tfrac{X}{t},\tfrac{P}{t}) * \tfrac{1}{\pi}e^{-(X^2+P^2)} \tag{E.81}$$

$$\tag{E.82}$$

The vacuum state Wigner function is Gaussian of variance 1/2. It can be written as a convolution of two normalized Gaussian functions such that the sum of their variances equals 1/2.

$$W_{\hat{\rho},\rm out}(X,P) = \frac{1}{\pi}e^{-(X^2+P^2)} * \frac{1}{\pi\times t^2}e^{\frac{-(X^2+P^2)}{t^2}} * \frac{1}{\pi\times(1-t^2)}e^{\frac{-(X^2+P^2)}{1-t^2}} \tag{E.83}$$

We notice that the first two factors above are the P-function of the input state and the vacuum state Wigner function scaled by the same factor t. Using the associativity of convolution and convolving these two factors, we obtain the scaled Wigner function of the input state:

$$W_{\hat{\rho},\mathrm{out}}(X,P) = \frac{1}{\pi t^2(1-t^2)} W_{\hat{\rho},\mathrm{in}}\left(\frac{X}{t},\frac{P}{t}\right) * e^{\frac{-(X^2+P^2)}{1-t^2}} \tag{E.84}$$

which is equivalent to Eq. (4.90).

Solution to Exercise 4.61. Since the beam splitter is balanced each of the field annihilation operators is transformed by the beam splitters as follows:

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}_{LO} + \hat{a}e^{i\theta})$$
$$\hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{a}_{LO} - \hat{a}e^{i\theta})$$

The factor of $e^{i\theta}$ accounts for a phase difference between the signal and the local oscillator (LO).

Each of the photodiodes measures the number of photons in its respective mode, and these measurements are thens subtracted from each other. This results in a measurement of $\hat{n}_-$:

$$\begin{aligned}
\hat{n}_- &= \hat{n}_1 - \hat{n}_2 \\
&= \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2 \\
&= \frac{1}{2}\left(\hat{a}_{LO}^\dagger\hat{a}_{LO} + \hat{a}^\dagger\hat{a} + \hat{a}_{LO}^\dagger\hat{a}e^{i\theta} + \hat{a}^\dagger\hat{a}_{LO}e^{-i\theta}\right) \\
&\quad - \frac{1}{2}\left(\hat{a}_{LO}^\dagger\hat{a}_{LO} + \hat{a}^\dagger\hat{a} - \hat{a}_{LO}^\dagger\hat{a}e^{i\theta} - \hat{a}^\dagger\hat{a}_{LO}e^{-i\theta}\right) \\
&= \hat{a}_{LO}^\dagger\hat{a}e^{i\theta} + \hat{a}^\dagger\hat{a}_{LO}e^{-i\theta}
\end{aligned}$$

Solution to Exercise 4.62. Due to the macroscopic size of the coherent state of the local oscillator, its annihilation operator can be approximated by its eigenvalue α_{LO}. Assuming it to be real, we have

$$\begin{aligned}
\hat{n}_- &= \alpha_{LO}(ae^{-i\theta} + a^\dagger e^{i\theta}) \\
&= \alpha_{LO}X_\theta\sqrt{2}.
\end{aligned}$$

Solution to Exercise 4.64.

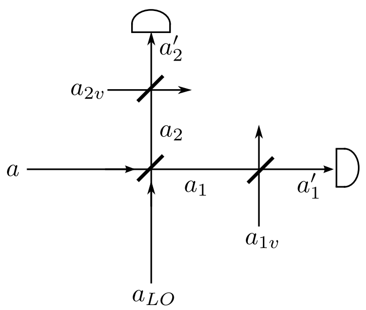


Figure E.1: A model of balanced homodyne detection with imperfect photodetectors.

(b) Imperfect photodiodes with quantum efficiency η can be modeled by perfect photodiodes preceded by beam splitters of transmissivity η, as shown in Fig. E.1. The output modes of the symmetric beam splitter can be written as,

$$a_1 = \frac{a_{LO}+a}{\sqrt{2}} \qquad \text{and} \qquad a_2 = \frac{a_{LO}-a}{\sqrt{2}}. \tag{E.85}$$

After the loss, the modes in each channel become

$$a_1' = \sqrt{\eta} a_1 + \sqrt{1-\eta} a_{1v} \qquad \text{and} \qquad a_2' = \sqrt{\eta} a_2 + \sqrt{1-\eta} a_{2v}, \tag{E.86}$$

where subscript v refers to a mode in the vacuum state. Substituting the operators in Eq. (E.85) into those into Eq. (E.86) yields

$$a_1' = \frac{\sqrt{\eta}(a_{LO}+a)}{\sqrt{2}} + \sqrt{1-\eta} a_{1v} \qquad \text{and} \qquad a_2' = \frac{\sqrt{\eta}(a_{LO}-a)}{\sqrt{2}} + \sqrt{1-\eta} a_{2v}. \tag{E.87}$$

We would like to show that this configuration (shown in Figure E.1) is equivalent to the one

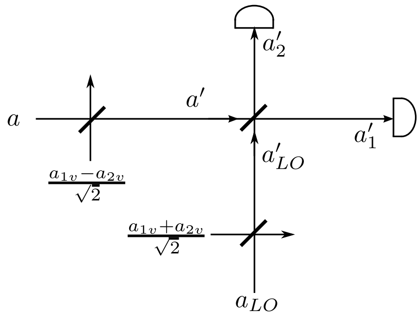


Figure E.2: Balanced homodyne detection with attenuated inputs.

in which photodiodes are perfect, but the signal and local oscillator are both attenuated before the central (50:50) BS (shown in Figure E.2). The vacuum modes that enter the attenuators can, without loss of generality, be written as

$$a_{1V} = \frac{(a_{1u}-a_{2u})}{\sqrt{2}} \qquad \text{and} \qquad a_{2V} = \frac{(a_{1u}+a_{2u})}{\sqrt{2}}, \tag{E.88}$$

where a_{1u} and a_{2u} also represent operators of modes in vacuum states. The signal and LO modes after the losses are then

$$a' = \sqrt{\eta} a + \sqrt{1-\eta} \frac{(a_{1u}-a_{2u})}{\sqrt{2}} \qquad \text{and} \qquad a_{LO}' = \sqrt{\eta} a_{LO} + \sqrt{1-\eta} \frac{(a_{1u}+a_{2u})}{\sqrt{2}}, \tag{E.89}$$

which are now the input modes of the central beam splitter. The output modes of that beam splitter can be written as

$$a_1' = \frac{a' + a_{LO}'}{\sqrt{2}} = \frac{\sqrt{\eta}(a_{LO}+a)}{\sqrt{2}} + \sqrt{1-\eta} a_{1u} \tag{E.90}$$

and

$$a_2' = \frac{a_{LO}' - a'}{\sqrt{2}} = \frac{\sqrt{\eta}(a_{LO}-a)}{\sqrt{2}} + \sqrt{1-\eta} a_{2u}. \tag{E.91}$$

We see that the expressions in (E.87) and (E.91) are identical. Thus the two configurations are equivalent.

