University of Calgary
Fall semester 2014

PHYS 615: Advanced Quantum Mechanics I

Final examination

December 18, 2014, 15:30, 3 hours, EEEL 349

Open books. No electronic equipment allowed.
Full credit = 100 points. Attempt all problems. Partial credit will be given.

Problem 1 (10 pts). Write the matrices of the observables associated with the x, y and z components of the spin for a spin-3/2 particle.

Problem 2 (25 pts). For a superposition of coherent states $|\psi\rangle = \mathcal{N}(|\alpha\rangle + |-\alpha\rangle)$, where α is real and positive and $\mathcal{N}$ is the normalization factor:

a) find $\mathcal{N}$;

b) find the mean and variance of the position and momentum observables;

c) find the mean and variance of the energy observable;

d) write the density operator in the position basis.

Problem 3 (15 pts). A von Neumann measurement of photon polarization state $|\psi\rangle = \alpha\,|H\rangle + \beta\,|V\rangle$ is performed in the diagonal basis.

a) Write the joint state of the system and the apparatus after the measurement in the measurement basis.

b) Write, in the canonical basis, the state of the system alone after the measurement.

c) What are the probabilities of the measurement outcomes?

The amplitudes α and β can be assumed real.

Problem 4 (25 pts). Two electrons shared between Alice and Bob are initially in state

$$|\Psi\rangle = (|j=1, m=0\rangle + |j=1, m=1\rangle)/\sqrt{2}.$$

a) Write the electron pair's state in the $|m_1, m_2\rangle$ basis.

b) What is the mean value of the observable $\hat{\sigma}_x \otimes \hat{\sigma}_z$ in this state?

c) What is the state of Bob's spin if Alice's electron is lost?

d) Alice performs a Stern-Gerlach measurement on her electron with the magnetic field gradient oriented along the vector $\hat{n}$ between the positive x and z semiaxes, at angle θ to the z axis. What are the probabilities of possible measurement outcomes and what state will be prepared at Bob's location in each case?

e) Instead of the above, at time $t = 0$, a magnetic field of magnitude B along the z axis is turned on that affects only Alice's electron. What is the probability to find the pair in the singlet state $|j=0, m=0\rangle$ at time t?

Problem 5 (25 pts). Two optical modes, initially in the vacuum state, interact under the Hamiltonian $\hat{H} = \alpha(\hat{X}_1\hat{P}_2 + \hat{P}_1\hat{X}_2)$, with a real positive α.

a) Write the differential equations for the evolution of observables $\hat{X}_{1,2}(t), \hat{P}_{1,2}(t)$ in the Heisenberg picture.

b) Solve these equations to express $\hat{X}_{1,2}(t), \hat{P}_{1,2}(t)$ through $\hat{X}_{1,2}(0), \hat{P}_{1,2}(0)$.

c) Verify explicitly that the commutation relations between $\hat{X}_{1,2}(t), \hat{P}_{1,2}(t)$ are consistent with the canonical. Write the uncertainty principle for those pairs of observables for which it applies.

d) Find the variances of $\hat{X}_1(t) \pm \hat{X}_2(t)$ and $\hat{P}_1(t) \pm \hat{P}_2(t)$ and compare them with those at $t = 0$. What physical phenomenon does this calculation demonstrate?

Useful relations

Clebsch-Gordan coefficients for two spin-1/2 particles:

$$\begin{aligned}
\langle m_1 = -1/2, m_2 = -1/2 | j = 1, m = -1 \rangle &= 1; \\
\langle m_1 = 1/2, m_2 = -1/2 | j = 0, m = 0 \rangle &= 1/\sqrt{2}; \\
\langle m_1 = 1/2, m_2 = -1/2 | j = 1, m = 0 \rangle &= 1/\sqrt{2}; \\
\langle m_1 = -1/2, m_2 = 1/2 | j = 0, m = 0 \rangle &= -1/\sqrt{2}; \\
\langle m_1 = -1/2, m_2 = 1/2 | j = 1, m = 0 \rangle &= 1/\sqrt{2}; \\
\langle m_1 = 1/2, m_2 = 1/2 | j = 1, m = 1 \rangle &= 1.
\end{aligned}$$

The Hamiltonian for the electron spin in the magnetic field:

$$\hat{H} = g\mu_B \vec{B} \cdot \vec{\hat{S}},$$

where μ_B is the Bohr magneton.