University of Calgary
Fall semester 2008

PHYS 673: Quantum and Nonlinear Optics

Take-home final examination

Due December 15, 2008 at 18:00 in SB 319. Computers are not allowed.

Total points: 100. You must solve all problems in order to receive full credit. Partial credit will be given except for the extra credit question.

Problem 1 (20 points). The average number of photons per pulse in the output of a pulsed laser is n_0. The laser's frequency is constant, but the phase is drifting randomly from pulse to pulse.

a) Find the density matrix of the state of light generated by the laser in the photon number basis. Assume that the state is measured using homodyne tomography with an ideal detector and an independent, phase stabilized local oscillator.

b) How will the answer change if the detector has a quantum efficiency η?

c) Write the analytic expression and and sketch the plot of the Wigner function of the state found in part (a). **Hint:** it is more convenient to use polar coordinates in the phase space.

Problem 2 (20 points). Consider a plane electromagnetic wave propagating through an isotropic medium with the linear index of refraction n_0 and the third-order effective nonlinear susceptibility $\chi^{(3)}$.

Write the wave equation for propagation inside this medium and show that the nonlinear behavior can be interpreted as a dependence of the index of refraction on the wave intensity. Write this dependence explicitly.

Problem 3 (15 points) Calculate $\chi^{(3)}$ for a gas of two-level atoms with a known set of parameters (number density N, transition dipole moment d, transition frequency ω_0, detuning Δ)

Problem 4 (15 points). Consider a cell of length L containing a gas of two-level atoms with the number density N, optical transition frequency ω and spontaneous emission rate Γ. A resonant electromagnetic field of amplitude E_0 is propagating through the gas.

a) Calculate the flux Φ of photons through the gas (number of photons per unit time, unit area).

b) From the solution of the master equations in the steady state, find the population density of the excited level. Using this result, determine the rate R of spontaneously emitted photons (per unit time, unit volume).

c) By comparing R with Φ and using the conservation of energy, find the absorption index α. Compare it with the expression found in class.

The following assumptions can be used.

- Doppler broadening can be neglected.
- The field is weak ($\Omega \ll \Gamma$).
- $\alpha L \ll 1$ so the light intensity can be assumed constant over the propagation length.

Problem 5 (10 points). White incoherent light of constant intensity propagates through a narrow Lorentzian-shaped bandpass filter with transmission function

$$T(\omega) = \frac{1}{1 + \frac{(\omega-\omega_0)^2}{\Delta^2}}.$$

Calculate the autocorrelation function $\Gamma(t,t') = \langle E(t)E(t')\rangle$ (neglect the overall coefficient) and the mutual coherence function

$$g^{(1)}(t,t') = \frac{|\Gamma(t,t')|}{\sqrt{\langle |E(t)|^2\rangle \langle |E(t')|^2\rangle}}$$

for the light after the filter.

How will the answer change if the input light is a flash at time $t = 0$ of duration $\tau \ll 1/\Delta$ (extra credit — 50 pts)?

Problem 6 (20 points). The "NOON" state $|\Psi\rangle = 1/\sqrt{2}(|N,0\rangle + |0,N\rangle)$ (the notation is in the photon number basis) enters a symmetric beam splitter. Before the beam splitter, the second input channel is subjected to retardation by optical phase φ. Find the probability of detecting N photons in the first beam splitter output channel as a function of φ.

Some useful integrals:

$$\int_0^{2\pi} e^{a\cos\phi}\mathrm{d}\phi = 2\pi J_0(a)$$

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^2}\mathrm{d}x = \pi e^{-|k|}$$