Appendix F

Solutions to Chapter 5 problems

Solution to Exercise 5.6. We start be writing down the density matrix $\hat{\rho}$

$$\hat{\rho}\begin{pmatrix}\rho_{HH} & \rho_{HV}\\ \rho_{VH} & \rho_{VV}\end{pmatrix}$$

A set of measurements in the canonical (HV) basis tells us the diagonal elements of the density matrix:

$$\begin{aligned}\rho_{HH} &= \mathrm{pr}(H) = \frac{N_H}{N_H + N_V};\\ \rho_{VV} &= \mathrm{pr}(V) = \frac{N_V}{N_H + N_V}.\end{aligned}$$

To get the off-diagonal elements of $\hat{\rho}$, we need measurements in the diagonal and the circular basis. From the measurement in the diagonal basis we get

$$\mathrm{pr}(+) = \frac{1}{2}\left(\langle H| + \langle V|\right)|\hat{\rho}|\left(|H\rangle + |V\rangle\right) = \frac{1}{2}\left(\rho_{HH} + \rho_{HV} + \rho_{VH} + \rho_{VV}\right) = \frac{1}{2}\left(1 + \rho_{HV} + \rho_{VH}\right) \tag{F.1}$$

and from the circular basis

$$\mathrm{pr}(R) = \frac{1}{2}\left(\langle H| - i\langle V|\right)|\hat{\rho}|\left(|H\rangle + i|V\rangle\right) = \frac{1}{2}\left(\rho_{HH} + \rho_{VV} + i\left(\rho_{HV} - \rho_{VH}\right)\right) = \frac{1}{2}\left(1 + i\left(\rho_{HV} - \rho_{VH}\right)\right). \tag{F.2}$$

Solving these two equations, we find

$$\begin{aligned}\rho_{HV} &= \frac{1}{2}(i-1) + \mathrm{pr}(+) - i\mathrm{pr}(R)\\ \rho_{VH} &= \frac{1}{2}(-i-1) + \mathrm{pr}(+) + i\mathrm{pr}(R)\end{aligned}$$

We can also rewrite the density matrix in terms of the count rates

$$\hat{\rho} = \begin{pmatrix}\frac{N_H}{N_H+N_V} & \frac{1}{2}(i-1) + \frac{N_+}{N_++N_-} - i\frac{N_R}{N_R+N_L}\\ \frac{1}{2}(-i-1) + \frac{N_+}{N_++N_-} + i\frac{N_R}{N_R+N_L} & \frac{N_V}{N_H+N_V}\end{pmatrix}$$

Solution to Exercise 5.11. According to the optical equivalence theorem (4.47), any density matrix $\hat{\rho}$ can be written as a linear combination of coherent-state density matrices:

$$\hat{\rho} = \int P_\rho(\alpha)|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha. \tag{F.3}$$

This result can be used for quantum process tomography. To characterize the quantum process $\mathcal{E}$ one measures its effect on the coherent states, $\mathcal{E}(\hat{\rho}_\alpha = |\alpha\rangle\langle\alpha|)$ over many (ideally, all) values of α.

Then, according to the linearity of quantum processes (Eq. (5.3)), we have the effect of the process on state $\hat{\rho}$:

$$\mathcal{E}(\hat{\rho}) = \int P_\rho(\alpha)\ \mathcal{E}(|\alpha\rangle\langle\alpha|)\ \mathrm{d}^2\alpha. \tag{F.4}$$

Solution to Exercise 5.13. The number n_{out} of photons per second that Bob receives after the signal traverses a distance L is related to n_{in} according to

$$\beta L = 10\log_{10}\frac{n_{out}}{n_{in}}.$$

Hence $n_{out} = n_{in}10^{-(\beta L/10)}$

The number of photons that Bob is able to detect is then

$$\begin{aligned} n_{\det} &= \eta \times n_{out} \\ &= \eta n_{in}10^{-(\beta L/10)} \end{aligned}$$

For the BB84 protocol, the transmission is secure if

$$\frac{\text{error rate}}{\text{Bob's total counts}} \le 25\%. \tag{F.5}$$

The count events at Bob's detector are caused by "legitimate" photons and the dark events; the errors occur due to the latter. Only those dark count events that occur within the time interval of τ of expected photon arrival times are indistinguishable from the events cause by legitimate photons. Such events comprise a fraction τn_{in} of the total dark counts, so the effective dark count rate $f_{d,\text{eff}} = f_d\tau n_{in}$.

Equation (F.5) can thus be rewritten as

$$\frac{f_d\tau n_{in}}{\eta n_{in}10^{-(\beta L/10)} + f_d\tau n_{in}} \le 0.25$$

Substituting the values and solving the above equation in the limiting case of 25% error gives $L \approx 175$ km which is the maximum distance for which the given communication channel is secure.

