University of Calgary
Fall semester 2013

PHYS 615: Advanced Quantum Mechanics I

Homework assignment 6

Due Friday December 06, 2013

Problem 6.1. Verify numerically the Landau-Zener formula $p_D = e^{-2\pi\Gamma}$ for the probability of a diabatic transition, where $\Gamma = \Omega^2/\dot{\Delta}$, and the Hamiltonian is $\hat{H} = -\hbar \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix}$. The detuning Δ runs from -10 to 10 s^{-1} at constant rate $\dot{\Delta}$. Set the initial amplitude of the excited state $c_a(0) = 1$, ground state $c_b(0) = 0$. Then solve the Schrödinger equation numerically and find the final values of the state amplitudes. Based on your data, plot p_D vs $\dot{\Delta}$ for $\dot{\Delta} < 15\ \mathrm{s}^{-2}$, with $\Omega = 0.2,\ 0.6,\ 1.0$ s^{-1} along with the theoretical predictions according to the Landau-Zener formula. Also plot the calculated $c_a(t)$ and $c_b(t)$ for $\Omega = 1.0\ \mathrm{s}^{-1}$ and $\dot{\Delta} = 1$ as well as 15 s^{-2}

Problem 6.2. The Bloch vector travels adiabatically along the route $|a\rangle \to (|a\rangle + |b\rangle)/\sqrt{2} \to (|a\rangle + i\,|b\rangle)/\sqrt{2} \to |a\rangle$, following the shortest path between each two stations. Find the geometric phase accumulated. Show that it is independent of the instantaneous velocity at each moment in time (as long as the adiabaticity criterion is fulfilled).

Problem 6.3. Consider an atom with a Λ-shaped energy level structure, as shown in Fig. 1. There are two ground levels $|b\rangle$, $|c\rangle$ and one excited level $|a\rangle$. Spontaneous emission rates from $|a\rangle$ into $|b\rangle$ and $|c\rangle$ are Γ_b and Γ_c, respectively. There are two electromagnetic fields: the *control* field with Rabi frequency Ω_c coupling $|c\rangle$ with $|a\rangle$ with optical frequency ω_c and detuning Δ_c and the *signal* field with Rabi frequency Ω_b coupling $|b\rangle$ with $|a\rangle$ with optical frequency ω_b and detuning Δ_b. There is ground state decoherence manifesting itself as decay of the matrix element ρ_{bc} with rate $\gamma \ll \Gamma_b, \Gamma_c$.

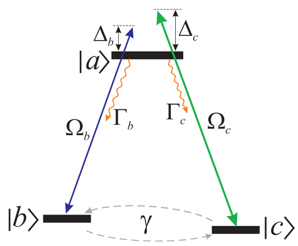

Figure 1: Λ-type atom.

a) Taking $|b\rangle$ as the zero energy state and assuming that the unperturbed Hamiltonian is given by $\hat{H}_0 = \hbar\omega_b\,|a\rangle\langle a| + \hbar(\omega_b - \omega_c)\,|c\rangle\langle c|$, show that the perturbation Hamiltonian in the interaction picture, rotating-wave approximation is given by

$$\hat{V} = \begin{pmatrix} -\Delta_b & -\Omega_b & -\Omega_c \\ -\Omega_b^* & 0 & 0 \\ -\Omega_c^* & 0 & -\Delta_b + \Delta_c \end{pmatrix} \tag{1}$$

b) Assume that the control field is much stronger than the signal. In this case, most of the atomic population is optically pumped into $|b\rangle$, so the stochastic wavefunction approximation can be

used. Write the Schrödinger equation for the amplitudes ψ_a and ψ_c.
Hint: Because, in the stochastic wavefunction approximation, $\rho_{cb} = \psi_c \psi_b^* = \psi_c$, we can treat the ground state decoherence as follows: $(\dot{\psi}_c)_{\text{dec}} = -\gamma \psi_c$.

c) Find the steady state amplitudes.

d) Show that the atomic medium's susceptibility with respect to the (weak) signal field is given by

$$\chi = \frac{N d_b^2}{\hbar \epsilon_0} \frac{\Delta_b - \Delta_c + i\gamma}{|\Omega_c|^2 - (\Delta_p - \Delta_c + i\gamma)(\Delta_b + i\frac{\Gamma_b + \Gamma_c}{2})} \tag{2}$$

e) Use software to plot the absorption index as a function of Δ_b (-3 MHz $< \Delta_b <$ 3 MHz) for $\Gamma_b = \Gamma_c = 1$ MHz, $\gamma = 0$, $\Delta_c = 0$ and $\Omega_c = 0.2$ as well as 2 MHz.

Problem 6.4. In class, we have written, but not derived, the general expression for the evolution of the two-level atom:

$$\psi_a(t) = \left(i \frac{\Omega}{W} \sin Wt \right) e^{i\Delta t/2}; \tag{3a}$$

$$\psi_b(t) = \left(\cos Wt - i \frac{\Delta}{2W} \sin Wt \right) e^{i\Delta t/2}. \tag{3b}$$

with $W = \sqrt{\Delta^2/4 + |\Omega|^2}$. We also found that the evolution of the Bloch vector under the Hamiltonian $\hat{H} = \hbar \vec{v} \cdot \hat{\vec{\sigma}}$ is precession according to

$$\dot{\hat{\vec{\sigma}}} = 2\vec{v} \times \hat{\vec{\sigma}}. \tag{4}$$

Find the Bloch vector using geometry from Eq. (4) and from quantum state (3) for $Wt = 0, \pi/2, \pi$ with real Ω. Verify that these results are consistent with each other. Plot the trajectory of the Bloch vector for $\Delta = -\Omega/2$.

Problem 6.5. In class, we have derived the energy spectrum of the Jaynes-Cummings Hamiltonian:

$$E_{n,\pm} = \hbar\omega \left(n + \frac{1}{2} \right) - \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2} \right)^2 + g^2(n+1)}.$$

Suppose a cavity initially contains an atom in the excited state and a coherent state $|\alpha\rangle$. Assuming $\Delta = 0$, write an analytic expression for the probability p_e of the atom's being in the excited state as a function of time. Using software, plot that probability for gt running between 0 and 100 with $\alpha^2 = 10$ and 100. Observe "collapse and revival" behavior. Estimate analytically the time of the first revival of p_e. Neglect spontaneous decay and cavity leakage.