Phys 673

Final examination

Solutions

$\boxed{1}$ a) Laser emits a coherent state with $\alpha = \sqrt{n_0} = 2$

$$|\alpha\rangle = e^{-\alpha^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\hat{\rho} = e^{-\alpha^2/2} \sum_n \frac{\alpha^n \alpha^m}{\sqrt{n! m!}} |n\rangle\langle m|$$

Phase drifts $\rightarrow$ off-diagonal elements lost

$$\hat{\rho} = e^{-\alpha^2/2} \sum_n \frac{\alpha^{2n}}{n!} |n\rangle\langle n|$$

b) $\alpha \to \alpha' = \sqrt{\eta n_0}$

$$\hat{\rho} = e^{-\alpha'^2/2} \sum_n \frac{\alpha'^{2n}}{n!} |n\rangle\langle n|$$

c) For the coherent state

$$W = \frac{1}{\pi} e^{-(x-x_0)^2 - (p-p_0)^2} \quad \text{where} \quad x_0 = \alpha\sqrt{2}\cos\Theta, \quad p_0 = \alpha\sqrt{2}\sin\Theta$$

In polar coordinats $x = r\cos\varphi$, $p = r\sin\varphi$

$$W = \frac{1}{\pi} e^{-r^2 - 2\alpha^2 + 2\sqrt{2}\alpha r\cos\Theta\cos\varphi + 2\sqrt{2}\alpha\sin\Theta\sin\varphi}$$

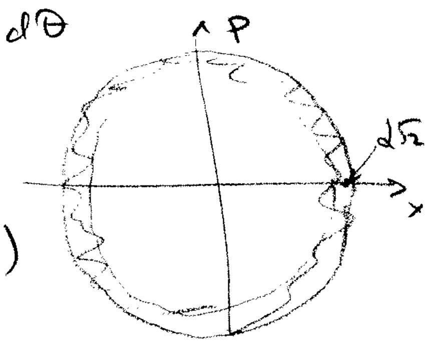
$$= \frac{1}{\pi} e^{-r^2 - 2\alpha^2 + 2\sqrt{2}\alpha r\cos(\Theta - \varphi)}$$

Randomize phase Θ

$$W = \frac{1}{\pi}\frac{1}{2\pi}\int_0^{2\pi} e^{-r^2 - 2\alpha^2 + 2\sqrt{2}\alpha r\cos(\Theta - \varphi)} d\Theta$$

$$= \frac{1}{\pi} e^{-r^2 - 2\alpha^2} I_0(2\sqrt{2}\alpha r)$$

(note error in the exam sheet!)



$\boxed{2}$

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 P}{\partial t^2}$$

$$P = \varepsilon_0 \chi^{(1)} E + \varepsilon_0 \chi^{(3)} E^3$$

$$E = E_0\, e^{ikz - i\omega t} + c.c.$$

Components of P oscillating at frequency ω

$$P = \left[\varepsilon_0 \chi^{(1)} E_0 + \varepsilon_0 \chi^{(3)} E_0^2 E_0^*\right] e^{ikz - i\omega t} + c.c.$$

Intensity $\quad I = 2\varepsilon_0 c E_0 E_0^*$

$$P = \varepsilon_0 \left(\chi^{(1)} + \frac{\chi^{(3)} I}{2\varepsilon_0 c}\right) E_0 e^{ikz - i\omega t} + c.c.$$

Wave equation becomes

$$-k^2 E_0 + \frac{\omega^2}{c^2} E_0 = -\frac{\omega^2}{\varepsilon_0 c^2}\varepsilon_0 \left(\chi^{(1)} + \frac{\chi^{(3)} I}{2\varepsilon_0 c}\right) E_0$$

$$k = \frac{\omega}{c}\sqrt{1 + \chi^{(1)} + \frac{\chi^{(3)} I}{2\varepsilon_0 c}}$$

$$n = \sqrt{n_0^2 + \frac{\chi^{(3)} I}{2\varepsilon_0 c}}$$

$\boxed{3}$

From class: $\quad \rho_{ba} = \dfrac{-\Omega\,(\Delta - i\Gamma/2)}{2\Omega^2 + \Delta^2 + \Gamma^2/4}$

$$P = \overbrace{\rho_{ba}\, d\, N}^{P_0}\, e^{ikz - i\omega t} + c.c.$$

$$\Omega = \frac{E_0 d}{\hbar}$$

$$P_0 = -\frac{d}{\hbar}\,\overbrace{\frac{\Delta - i\Gamma/2}{\Delta^2 + \Gamma^2/4}}^{=\frac{1}{\Delta + i\Gamma/2}}\,\frac{E_0}{2\frac{d^2}{\hbar^2}\frac{E_0^2}{\Delta^2+\Gamma^2/4} + 1}$$