Solution to Exercise 5.22. Consider the basis formed by states

$$\begin{aligned} |\psi\rangle &= \alpha\,|H\rangle + \beta\,|V\rangle\,; \\ |\psi_\perp\rangle &= \beta^*\,|H\rangle - \alpha^*\,|V\rangle\,. \end{aligned} \tag{F.6}$$

Suppose Alice and Bob share the entangled state $|\Psi^-\rangle$ and Alice measures her portion in the $\{|\psi\rangle, |\psi_\perp\rangle\}$ basis.

a) Alice's reduced density matrix is

$$\begin{aligned} \rho_A &= \mathrm{Tr}_B\,|\Psi^-\rangle\langle\Psi^-| \\ &= \frac{1}{2}\left(|H\rangle\langle H| + |V\rangle\langle V|\right), \end{aligned}$$

so her density matrix is completely mixed. Thus she has precisely a $\frac{1}{2}$ probability of obtaining each result no matter in which basis she measures.

b) If Alice measures her portion to be in the state $|\psi\rangle$, then Bob's portion will be

$$\begin{aligned} \langle\psi|\ \Psi^-\rangle &= \frac{1}{\sqrt{2}}\left(\alpha^*\langle H| + \beta^*\langle V|\right)\left(|HV\rangle - |VH\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\alpha^*\,|V\rangle - \beta^*\,|H\rangle\right) \\ &= -\frac{1}{\sqrt{2}}\,|\psi_\perp\rangle\,. \end{aligned}$$

However, if Alice measures her portion to be in the state $|\psi_\perp\rangle$, Bob has

$$\begin{aligned}\langle\psi_\perp|\,\Psi^-\rangle &= \frac{1}{\sqrt{2}}\left(\beta\langle H| - \alpha\langle V|\right)\left(|HV\rangle - |VH\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\beta|V\rangle + \alpha|H\rangle\right)\\ &= \frac{1}{\sqrt{2}}|\psi\rangle\,.\end{aligned}$$

After Alice communicates to Bob which state she measured, Bob will know whether he has $|\psi\rangle$ or $|\psi_\perp\rangle$.

Solution to Exercise 5.23. Suppose Alice has no communication with Bob. He does not know which state Alice has obtained in her measurement. Therefore he has a probability of $\frac{1}{2}$ to have state $|\psi\rangle$ or $|\psi_\perp\rangle$. This means we can write Bob's reduced density matrix as the mixture

$$\begin{aligned}\rho_B &= \frac{1}{2}\left(|\psi\rangle\langle\psi| + |\psi_\perp\rangle\langle\psi_\perp|\right)\\ &= \frac{1}{2}\Big(\left(|\alpha|^2 + |\beta|^2\right)|H\rangle + \left(|\alpha|^2 + |\beta|^2\right)|V\rangle\Big)\\ &= \frac{1}{2}\left(|H\rangle\langle H| + |V\rangle\langle V|\right),\end{aligned}$$

which again is completely mixed.

Solution to Exercise 5.31.

(a) Let $|\Psi^-\rangle$ be the input state of the beam splitter. Referring to Problem 5.2, we write this state as

$$\begin{aligned}|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)\\ &= \frac{1}{\sqrt{2}}\left(\underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|0110\rangle}_{H1\,V1\,H2\,V2}\right)\end{aligned} \tag{F.7}$$

(where $H1$ refers to the horizontal polarization of spatiotemporal mode 1, etc.). In order to find out how this state transforms on the beam splitter, we need to apply the beam splitter equation (4.83) separately to the states in the vertically and horizontally polarized pair of modes. For example, we can write the first term of Eq. (F.7) as

$$\underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} = \underbrace{|10\rangle}_{H1\,H2} \otimes \underbrace{|01\rangle}_{V1\,V2}\,.$$

According to Eq. (4.83), these states transform on the beam splitter as follows:

$$|10\rangle \xrightarrow{\hat{U}} \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \tag{F.8}$$

$$|01\rangle \xrightarrow{\hat{U}} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{F.9}$$

so we have

$$\underbrace{|10\rangle}_{H1\,H2} \otimes \underbrace{|01\rangle}_{V1\,V2} \xrightarrow{\hat{U}} \frac{1}{2}\left(\underbrace{|10\rangle}_{H1\,H2} \otimes \underbrace{|01\rangle}_{V1\,V2} + \underbrace{|01\rangle}_{H1\,H2} \otimes \underbrace{|01\rangle}_{V1\,V2} - \underbrace{|10\rangle}_{H1\,H2} \otimes \underbrace{|10\rangle}_{V1\,V2} - \underbrace{|01\rangle}_{H1\,H2} \otimes \underbrace{|10\rangle}_{V1\,V2}\right). \tag{F.10}$$

This can be rewritten as

$$\underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} \xrightarrow{\hat{U}} \frac{1}{2}\left(\underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} + \underbrace{|0011\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|1100\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|0110\rangle}_{H1\,V1\,H2\,V2} \right).$$

For the second term of Eq. (F.7) we find, in a similar fashion,

$$|VH\rangle = \underbrace{|0110\rangle}_{H1\,V1\,H2\,V2} \xrightarrow{\hat{U}} \frac{1}{2}\left(\underbrace{|0110\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|1100\rangle}_{H1\,V1\,H2\,V2} + \underbrace{|0011\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} \right)$$

and hence

$$|\Psi^-\rangle \xrightarrow{\hat{U}} \frac{1}{\sqrt{2}}\left(\underbrace{|1001\rangle}_{H1\,V1\,H2\,V2} - \underbrace{|0110\rangle}_{H1\,V1\,H2\,V2} \right) = |\Psi^-\rangle\,.$$

In other words, Bell state $|\Psi^-\rangle$ is invariant with respect to the beam splitter transformation.

Conducting the same argument for state $|\Psi^+\rangle$, we obtain

$$|\Psi^+\rangle \xrightarrow{\hat{U}}= \frac{1}{\sqrt{2}}(|0011\rangle - |1100\rangle) \tag{F.11}$$

Thus the photons after leaving the beam splitter are in same output spatiotemporal mode.

For states $|\Phi^\pm\rangle$, we have either HH or VV in the input modes. This means, two photons of the same polarization are meeting on a beam splitter. Such photons will experience the Hong-Ou-Mandel effect, which means that after the beam splitter they will be found in the same optical mode.

(b) It is possible to distinguish between $|\Psi^+\rangle$ and $|\Phi^\pm\rangle$. In the figure, if we get clicks from D1 and D2 or D3 and D4, we know the photons bunched together in one spatiotemporal mode has orthogonal polarization, i.e. the input Bell state was $|\Psi^+\rangle$. Otherwise, if we get both clicks from D1 or D2 or D3 or D4, we know the photons have the same polarization, i.e. the Bell states are $|\Phi^\pm\rangle$.

