Phys 615 (Fall 2014)

Final Exam

Solutions

$\boxed{1}$

$$S_z = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix}\hbar$$

$$S_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix} \qquad S_- = \hbar\begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2}i\begin{pmatrix} 0 & -\sqrt{3} & & \\ \sqrt{3} & 0 & -2 & \\ & 2 & 0 & -\sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$\boxed{2}$ a) $\langle\psi|\psi\rangle = N^2\left(\langle\alpha|\alpha\rangle + \langle\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle\right) = 1$

$$\langle\alpha|\alpha\rangle = \langle-\alpha|-\alpha\rangle = 1$$

$$\langle\alpha|-\alpha\rangle = \langle-\alpha|\alpha\rangle = e^{-2\alpha^2}$$

$$N = \left[2\left(1+e^{-2\alpha^2}\right)\right]^{-1/2}$$

b) $$\langle X\rangle = \langle\psi|\frac{a+a^+}{\sqrt{2}}|\psi\rangle = \frac{N^2}{\sqrt{2}}\left(\langle\alpha| + \langle-\alpha|\right)\left(a+a^+\right)\left(|\alpha\rangle + |-\alpha\rangle\right)$$

$$= \frac{N^2}{\sqrt{2}}\left[\left(\langle\alpha| + \langle-\alpha|\right)\left(\alpha|\alpha\rangle - \alpha|-\alpha\rangle\right) + \left(\alpha\langle\alpha| - \alpha\langle-\alpha|\right)\left(|\alpha\rangle + |-\alpha\rangle\right)\right]$$

$$= \frac{N^2}{\sqrt{2}}\left[2\alpha^2\langle\alpha|\alpha\rangle - 2\alpha^2\langle-\alpha|-\alpha\rangle\right] = 0$$

$$\langle X^2\rangle = \langle\psi|\frac{a^2 + a^{+2} + 2a^+a + 1}{2}|\psi\rangle$$

$$= N^2\left[\left(\langle\alpha| + \langle-\alpha|\right)\frac{a^2 + a^{+2} + 2a^+a}{2}\left(|\alpha\rangle + |-\alpha\rangle\right)\right] + \frac{1}{2}$$

$$= \frac{N^2}{2}2\alpha^2\left(\langle\alpha| + \langle-\alpha|\right)\left(|\alpha\rangle + |-\alpha\rangle\right) + \frac{N^2}{2}2\alpha^2\left(\langle\alpha| - \langle-\alpha|\right)\left(|\alpha\rangle - |-\alpha\rangle\right) + \frac{1}{2}$$

$$= \alpha^2 + \alpha^2\frac{1-e^{-2\alpha^2}}{1+e^{-2\alpha^2}} + \frac{1}{2} = \frac{2\alpha^2}{1+e^{-2\alpha^2}} + \frac{1}{2} = \frac{4\alpha^2 + 1 + e^{-2\alpha^2}}{2\left(1+e^{-2\alpha^2}\right)}$$

$$\langle P \rangle = \langle \Psi | \frac{a - a^+}{\sqrt{2} i} | \Psi \rangle = 0$$

$$\langle P^2 \rangle = \langle \Psi | \frac{a^2 + a^{+2} - 2a^+ a - 1}{-2} | \Psi \rangle$$

$$= \mathcal{N}^2 \left[(\langle \alpha | + \langle -\alpha |) \frac{-a^2 - a^{+2} + 2a^+ a}{2} (|\alpha\rangle + |-\alpha\rangle) \right] + \frac{1}{2}$$

$$= -\frac{\mathcal{N}^2}{2} 2\alpha^2 (\langle \alpha | + \langle -\alpha |)(|\alpha\rangle + |-\alpha\rangle) + \frac{\mathcal{N}^2}{2} 2\alpha^2 (\langle \alpha | - \langle -\alpha |)(|\alpha\rangle - |-\alpha\rangle) + \frac{1}{2}$$

$$= -\alpha^2 + \alpha^2 \frac{1 - e^{-2\alpha^2}}{1 + e^{-2\alpha^2}} + \frac{1}{2} = \frac{-2e^{-2\alpha^2}}{1 + e^{-2\alpha^2}} \alpha^2 + \frac{1}{2} = \frac{1 + e^{-2\alpha^2} - 4\alpha^2 e^{-2\alpha^2}}{2(1 + e^{-2\alpha^2})}$$