Taylor expansion with respect to E_0

$$P_0 = \overbrace{-\frac{Nd^2}{\hbar}\frac{1}{\Delta + i\Gamma/2}}^{\varepsilon_0 \chi^{(1)}} E_0 + \overbrace{2\frac{Nd^4}{\hbar^3}\frac{1}{\Delta + i\Gamma/2}\frac{1}{\Delta^2 + \Gamma^2/4}}^{\varepsilon_0 \chi^{(3)}} E_0^3 + O(E_0^5)$$

$\boxed{4}$ a) Intensity $I = 2\varepsilon_0 c E_0^2$ $[J/cm^2 \cdot s]$

$$\Phi = \frac{I}{\hbar\omega} \quad [1/cm^2 \cdot s]$$

b) From class: (on resonance)
$$\rho_{ee} = \frac{\Omega^2}{2\Omega^2 + \Gamma^2/4} \approx 4\,\Omega^2/\Gamma^2 = 4\,\frac{E_0^2 d^2}{\hbar^2 \Gamma^2}$$

$$R = \Gamma N \rho_{ee} \quad [1/cm^3\, s]$$

c) $$\alpha = R/\Phi = 4\Gamma N \frac{E_0^2 d^2}{\hbar^2 \Gamma^2} \frac{\hbar\omega}{2\varepsilon_0 c E_0^2} = 2N \frac{d^2 \omega}{\varepsilon_0 c \hbar \Gamma}$$

$\boxed{5}$ a) According to the Wiener-Khinchin theorem

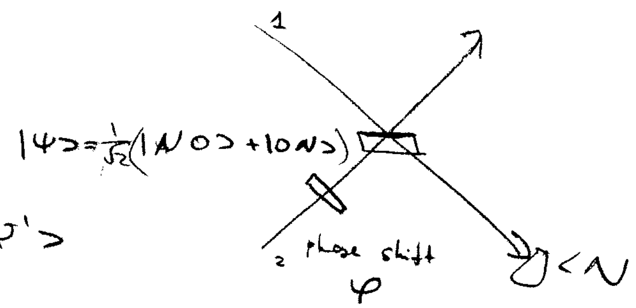
$$\Gamma(t,t') = \int_{-\infty}^{\infty} T(\omega)\, e^{-i\omega(t-t')}\, d\omega =$$

$$= e^{-2\Delta|t-t'|}\, e^{-i\omega_0(t-t')}$$

$$g^{(1)} = \frac{\Gamma(t\ t')}{\Gamma(t,t)} = e^{-2\Delta|t-t'|}$$

6. Effect of phase shifter

$$|N\rangle \to e^{i\varphi N}|N\rangle$$



$$\frac{1}{\sqrt{2}}(|N\,0\rangle + |0\,N\rangle) \to$$

$$\to \frac{1}{\sqrt{2}}\left(|N\,0\rangle + e^{i\varphi N}|0\,N\rangle\right) \equiv |\psi'\rangle$$

Effect of the beam splitter

$$e^{-iHt}|N\,0\rangle = \sum_k \binom{N}{k}^{1/2}\left(\frac{1}{\sqrt{2}}\right)^N |k,\,N-k\rangle$$

$$e^{-iHt}|0\,N\rangle = \sum_k \binom{N}{k}^{1/2}\left(\frac{1}{\sqrt{2}}\right)^N (-1)^{N-k}|k,\,N-k\rangle$$

Therefore

$$\langle N,0|\,e^{-iHt}|\psi'\rangle = \left(\frac{1}{\sqrt{2}}\right)^{N+1}\left(1+e^{iN\varphi}\right)$$

Probability of detecting N photons

$$pr(|N,0\rangle) = |\langle N,0|\,e^{-iHt}|\psi'\rangle|^2 = \frac{1}{2^{N-1}}\cos^2 N\varphi$$

Interference fringes are N times more frequent as compared to classical interference.