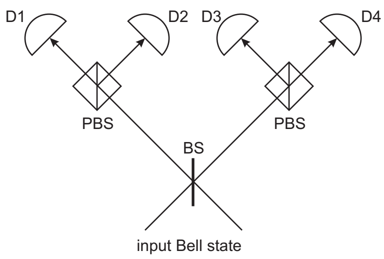
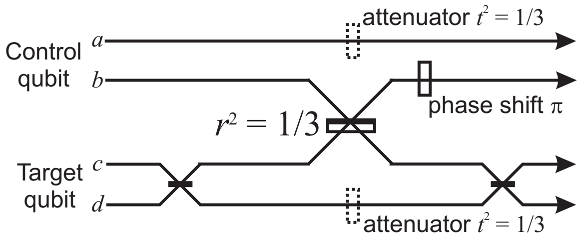


Figure F.1: A scheme that allows distinguishing Bell states $|\Psi^+\rangle$, $|\Psi^-\rangle$, and $|\Phi^\pm\rangle$

Solution to Exercise 5.35. Consider the case where modes a and b together form a qubit with only one photon, and similarly modes c and d form a qubit. By postselecting on the events where the number of photons in each of the output cubits is the same as the number of photons in each of the input cubits, we can use this scheme to probabilistically implement the CPHASE gate (5.11). A photon in mode b would correspond to the logical value 1 for the first qubit, and similarly a photon in mode c corresponds to the logical value 1 for the second qubit. Now consider the action of this



scheme on each possible input state (without the attenuators in a and d):

$$|1001\rangle \Longrightarrow |1001\rangle$$

$$|0101\rangle \Longrightarrow \sqrt{\frac{1}{3}}|0101\rangle + \sqrt{\frac{2}{3}}|0011\rangle \quad \text{(due to phase of } \pi \text{ from BS + } \pi \text{ from PS)}$$

$$|1010\rangle \Longrightarrow \sqrt{\frac{1}{3}}|1010\rangle - \sqrt{\frac{2}{3}}|1100\rangle \quad \text{(due to phase } \pi \text{ from PS on transmitted)}$$

$$|0110\rangle \Longrightarrow \underbrace{\frac{1}{3}|0110\rangle}_{b\to b',\, c\to c'} - \underbrace{\frac{2}{3}|0110\rangle}_{b\to c',\, c\to b'} - \underbrace{\frac{\sqrt{2}}{3}|0200\rangle}_{b\to b',\, c\to b'} + \underbrace{\frac{\sqrt{2}}{3}|0020\rangle}_{b\to c',\, c\to c'}$$

By disregarding all of the events where the number of photons in each qubit has changed, we can cut out all of the events where there are 0 or 2 photons in qubits 1 or 2:

$$|1001\rangle \Longrightarrow |1001\rangle$$

$$|0101\rangle \Longrightarrow \sqrt{\frac{1}{3}}|0101\rangle$$

$$|1010\rangle \Longrightarrow \sqrt{\frac{1}{3}}|1010\rangle$$

$$|0110\rangle \Longrightarrow -\frac{1}{3}|0110\rangle$$

The probability of successful operation here depends on the input state. Considering the transformation of modes $|bc\rangle \to |b'c'\rangle$ for the above results we get $|00\rangle \Longrightarrow |00\rangle$ with probability 1, $|10\rangle \Longrightarrow |10\rangle$ and $|01\rangle \Longrightarrow |01\rangle$ with probability 1/3, and $|11\rangle \Longrightarrow -|11\rangle$ with probability 1/9.

We can now compensate for this unequal probability of success by adding attenuators with $t^2 = 1/3$ in channels a and d. Then we have:

$$|1001\rangle \Longrightarrow \frac{1}{3}|1001\rangle$$

$$|0101\rangle \Longrightarrow \frac{1}{3}|0101\rangle$$

$$|1010\rangle \Longrightarrow \frac{1}{3}|1010\rangle$$

$$|0110\rangle \Longrightarrow -\frac{1}{3}|0110\rangle$$

and the success probability for each input state is 1/9 in all cases.

Appendix G

Solutions to Chapter 6 problems

Solution to Exercise 6.1.

a) To prove that

$$\partial_t|\psi_I(t)\rangle = -\frac{i}{\hbar}\hat{V}(t)|\psi_I(t)\rangle, \tag{G.1}$$

we expand the derivative as

$$\begin{aligned}\partial_t|\psi_I(t)\rangle &= \partial_t\left(e^{i(\hat{H}_0/\hbar)t}|\psi_S(t)\rangle\right)\\ &= \partial_t\left(e^{i(\hat{H}_0/\hbar)t}\right)|\psi_S(t)\rangle + e^{i(\hat{H}_0/\hbar)t}\partial_t\left(|\psi_S(t)\rangle\right)\\ &= i(\hat{H}_0/\hbar)e^{i(\hat{H}_0/\hbar)t}|\psi_S(t)\rangle + e^{i(\hat{H}_0/\hbar)t}\partial_t\left(|\psi_S(t)\rangle\right).\end{aligned} \tag{G.2}$$

We shall use the Schrödinger equation for the value of the second partial derivative:

$$\partial_t|\psi_S(t)\rangle = -\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_I(t))|\psi_S(t)\rangle, \tag{G.3}$$

and thus

$$\partial_t|\psi_I(t)\rangle = i(\hat{H}_0/\hbar)e^{i(\hat{H}_0/\hbar)t}|\psi_S(t)\rangle - e^{i(\hat{H}_0/\hbar)t}\frac{i}{\hbar}(\hat{H}_0 + \hat{H}_I(t))|\psi_S(t)\rangle. \tag{G.4}$$

The first two terms cancel since H_0 commutes with $e^{i(\hat{H}_0/\hbar)t}$. Hence,

$$\partial_t|\psi_I(t)\rangle = -\frac{i}{\hbar}e^{i(\hat{H}_0/\hbar)t}\hat{H}_I(t)|\psi_S(t)\rangle. \tag{G.5}$$

$$= -\frac{i}{\hbar}e^{i(\hat{H}_0/\hbar)t}\hat{H}_I(t)e^{-i(\hat{H}_0/\hbar)t}|\psi_I(t)\rangle. \tag{G.6}$$

$$= -\frac{i}{\hbar}\hat{V}(t)|\psi_I(t)\rangle. \tag{G.7}$$

b) This can be proved by simple substitution:

$$\langle\psi_I(t)|\hat{A}_I(t)|\psi_I(t)\rangle = \langle\psi_I(t)|e^{i(\hat{H}_0/\hbar)t}\hat{A}_0e^{-i(\hat{H}_0/\hbar)t}|\psi_I(t)\rangle \tag{G.8}$$

$$= \langle\psi_S(t)|\hat{A}_0|\psi_S(t)\rangle \tag{G.9}$$

which is the expectation value from the Schrödinger picture.