c) $\langle E \rangle = \hbar\omega \left(\frac{1}{2} + \langle \psi | a^+ a | \psi \rangle \right)$

$$= \hbar\omega \left[\frac{1}{2} + \left(\langle \alpha | + \langle -\alpha | \right) a^+ a \left(|\alpha\rangle + |-\alpha\rangle \right) N^2 \right]$$

$$= \hbar\omega \left[\frac{1}{2} + \alpha^2 \left(\langle \alpha | - \langle -\alpha | \right) \left(|\alpha\rangle - |-\alpha\rangle \right) N^2 \right]$$

$$= \hbar\omega \left[\frac{1}{2} + 2\alpha^2 \left(1 - e^{-2\alpha^2} \right) N^2 \right]$$

$$= \hbar\omega \left[\frac{1}{2} + \alpha^2 \tanh \alpha^2 \right]$$

$$\langle E^2 \rangle = \hbar^2\omega^2 \left(\frac{1}{4} + \langle a^+ a \rangle + \langle a^+ a a^+ a \rangle \right)$$

$$= \hbar^2\omega^2 \left(\frac{1}{4} + \langle a^+ a \rangle + \langle a^+ (a^+ a + 1) a \rangle \right)$$

$$= \hbar^2\omega^2 \left(\frac{1}{4} + 2\langle a^+ a \rangle + \langle a^{+2} a^2 \rangle \right)$$

$$= \hbar^2\omega^2 \left(\frac{1}{4} + 2\alpha^2 \tanh \alpha^2 + \alpha^4 \left(\langle \alpha | + \langle -\alpha | \right) \left(|\alpha\rangle + |-\alpha\rangle \right) N^2 \right)$$

$$= \hbar^2\omega^2 \left(\frac{1}{4} + 2\alpha^2 \tanh \alpha^2 + \alpha^4 \right)$$

d) $\psi(x) = N \left(\psi_0 (x - \alpha\sqrt{2}) + \psi_0 (x + \alpha\sqrt{2}) \right)$, with $\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$

$$\rho(x, x') = N^2 \left(\psi_0 (x - \alpha\sqrt{2}) \psi_0 (x' - \alpha\sqrt{2}) + \psi_0 (x - \alpha\sqrt{2}) \psi_0 (x' + \alpha\sqrt{2}) \right.$$

$$\left. + \psi_0 (x + \alpha\sqrt{2}) \psi_0 (x' - \alpha\sqrt{2}) + \psi_0 (x + \alpha\sqrt{2}) \psi_0 (x' + \alpha\sqrt{2}) \right)$$

$$= \frac{1}{2(1 + e^{-2\alpha^2})} \left[e^{-[(x - \alpha\sqrt{2})^2 - (x' - \alpha\sqrt{2})^2]/2} + e^{-[(x - \alpha\sqrt{2})^2 - (x' + \alpha\sqrt{2})^2]/2} \right.$$

$$\left. + e^{-[(x + \alpha\sqrt{2})^2 - (x' - \alpha\sqrt{2})^2]/2} + e^{-[(x + \alpha\sqrt{2})^2 - (x' + \alpha\sqrt{2})^2]/2} \right.$$

$\boxed{5}$

$$\alpha|H\rangle + \beta|V\rangle = \frac{\alpha+\beta}{\sqrt{2}} \frac{|H\rangle + |V\rangle}{\sqrt{2}} + \frac{\alpha-\beta}{\sqrt{2}} \frac{|H\rangle - |V\rangle}{\sqrt{2}}$$

$$= \frac{\alpha+\beta}{\sqrt{2}} |+\rangle + \frac{\alpha-\beta}{\sqrt{2}} |-\rangle$$

a) $$\frac{\alpha+\beta}{\sqrt{2}} |+_S +_A\rangle + \frac{\alpha-\beta}{\sqrt{2}} |-_S -_A\rangle$$

b) $$\frac{|\alpha+\beta|^2}{2} |+\rangle\langle+| + \frac{|\alpha-\beta|^2}{2} |-\rangle\langle-|$$

$$= \frac{|\alpha+\beta|^2}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{|\alpha-\beta|^2}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha^2+\beta^2}{2} & \alpha\beta \\ \alpha\beta & \frac{\alpha^2+\beta^2}{2} \end{pmatrix} = \begin{pmatrix} 1/2 & \alpha\beta \\ \alpha\beta & 1/2 \end{pmatrix}$$

c) $$pr_+ = \frac{|\alpha+\beta|^2}{2}, \qquad pr_- = \frac{|\alpha-\beta|^2}{2}$$

