Phys 615 Fall 2014

Midterm examination

Solutions

$\boxed{1}$

$$[(\sigma_x+\sigma_y)^2\ \ \sigma_z] = (\sigma_x+\sigma_y)[\sigma_x+\sigma_y\ \ \sigma_z] + [\sigma_x+\sigma_y, \sigma_z](\sigma_x+\sigma_y)$$

$$= (\sigma_x+\sigma_y)(-2i\sigma_y+2i\sigma_x) + (-2i\sigma_y+2i\sigma_x)(\sigma_x+\sigma_y)$$

$$= 4i\sigma_x^2 - 4i\sigma_y^2 = 0+0=0$$

$\boxed{2}$

$$\det(\rho - \lambda \mathbb{1}) = 0$$

$$(1/2-\lambda)^2 - 1/36 = 0$$

$$\lambda^2 - \lambda + 2/9 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-8/9}}{2} = \begin{cases} 2/3 \\ 1/3 \end{cases}$$

1) $(\rho - 2/3\,\mathbb{1})\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\begin{pmatrix} -1/6 & i/6 \\ -i/6 & -1/6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = |L\rangle$$

2) $(\rho - 1/3\,\mathbb{1})\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |R\rangle$$

$$\rho = \frac{1}{3}|R\rangle\langle R| + \frac{2}{3}|L\rangle\langle L|$$

$\boxed{3}$ $|\uparrow\rangle = \dfrac{|+\rangle + |-\rangle}{\sqrt{2}}$

In diagonal basis:

$$\rho_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-\gamma t} \\ e^{-\gamma t} & 1 \end{pmatrix}$$

In canonical basis

$$\rho(t) = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & e^{-\gamma t} \\ e^{-\gamma t} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1+e^{-\gamma t} & 1-e^{-\gamma t} \\ 1+e^{-\gamma t} & -1+e^{-\gamma t} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2+2e^{-\gamma t} & 0 \\ 0 & 2-2e^{-\gamma t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}e^{-\gamma t} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}e^{-\gamma t} \end{pmatrix}$$

$\boxed{4}$

$$\Pi_R = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\Pi_L = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$F_R = \eta\, \Pi_R = \frac{\eta}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$F_L = \eta\, \Pi_L = \frac{\eta}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$F_0 = (1-\eta)\Pi_R + (1-\eta)\Pi_L = (1-\eta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

$$pr_R = Tr\left(F_R\, \rho\right) = \frac{\eta}{2} Tr\begin{pmatrix} |\alpha|^2 - i\alpha^*\beta & \\ & i\alpha\beta^* + |\beta|^2 \end{pmatrix} = \frac{\eta}{2}|\alpha - i\beta|^2$$

$$pr_L = \frac{\eta}{2}|\alpha + i\beta|^2$$

$$pr_0 = 1 - pr_R - pr_L = 1-\eta$$

$\boxed{5}$ a) $\langle +_A | \Psi \rangle = \frac{1}{\sqrt{6}} \left( \frac{1}{\sqrt{2}} |V\rangle + \frac{1}{\sqrt{2}} |H\rangle + \frac{2}{\sqrt{2}} |V\rangle \right) = \frac{1}{\sqrt{12}} \left( |H\rangle + 3|V\rangle \right)$

$\langle -_A | \Psi \rangle = \frac{1}{\sqrt{6}} \left( \frac{1}{\sqrt{2}} |V\rangle - \frac{1}{\sqrt{2}} |H\rangle - \frac{2}{\sqrt{2}} |V\rangle \right) = \frac{-1}{\sqrt{12}} \left( |H\rangle + |V\rangle \right)$

b) $pr_+ = \frac{10}{12} = \frac{5}{6}$

$pr_- = \frac{1}{6}$

c) $\frac{1}{12} \left( |H\rangle + 3|V\rangle \right)\left( \langle H| + 3\langle V| \right) + \frac{1}{12} \left( |H\rangle + |V\rangle \right)\left( \langle H| + \langle V| \right)$

$$= \frac{1}{12} \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} + \frac{1}{12} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$