Solution to Exercise 6.3. The interaction Hamiltonian equals

$$\hat{H}_I = \hat{H} - \hat{H}_0 = \begin{pmatrix} \hbar\omega_0 & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix} - \begin{pmatrix} \hbar\omega & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix}$$

with $\Delta = \omega - \omega_0$. Transforming this according to the equation (6.7) for the evolution of operators in the interaction picture we find

$$\hat{V}(t) = \exp\left[i\begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}t\right]\begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix}\exp\left[-i\begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}t\right]$$

We recall that the exponent of a diagonal matrix $e^{\hat{A}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is also a diagonal matrix: $e^{\hat{A}} = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$. Accordingly,

$$\begin{aligned}\hat{V}(t) &= \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\hbar\Delta e^{-i\omega t} & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* e^{-i\omega t} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d}e^{i\omega t} \\ -\vec{E}\vec{d}^* e^{-i\omega t} & 0 \end{pmatrix}.\end{aligned}$$

Solution to Exercise 6.5. The evolution of the state in the rotating frame is given by $|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$ where the Hamiltonian is

$$\hat{H} = \hbar\begin{pmatrix} -\Delta & -\Omega \\ -\Omega^* & 0 \end{pmatrix}.$$

To evaluate $e^{-\frac{i}{\hbar}\hat{H}t}$ we first determine the eigenvalue decomposition of $\hat{H} = \lambda_1 |v_1\rangle\langle v_1| + \lambda_2 |v_2\rangle\langle v_2|$ where λ_1 and λ_2 are the eigenvalues of $\hat{H}$ with corresponding (normalized) eigenvectors $|v_1\rangle$ and $|v_2\rangle$. In fact, we have

$$\lambda_1 = -\hbar\left(\frac{\Delta}{2} + \sqrt{\frac{\Delta^2}{4} + |\Omega|^2}\right) \quad \text{and} \quad \lambda_2 = -\hbar\left(\frac{\Delta}{2} - \sqrt{\frac{\Delta^2}{4} + |\Omega|^2}\right)$$

with corresponding eigenvectors

$$|v_1\rangle = \frac{1}{N_1}\begin{pmatrix} \frac{\Delta}{2} + \sqrt{\frac{\Delta^2}{4} + |\Omega|^2} \\ \Omega^* \end{pmatrix} \quad \text{and} \quad |v_2\rangle = \frac{1}{N_2}\begin{pmatrix} \frac{\Delta}{2} - \sqrt{\frac{\Delta^2}{4} + |\Omega|^2} \\ \Omega^* \end{pmatrix}$$

and normalization constants N_1 and N_2 such that $\langle v_1|v_1\rangle = \langle v_2|v_2\rangle = 1$.

To find the coefficients $\psi_a(t)$ and $\psi_b(t)$ as functions of time, we need to calculate $\psi_a(t) = \langle a|\psi(t)\rangle = \langle a|e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$ and $\psi_b(t) = \langle b|e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$. The ground and excited states are denoted by the vectors

$$|b\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad |a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The initial state is simply the ground state $|\psi(0)\rangle = |b\rangle$ and we can write $e^{-\frac{i}{\hbar}\hat{H}t}$ in the eigenbasis of $\hat{H}$

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}\lambda_1 t}|v_1\rangle\langle v_1| + e^{-\frac{i}{\hbar}\lambda_2 t}|v_2\rangle\langle v_2|.$$

Now we calulate $\psi_a(t)$ as

$$\begin{aligned}\psi_a(t) &= e^{-\frac{i}{\hbar}\lambda_1 t}\langle a|v_1\rangle\langle v_1|b\rangle + e^{-\frac{i}{\hbar}\lambda_2 t}\langle a|v_2\rangle\langle v_2|b\rangle \\ &= e^{i\frac{\Delta}{2}t}\,\Omega\left(e^{i\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}t}\,\frac{\frac{\Delta}{2}+\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}}{|N_1|^2} + e^{-i\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}t}\,\frac{\frac{\Delta}{2}-\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}}{|N_2|^2}\right).\end{aligned}$$

We define $W = \sqrt{\frac{\Delta^2}{4} + |\Omega|^2}$ and note that squaring the normalization constants yields $|N_1|^2 = 2W(\frac{\Delta}{2} + W)$ and $|N_2|^2 = -2W(\frac{\Delta}{2} - W)$. Thus we have

$$\begin{aligned}\psi_a(t) &= \Omega\left(e^{iWt}\frac{\frac{\Delta}{2}+W}{2W(\frac{\Delta}{2}+W)} + e^{-iWt}\frac{\frac{\Delta}{2}-W}{-2W(\frac{\Delta}{2}-W)}\right)e^{i\frac{\Delta}{2}t}\\ &= \frac{\Omega}{2W}\left(e^{iWt} - e^{-iWt}\right)e^{i\frac{\Delta}{2}t}\\ &= i\frac{\Omega}{W}\sin(Wt)\,e^{i\frac{\Delta}{2}t}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi_b(t) &= e^{-\frac{i}{\hbar}\lambda_1 t}|\langle b|v_1\rangle|^2 + e^{-\frac{i}{\hbar}\lambda_2 t}|\langle b|v_2\rangle|^2\\ &= \left(e^{iWt}\frac{|\Omega|^2}{|N_1|^2} + e^{-iWt}\frac{|\Omega|^2}{|N_2|^2}\right)e^{i\frac{\Delta}{2}t}.\end{aligned} \tag{G.10}$$

Noting that $(\frac{\Delta}{2} + W)(\frac{\Delta}{2} - W) = -|\Omega|^2$, we can rewrite $\frac{|\Omega|^2}{|N_1|^2}$ as

$$\begin{aligned}\frac{|\Omega|^2}{|N_1|^2} = |\Omega|^2\frac{1}{2W(\frac{\Delta}{2} + W)} &= -|\Omega|^2\frac{(\frac{\Delta}{2} - W)}{2W|\Omega|^2}\\ &= \frac{1}{2} - \frac{\Delta}{4W}\end{aligned}$$

and $\frac{|\Omega|^2}{|N_2|^2}$ as

$$\frac{|\Omega|^2}{|N_2|^2} = -|\Omega|^2\frac{(\frac{\Delta}{2} + W)}{2W|\Omega|^2} = \frac{1}{2} + \frac{\Delta}{4W}.$$