$\boxed{4}$ a) $\left(|j=1, m=0\rangle + |j=1, m=1\rangle\right)/\sqrt{2}$

$$= \frac{1}{2}\left(|m_1=\tfrac12, m_2=-\tfrac12\rangle + |m_1=-\tfrac12, m_2=\tfrac12\rangle\right) + \frac{1}{\sqrt2}|m_1=\tfrac12, m_2=\tfrac12\rangle$$

b) $\sigma_x = |\tfrac12\rangle\langle -\tfrac12| + |-\tfrac12\rangle\langle \tfrac12|$; $\sigma_z = |\tfrac12\rangle\langle\tfrac12| - |-\tfrac12\rangle\langle-\tfrac12|$

$$\sigma_x \otimes \sigma_z |\Psi\rangle = \frac12\left(-|-\tfrac12, -\tfrac12\rangle + |\tfrac12, +\tfrac12\rangle\right) + \frac{1}{\sqrt2}|-\tfrac12, +\tfrac12\rangle$$

$$\langle\Psi|\sigma_x \otimes \sigma_z|\Psi\rangle = 1/2\sqrt2 + 1/2\sqrt2 = 1/\sqrt2$$

c) $|\Psi\rangle = |\tfrac12\rangle \otimes \left(\frac12|-\tfrac12\rangle + \frac{1}{\sqrt2}|\tfrac12\rangle\right) + |-\tfrac12\rangle \otimes \frac12|+\tfrac12\rangle$

$$Tr_A |\Psi\rangle\langle\Psi| = \left(\frac12|-\tfrac12\rangle + \frac{1}{\sqrt2}|\tfrac12\rangle\right)\left(\frac12\langle-\tfrac12| + \frac{1}{\sqrt2}\langle\tfrac12|\right) + \frac14|\tfrac12\rangle\langle\tfrac12|$$

$$= \begin{pmatrix} 1/2 & 1/2\sqrt2 \\ 1/2\sqrt2 & 1/4 \end{pmatrix} + \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/2\sqrt2 \\ 1/2\sqrt2 & 1/4 \end{pmatrix}$$

d) $|m_{\hat n} = \tfrac12\rangle = e^{-i\frac{S_y}{\hbar}\theta}|m_z=\tfrac12\rangle$

$$= e^{-i\frac{\theta}{2}\sigma_y}|m_z=\tfrac12\rangle = \left(\cos\frac\theta2\,\hat{1} - i\sin\frac\theta2\,\sigma_y\right)|m_z=\tfrac12\rangle$$

$$= \cos\frac\theta2\,\hat{1} + \sin\frac\theta2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}|m_z=\tfrac12\rangle$$

$$= \cos\frac\theta2|m_z=\tfrac12\rangle + \sin\frac\theta2|m_z=-\tfrac12\rangle$$

$$|m_{\hat n}=-\tfrac12\rangle = \cos\frac\theta2|m_z=-\tfrac12\rangle - \sin\frac\theta2|m_z=\tfrac12\rangle$$

$$\langle m_{\hat n}=\tfrac12|\Psi\rangle = \cos\frac\theta2\left(\frac12|-\tfrac12\rangle + \frac{1}{\sqrt2}|\tfrac12\rangle\right) + \sin\frac\theta2\,\frac12|\tfrac12\rangle$$

$$= \frac12\cos\frac\theta2|-\tfrac12\rangle + \left(\frac{1}{\sqrt2}\cos\frac\theta2 + \frac12\sin\frac\theta2\right)|\tfrac12\rangle$$

$$pr_{m_{\hat n}=1/2} = \frac34\cos^2\frac\theta2 + \frac{1}{\sqrt2}\cos\frac\theta2\sin\frac\theta2 + \frac14\sin^2\frac\theta2$$

$$\langle m_{\hat n}=-\tfrac12|\Psi\rangle = \cos\frac\theta2\,\frac12|\tfrac12\rangle - \sin\frac\theta2\left(\frac12|-\tfrac12\rangle + \frac{1}{\sqrt2}|\tfrac12\rangle\right)$$

$$= \left(\frac12\cos\frac\theta2 - \frac{1}{\sqrt2}\sin\frac\theta2\right)|\tfrac12\rangle - \frac12\sin\frac\theta2|-\tfrac12\rangle$$

