University of Calgary
Fall semester 2014

PHYS 615: Advanced Quantum Mechanics I

Homework assignment 5

Due Wednesday November 26, 2014

For all problems, it is allowed to use tables of Clebsch-Gordan coefficients (*ClebschGordan* function in *Mathematica*) as well as the Wigner rotation matrix for $j = 1$ from the textbook (Sec. 3.5).

Problem 5.1. Calculate $d^j_{m'm}(\beta) = \left\langle jm' \middle| e^{-i\beta\hat{J}_y/\hbar} \middle| jm \right\rangle$ for $j = 1/2$ using the Wigner formula and compare your result with the equation $e^{i\theta\vec{v}\hat{\vec{\sigma}}} = \cos\theta\,\hat{\mathbf{1}} + i\sin\theta\,\vec{v}\,\hat{\vec{\sigma}}$ derived in class.

Problem 5.2. Alice has a spin-1 particle and Bob has a spin-1/2 particle. The added angular momentum of the two particles is in state $|j = 1/2, m = 1/2\rangle$.

a) What is the state of Bob's particle if Alice's particle is lost?

b) The field gradient in Alice's Stern-Gerlach apparatus is along vector $\vec{n}$ that is at 45° between the positive x and y axes. Alice performs a measurement and observes that the projection of her particle's spin onto $\vec{n}$ is $+\hbar$. What state does this measurement prepare at Bob's location? What is the probability of this event?

The answers should be given in the eigenbasis of the angular momentum operator of Bob's particle.

Problem 5.3. A spin-1 particle is in initial state $|m_1 = 1\rangle$ and a spin-1/2 particle is in initial state $|m_2 = -1/2\rangle$. Interaction Hamiltonian $\hat{H} = \gamma\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$ is turned on at $t = 0$. Find the state of the two particles in the $\{|m_1, m_2\rangle\}$ basis as a function of time.

Problem 5.4. Angular momenta are added for three spin-1/2 particles with magnetic quantum numbers $m_1 = m_2 = 1/2$ and $m_3 = -1/2$. This problem can be solved in two ways. If we first add the momenta of particles 1 and 2, we obtain the state $|j = 1, m = 1\rangle$. Now adding the third particle and using the appropriate Clebsch-Gordan coefficients, we obtain the state

$$\sqrt{\frac{1}{3}}\left|j = \frac{3}{2}, m = \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle.$$

On the other hand, if we first add the momenta of particles 1 and 3, we obtain the state $(|j = 1, m = 0\rangle + |j = 0, m = 0\rangle)/\sqrt{2}$, and adding particle 2 to this result, we have

$$\begin{aligned} &\sqrt{\frac{1}{2}}\left(\sqrt{\frac{2}{3}}\left|j = \frac{3}{2}, m = \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle\right) + \sqrt{\frac{1}{2}}\left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle \\ =&\sqrt{\frac{1}{3}}\left|j = \frac{3}{2}, m = \frac{1}{2}\right\rangle + \left(\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{6}}\right)\left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle. \end{aligned}$$

Explain why these approaches produce different answers.

Problem 5.5. We found in class that the infinitesimal rotation by angle ϵ around the z axis acting on a position eigenstate transforms it as follows:

$$e^{-i\epsilon\hat{L}_z/\hbar}\left|x, y, z\right\rangle \approx \left|x - \epsilon y, y + \epsilon x, z\right\rangle ,$$

which in the spherical coordinates corresponds to

$$e^{-i\epsilon\hat{L}_z/\hbar}\left|r, \theta, \phi\right\rangle \approx \left|r, \theta, \phi + \epsilon\right\rangle ,$$

from which we found that the action of operator $\hat{L}_z$ on the wavefunction $\psi(r, \theta, \phi) = \langle r, \theta, \phi|\psi\rangle$ of an arbitrary state $|\psi\rangle$ is given by

$$\left\langle r, \theta, \phi\right| \hat{L}_z \left| \psi\right\rangle = -i\hbar\frac{\partial}{\partial\phi}\left\langle r, \theta, \phi|\psi\right\rangle .$$

Conduct a similar argument for $\hat{L}_x$ and $\hat{L}_y$ to show that

$$\left\langle r, \theta, \phi\right| \hat{L}_x \left| \psi\right\rangle = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\left\langle r, \theta, \phi|\psi\right\rangle$$

and

$$\left\langle r, \theta, \phi\right| \hat{L}_y \left| \psi\right\rangle = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\left\langle r, \theta, \phi|\psi\right\rangle .$$

Then show that

$$\left\langle r, \theta, \phi\right| \hat{L}^2 \left| \psi\right\rangle = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right]\left\langle r, \theta, \phi|\psi\right\rangle .$$