Inserting this into (G.10) above, we get

$$\begin{aligned}\psi_b(t) &= \left(\frac{1}{2}\left(e^{iWt} + e^{iWt}\right) - \frac{\Delta}{4W}\left(e^{iWt} - e^{-iWt}\right)\right)e^{i\frac{\Delta}{2}t}\\ &= \left(\cos(Wt) - i\frac{\Delta}{2W}\sin(Wt)\right)e^{i\frac{\Delta}{2}t}.\end{aligned}$$

Solution to Exercise 6.11. The atom is initially in the ground state

$$|\psi\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{G.11}$$

At resonance, the evolution matrix associated with a pulse of area of A is given by

$$\hat{U}_1 = \begin{pmatrix}\cos(A/2) & i\sin(A/2)\\ i\sin(A/2) & \cos(A/2)\end{pmatrix}, \tag{G.12}$$

so after a pulse area of $\frac{\pi}{2}$, the wavefunction of the atom is

$$\hat{U}_1|\psi\rangle = \begin{pmatrix}\cos(\frac{\pi}{4}) & i\sin(\frac{\pi}{4})\\ i\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4})\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}. \tag{G.13}$$

After this, the atom is left alone for a time t. In the absence of the laser field, the interaction Hamiltonian (6.16) becomes

$$\hat{V}_{\Omega=0} = \hbar\begin{pmatrix}-\Delta & 0\\ 0 & 0\end{pmatrix}, \tag{G.14}$$

and the evolution under this Hamiltonian is

$$\hat{U}_2 = e^{-i\hat{V}_{\Omega=0}t/\hbar} = \hbar\begin{pmatrix}e^{i\Delta t} & 0\\ 0 & 1\end{pmatrix}, \tag{G.15}$$

which leads to the state

$$\hat{U}_2\hat{U}_1 |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{i\Delta t} \\ 1 \end{pmatrix}. \tag{G.16}$$

Subsequently, another pulse of area $\frac{\pi}{2}$ is applied, producing the state

$$\hat{U}_1\hat{U}_2\hat{U}_1 |\psi\rangle = \begin{pmatrix} \cos(\frac{\pi}{4}) & i\sin(\frac{\pi}{4}) \\ i\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}\begin{pmatrix} ie^{i\Delta t} \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} ie^{i\Delta t} + i \\ 1 - e^{i\Delta t} \end{pmatrix} \tag{G.17}$$

So the final excited state population is $|\psi_a|^2 = \frac{1}{4}|ie^{i\Delta t} + i|^2 = \cos^2 \Delta t/2$.

Solution to Exercise 6.13. To find the eigenvalues of the Hamiltonian (6.16) we can write the characteristic equation as:

$$\det\left(\hbar\begin{pmatrix} -\Delta & -\Omega \\ -\Omega^* & 0 \end{pmatrix} - \lambda\hat{\mathbf{1}}\right) = 0$$
$$(\Delta + \lambda)\lambda - |\Omega|^2\hbar^2 = 0$$
$$\Rightarrow \lambda_{1,2} = \hbar\frac{-\Delta \pm \sqrt{\Delta^2 + 4|\Omega|^2}}{2}$$

The eigenstates corresponding to the above eigenvalues can be obtained by solving the eigenvalue equation

$$\hat{V}\left(\begin{array}{c} 1 \\ X \end{array}\right) = \lambda_{1,2}\left(\begin{array}{c} 1 \\ X \end{array}\right) \tag{G.18}$$

where the normalized excited state amplitude is

$$\psi_a = \frac{1}{\sqrt{1 + X^2}} \tag{G.19}$$

and for the ground state

$$\psi_b = \frac{X}{\sqrt{1 + X^2}}. \tag{G.20}$$

By solving this equation for both $\lambda_{1,2}$ we get

$$X_{1,2} = \frac{-\Delta \mp \sqrt{\Delta^2 + 4\Omega^2}}{2\Omega} \tag{G.21}$$

In case the value Δ of the detuning is much higher than the Rabi Frequency Ω, we can make the following simplifications in the expressions for energy eigenvalues:

$$\lambda_{1,2} \approx \hbar\left(\frac{-\Delta + (|\Delta| + 2\Omega^2/|\Delta|)}{2}\right)$$

where we used the fact that $\sqrt{1+\alpha} \approx 1 + \alpha/2$ for a small α. There are two cases that one must consider: large positive and large negative values of Δ.

In the case of *blue detuning (large positive* Δ*)*, eigenvalue λ_1 is close to zero: $\lambda_1 \approx \hbar\Omega^2/\Delta$. The corresponding value of X (The probability amplitude for the excited state) is given by:

$$\begin{aligned} X_1 &= \frac{-\Delta - \sqrt{\Delta^2 + 4\Omega^2}}{2\Omega} \\ &\approx \frac{-\Delta - (\Delta + 2\Omega^2/\Delta)}{2\Omega} \\ &\approx -\Delta/\Omega - \Omega/\Delta \end{aligned}$$

which has a large magnitude. Equation (G.20) implies in this case that the population is primarily in the ground state. The population in the excited state is given by:

$$\begin{aligned}
|\psi_a|^2 &= \frac{1}{1+X^2} \\
&\approx \frac{1}{X^2} \\
&\approx \frac{1}{(-\Delta/\Omega - \Omega/\Delta)^2} \\
&\approx \frac{\Omega^2}{\Delta^2}
\end{aligned}$$

For *red detuning (large negative* Δ*)*, the second eigenvalue is close to zero: $\lambda_2 \approx -\hbar\Omega^2/\Delta$. The corresponding value of X is given by

$$\begin{aligned}
X_2 &= \frac{+\Delta - \sqrt{\Delta^2 + 4\Omega^2}}{2\Omega} \\
&\approx \frac{+\Delta - |(\Delta + 2\Omega^2/\Delta)}{2\Omega}| \\
&\approx \frac{+\Delta + (\Delta + 2\Omega^2/\Delta)}{2\Omega} \\
&\approx \Delta/\Omega + \Omega/\Delta,
\end{aligned}$$

and, again, has a large magnitude. Thus, in this case too, the eigenvalue which is close to zero consists, primarily, of the ground state. The population in the excited state is $|\psi_a|^2 \approx \frac{\Omega^2}{\Delta^2}$.