$$pr_{m_{\hat n}=-1/2} = \frac14\cos^2\frac\theta2 - \frac{1}{\sqrt2}\cos\frac\theta2\sin\frac\theta2 + \frac34\sin^2\frac\theta2$$

e) $|j=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|1/2, -1/2\rangle - |-1/2, 1/2\rangle\right)$

$$|\Psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega t}|1/2, -1/2\rangle + e^{i\omega t}|-1/2, 1/2\rangle\right) + \frac{1}{\sqrt{2}} e^{-i\omega t}|1/2, 1/2\rangle,$$

where $\omega = \frac{g}{2}\frac{\mu_B B}{\hbar}$

$$\langle j=0, m=0|\Psi(t)\rangle = \frac{1}{2\sqrt{2}}\left(e^{-i\omega t} - e^{i\omega t}\right) = \frac{-i}{\sqrt{2}} \sin \omega t$$

$$Pr_{|j=0, m=0\rangle} = \frac{1}{2}\sin^2 \omega t$$

$\boxed{5}$ a) $H = \alpha(x_1 p_2 + p_1 x_2)$

$$\begin{cases} \dot{x}_1 = i[H\ x_1] = i\alpha\, x_2 [p_1\ x_1] = \alpha x_2 \\ \dot{x}_2 = \alpha x_1 \end{cases}$$

$$\begin{cases} \dot{p}_1 = i[H\ p_1] = i\alpha\, p_2 [x_1\ p_1] = -\alpha p_2 \\ \dot{p}_2 = -\alpha p_1 \end{cases}$$

b)

$$\begin{cases} x_1 = Ae^{\alpha t} + Be^{-\alpha t} \\ x_2 = Ae^{\alpha t} - Be^{-\alpha t} \end{cases} \quad \begin{matrix} x_1(0) = A+B \\ x_2(0) = A-B \end{matrix} \quad \begin{matrix} A = \frac{x_1(0)+x_2(0)}{2} \\ B = \frac{x_1(0)-x_2(0)}{2} \end{matrix}$$

$$\begin{cases} p_1 = Ce^{\alpha t} + De^{-\alpha t} \\ p_2 = -Ce^{\alpha t} + De^{-\alpha t} \end{cases} \quad \begin{matrix} p_1(0) = C+D \\ p_2(0) = D-C \end{matrix} \quad \begin{matrix} C = \frac{p_1(0)-p_2(0)}{2} \\ D = \frac{p_1(0)+p_2(0)}{2} \end{matrix}$$

$$\begin{cases} x_1(t) = x_1(0)\cosh\alpha t + x_2(0)\sinh\alpha t \\ x_2(t) = x_2(0)\cosh\alpha t + x_1(0)\sinh\alpha t \end{cases}$$

$$\begin{cases} p_1(t) = p_1(0)\cosh\alpha t - p_2(0)\sinh\alpha t \\ p_2(t) = p_2(0)\cosh\alpha t - p_1(0)\sinh\alpha t \end{cases}$$

c) $[x_1(t)\ p_1(t)] = [x_1(0)\ p_1(0)]\cosh^2\alpha t + [x_2(0)\ p_2(0)]\sinh^2\alpha t$

$= i\cosh^2\alpha t - i\sinh^2\alpha t = i$

$[x_2(t)\ p_2(t)] = i$

$\langle\Delta x_1^2\rangle\langle\Delta p_1^2\rangle = 1/4$

$\langle\Delta x_2^2\rangle\langle\Delta p_2^2\rangle = 1/4$

d)

$$\begin{cases} x_1(t) + x_2(t) = (x_1(0)+x_2(0))e^{\alpha t} \\ x_1(t) - x_2(t) = (x_1(0)-x_2(0))e^{-\alpha t} \end{cases}$$

$$\begin{cases} p_1(t) + p_2(t) = (p_1(0)+p_2(0))e^{-\alpha t} \\ p_1(t) - p_2(t) = (p_1(0)-p_2(0))e^{\alpha t} \end{cases}$$

$\langle\Delta(x_1 \pm x_2)^2\rangle = \langle\Delta(x_1(0)\pm x_2(0))^2\rangle e^{\pm 2\alpha t} =$

$= [\langle\Delta x_1(0)^2\rangle + \langle\Delta x_2(0)^2\rangle]e^{\pm 2\alpha t} = e^{\pm 2\alpha t}$

$\langle\Delta(p_1 \pm p_2)^2\rangle = e^{\mp 2\alpha t}$