University of Calgary
Fall semester 2013

PHYS 615: Advanced Quantum Mechanics I

Homework assignment 1

Due Monday September 22, 2013

Problem 1.1. Consider a quarter-waveplate with its optical axis oriented at angle ϕ to horizontal.

a) Find the operator associated that waveplate. Write it in the Dirac and matrix notations.

b) Verify that this operator is unitary.

c) Verify that, for $\phi = 45°$, this operator turns horizontal and vertical polarizations into circular.

d) A horizontally polarized photon enters the waveplate and then is subjected to a measurement in the canonical basis. Find and plot the detection probabilities as a function of ϕ.

Hint: the quarter-waveplate oriented horizontally is associated with operator $|H\rangle\langle H| + i\,|V\rangle\langle V|$.

Problem 1.2. The components of the spin operator $\hat{L}$ of a spin-1 particle have the following matrices in the eigenbasis $\{|1\rangle, |0\rangle, |-1\rangle\}$ of $\hat{L}_z$:

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) Show that these operators are Hermitian (i.e. they can be interpreted as physical observables).

b) Are these operators unitary?

c) Find the eigenvalues and eigenstates of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$.

d) Find the commutation relations of these observables.

e) The observable $\hat{L}_x$ is measured in the state $|\psi_0\rangle = (3i\,|1\rangle + 4\,|0\rangle)/5$. What results can be obtained and with which probabilities?

f) Find the expectation values and uncertainties of the measurements of $\hat{L}_x$ and $\hat{L}_y$ in the state $|\psi\rangle$.

g) Verify that the uncertainty principle holds for the measurements in part (f).

h) The particle initially ($t = 0$) in state $|-1\rangle$ is placed into a magnetic field $\vec{B}$ oriented along the y axis, so the Hamiltonian $\hat{H} = -\mu\vec{\hat{L}}\vec{B}$. Find the state of the system at an arbitrary time t using two methods: solving the differential equation for the state vector and calculating the evolution operator. What is the probability that the system will remain in its initial state at the moment $\omega t = \pi/2$? $\omega t = \pi$? $\omega t = 2\pi$ (where $\omega = \mu\hbar$)?

i) Find the mean and uncertainty of the state's energy as a function of time.

Problem 1.3. Consider an operator $\hat{A}$ that performs the following transformation.

$$|V\rangle \quad \to \quad \frac{|V\rangle + 3i\,|H\rangle}{\sqrt{10}}; \tag{1}$$

$$|+\rangle \quad \to \quad \frac{2+i}{\sqrt{5}}\,|-\rangle\,. \tag{2}$$

a) How is the horizontal polarization state mapped by $\hat{A}$?

b) Write the matrix of $\hat{A}$ in the canonical basis.

c) Determine how $\hat{A}$ acts upon the circular polarization states.

d) Using the previous result, find the matrix of $\hat{A}$ in the circular polarization basis;

e) Find the matrix of $\hat{A}$ in the canonical basis from its matrix in the circular basis using the method of "inserting $\hat{\mathbf{1}}$". Is your result consistent with that of part (b)?

f) Find the traces of the matrices of $\hat{A}$ in the canonical and circular bases. Are they identical?

g) Express $\hat{A}$ in the Dirac notation in terms of outer products of states $|H\rangle$ and $|V\rangle$;

h) Is $\hat{A}$ Hermitian? If not, what is its adjoint? Is $\hat{A}$ unitary?

Problem 1.4. Consider an apparatus for measuring the photon polarization that has the following properties:

- whenever a linearly polarized photon at angle θ enters the apparatus, it displays "2";
- whenever a linearly polarized photon at angle $\pi/2+\theta$ enters the apparatus, it displays "3";

Find the matrices of $\hat{A}$ in its eigenbasis and in the $\{|H\rangle, |V\rangle\}$ basis.