Solution to Exercise 6.14. In the previous problem, we found the energy eigenvalues to be:

$$\hbar\lambda_{1,2} = \hbar\frac{-\Delta \pm \sqrt{\Delta^2 + 4\Omega^2}}{2}$$

We also found the eigenstates to be:

$$\begin{aligned}
|\lambda_{1,2}\rangle &= \frac{1}{\sqrt{1 + x_{1,2}}}\begin{pmatrix} 1 \\ x_{1,2} \end{pmatrix} \\
\text{where} \quad x_{1,2} &= \frac{1}{\Omega}\left(\frac{-\Delta \mp \sqrt{\Delta^2 + 4\Omega^2}}{2}\right)
\end{aligned}$$

For large negative detunings this can be approximated by:

$$\begin{aligned}
\hbar\lambda_1 &= \hbar(-\Delta - \Omega^2/\Delta) \\
\hbar\lambda_2 &= \hbar(\Omega^2/\Delta) \\
x_1 &= -\frac{|\Omega|}{|\Delta|} \\
x_2 &= \frac{|\Delta|}{|\Omega|}
\end{aligned}$$

a) Since $\hbar\lambda_2 = \hbar(\Omega^2/\Delta)$ is the lower energy eigenstate, the depth of the trap is given by:

$$\begin{aligned}
\frac{\Omega^2}{\Delta} &= \left(\frac{\vec{E}\cdot\vec{d}}{\hbar}\right)^2 \frac{1}{\Delta} \\
&= \frac{I}{2\epsilon_0 c}\frac{e^2 a_0^2}{\hbar^2}\frac{1}{\Delta} \\
&= \frac{P}{\pi r^2}\frac{e^2 a_0^2}{2\epsilon_0 c\hbar^2}\frac{1}{\Delta}
\end{aligned}$$

Converting to λ and λ_0 to an optical frequency we get $\Delta = -6.5 \times 10^{14}$ s^{-1}. We also know $e = 1.6 * 10^{-19}$C and $a_0 = 0.248 * 10^{-9}$m for Rubidium. We then get a numerical estimate of the trap depth:

$$\frac{\Omega^2}{\Delta} = 7 * 10^7 \text{ Hz}$$

b) The state is in the lower energy eigenstate:

$$|\lambda_2\rangle = \frac{1}{\sqrt{1+x_2}}\begin{pmatrix}1\\ x_2\end{pmatrix}$$

with $x_2 = |\Delta|/|\Omega|$. The population in the excited state $|a\rangle$ is given by:

$$|\psi_a|^2 = \frac{1}{1+x_2^2} \approx \frac{\Omega^2}{\Delta^2}$$

For a spontaneous decay rate $\Gamma = 3.6 * 10^7$ s^{-1} the photon scattering rate is given by:

$$\Gamma_{sc} = \Gamma|\psi_a|^2 \approx \Gamma\frac{\Omega^2}{\Delta^2} = 4 \text{ s}^{-1}$$

Solution to Exercise 6.16. For the density matrix

$$\hat{\rho} = \begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}, \tag{G.22}$$

the evolution under Hamiltonian (6.16) is given by

$$\begin{aligned}(\partial_t\hat{\rho})_{\rm int} &= -\frac{i}{\hbar}[V,\hat{\rho}]\\ &= -\frac{i}{\hbar}(H_I\hat{\rho}-\hat{\rho}H_I)\\ &= -\frac{i}{\hbar}.-\hbar(\begin{pmatrix}\Delta & \Omega\\ \Omega & 0\end{pmatrix}\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}-\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}\begin{pmatrix}\Delta & \Omega\\ \Omega & 0\end{pmatrix})\\ &= \begin{pmatrix} i\Omega(\rho_{ba}-\rho_{ab}) & i\Delta\rho_{ab}+i\Omega(\rho_{bb}-\rho_{aa})\\ -i\Delta\rho_{ba}+i\Omega(\rho_{aa}-\rho_{bb}) & i\Omega(\rho_{ab}-\rho_{ba})\end{pmatrix}.\end{aligned}$$

Solution to Exercise 6.17.

a) Let us rewrite Eq. (6.25) in the density operator form:

$$\begin{aligned}\hat{\rho}_{\rm atom,res}(\delta t) &= |\psi_a|^2[(1-\delta p)\,|a0\rangle\langle a0| + \delta p\,|b1\rangle\langle b1| + \sqrt{\delta p(1-\delta p)}(|a0\rangle\langle b1| + |b1\rangle\langle a0|)]\\ &+ \psi_a\psi_b^*[\sqrt{1-\delta p}\,|a0\rangle\langle b0| + \sqrt{\delta p}\,|b1\rangle\langle b0|]\\ &+ \psi_a^*\psi_b[\sqrt{1-\delta p}\,|b0\rangle\langle a0| + \sqrt{\delta p}\,|b0\rangle\langle b1|]\\ &+ |\psi_b|^2\,|b0\rangle\langle b0|,\end{aligned} \tag{G.23}$$

where in each ket and bra the first symbol corresponds to the atomic state and the second to the reservoir. Now let us take the partial trace over the reservoir to obtain the atomic density matrix:

$$\begin{aligned}\hat{\rho}(\delta t) &= |\psi_a|^2[(1-\delta p)\,|a\rangle\langle a| + \delta p\,|b\rangle\langle b|]\\ &+ \psi_a\psi_b^*[\sqrt{1-\delta p}\,|a\rangle\langle b|]\\ &+ \psi_a^*\psi_b[\sqrt{1-\delta p}\,|b\rangle\langle a|]\\ &+ |\psi_b|^2\,|b\rangle\langle b|,\end{aligned} \tag{G.24}$$

We can rewrite this in the matrix form:

$$\hat{\rho}(\delta t) = \begin{pmatrix} (1-\delta p)|\psi_a|^2 & \sqrt{1-\delta p}\psi_a\psi_b^* \\ \sqrt{1-\delta p}\psi_a^*\psi_b & \delta p|\psi_a|^2 + |\psi_b|^2 \end{pmatrix} \tag{G.25}$$

b) Replacing in Eq. (G.25) $|\psi_a|^2 \to \rho_{aa}$, $\psi_a\psi_b^* \to \rho_{ab}$, $\psi_a^*\psi_b \to \rho_{ba}$, $|\psi_b|^2 \to \rho_{bb}$ and approximating $\sqrt{1-\delta p} = (1-\delta p/2)$ for infinitesimal δp, we obtain Eq. (6.26).

c) If $\delta p = \Gamma\delta t$, then Eq. (6.26) takes the form

$$\hat{\rho}(\delta t) = \begin{pmatrix} (1-\Gamma\delta t)\rho_{aa} & (1-\Gamma\delta t/2)\rho_{ab} \\ (1-\Gamma\delta t/2)\rho_{ba} & \Gamma\delta t\rho_{aa} + \rho_{bb} \end{pmatrix}, \tag{G.26}$$

from which we have that

$$(\partial_t\hat{\rho})_{\text{spont}} = \frac{\rho(\delta t) - \rho(0)}{\delta t} = \begin{pmatrix} -\Gamma\rho_{aa} & -\Gamma/2\rho_{ab} \\ -\Gamma/2\rho_{ba} & \Gamma\rho_{aa} \end{pmatrix}. \tag{G.27}$$

Solution to Exercise 6.18. Plugging Eqs. (6.24) and (6.27) into Eq. (6.28) the steady state equation, we obtain the following equations:

$$\rho_{ab} = (\rho_{bb} - \rho_{aa})\frac{\Omega}{\Delta + i\Gamma/2} \tag{G.28}$$

$$\rho_{ba} = (\rho_{bb} - \rho_{aa})\frac{\Omega}{\Delta - i\Gamma/2} \tag{G.29}$$

$$\rho_{bb} - \rho_{aa} = 2i\Omega(\rho_{ab} - \rho_{ba}) + 2\Omega\rho_{aa} \tag{G.30}$$

Substituting the first two of the above equations into the third one, we obtain the value of $\rho_{bb} - \rho_{aa}$:

$$\rho_{bb} - \rho_{aa} = \frac{\Gamma}{\Gamma + \frac{2\Omega^2\Gamma}{\Gamma^2/4+\Delta^2}} = \frac{\Gamma^2/4 + \Delta^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \tag{G.31}$$

Now, we use obtain the value of $\rho_a a$ using the above equation and the fact that that trace of a density matrix is one.

$$\rho_{aa} = \frac{1}{2}((\rho_{aa} + \rho_{bb}) - (\rho_{aa} - \rho_{bb})) \tag{G.32}$$

$$= \frac{1}{2}(1 - (\rho_{aa} - \rho_{bb})) \tag{G.33}$$

$$= \frac{\Omega^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \tag{G.34}$$

The value of ρ_{bb} is obtained as

$$\rho_{bb} = (\rho_{bb} - \rho_{aa}) + \rho_{aa} \tag{G.35}$$

$$= \frac{\Gamma^2/4 + \Delta^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} + \frac{\Omega^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \tag{G.36}$$

$$= \frac{\Gamma^2/4 + \Delta^2 + \Omega^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \tag{G.37}$$

The values of ρ_{ab} and ρ_{ba} are solved by using the value of $\rho_{bb} - \rho_{aa}$ and substituting them into their expressions above:

$$\rho_{ab} = \frac{-\Omega(\Delta - i\Gamma/2)}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \tag{G.38}$$

$$\rho_{ba} = \frac{-\Omega(\Delta + i\Gamma/2)}{\Gamma^2/4 + \Delta^2 + 2\Omega^2}. \tag{G.39}$$

Solution to Exercise 6.21. The dipole moment in Schrödinger's picture is

$$d_S = \begin{pmatrix} 0 & \vec{d} \\ \vec{d}^* & 0 \end{pmatrix} \tag{G.40}$$

Since we are working in the interaction picture, the evolution of the dipole moment must be taken in to account:

$$\begin{aligned} d_I &= \underbrace{\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & 1 \end{pmatrix}}_{e^{iH_0t/\hbar}} \begin{pmatrix} 0 & \vec{d} \\ \vec{d}^* & 0 \end{pmatrix} \underbrace{\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & 1 \end{pmatrix}}_{e^{-iH_0t/\hbar}} \\ &= \begin{pmatrix} 0 & \vec{d}e^{i\omega t} \\ \vec{d}^* e^{-i\omega t} & 0 \end{pmatrix} \end{aligned}$$

The expectation value of the dipole moment is therefore

$$\begin{aligned} \langle \vec{d} \rangle &= \mathrm{Tr}(\hat{\rho}\hat{d}_I) \\ &= \mathrm{Tr}\left[\begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb}) \end{pmatrix} \begin{pmatrix} 0 & \vec{d}e^{i\omega t} \\ \vec{d}^* e^{-i\omega t} & 0) \end{pmatrix}\right] \\ &= \rho_{ab}\vec{d}^* e^{-i\omega t} + cc \end{aligned}$$

Further, if N is the density of atoms in the gas then its polarization is equal to

$$\begin{aligned} \vec{P}(t) &= N\langle d \rangle \\ &= N\rho_{ab} d^* e^{-i\omega t} + c.c. \end{aligned}$$

On the other hand, the polarization is related to the field amplidude according to

$$\begin{aligned} \vec{P} &= \epsilon_0 \chi \vec{E}(t) \\ &= \epsilon_0 \chi \vec{E}_0(t) + c.c. \end{aligned}$$

Comparing the above two equations gives us

$$\chi = \frac{N\rho_{ab} d^*}{\epsilon_0 E_0}.$$

Substituting the value of $\rho_{ab} = \frac{i\Omega}{\Gamma/2 + i\Delta}$ from Eq. (6.30) and using $\Omega = E_0 d/\hbar$, we get

$$\begin{aligned} \chi &= \frac{N\Omega d^*}{\epsilon_0 E_0} \frac{i}{\Gamma/2 + i\Delta} \\ &= \frac{N|d|^2}{\hbar\epsilon_0} \frac{i}{\Gamma/2 + i\Delta} \end{aligned}$$

Solution to Exercise 6.22. We start with the classical dispersion relation:

$$\chi_d = -\frac{Ne^2}{\epsilon_0 m\omega_0} \frac{1}{2\Delta + i\Gamma}.$$

From the Rydberg energy $Ry = \frac{\hbar^2}{2ma_0^2}$ we estimate $\omega_0 = \frac{\hbar}{2ma_0^2}$. If we insert that into the dispersion relation we get

$$\chi_d = -\frac{N}{\epsilon_0\hbar} 2a_0^2 e^2 \frac{1}{2\Delta + i\Gamma}$$

Now we recall that a good approximation for the dipole moment is $< d >= ea_0$. So we can write χ exactly like Eq. (6.32).

$$\chi_d = \frac{N}{\epsilon_0 \hbar} d^2 \frac{i}{-i\Delta + \frac{\Gamma}{2}}$$

This precise coincidence is, of course, fortuitous. Yet it is impressive how the classical and quantum treatments, while based on completely different physics, result in identical expressions.