University of Calgary
Fall semester 2013

PHYS 615: Advanced Quantum Mechanics I

Homework assignment 5

Due Monday November 25, 2013

Problem 5.1. Using the Bogoliubov transformation for two-mode squeezing,

$$\hat{S}_2(\zeta)\hat{a}_1\hat{S}_2^\dagger(\zeta) = \hat{a}_1\cosh\zeta - \hat{a}_2^\dagger\sinh\zeta; \tag{1}$$
$$\hat{S}_2(\zeta)\hat{a}_2 S_2^\dagger(\zeta) = \hat{a}_2\cosh\zeta - \hat{a}_1^\dagger\sinh\zeta, \tag{2}$$

find the effect of two-mode squeezing $\hat{S}_2(\zeta)$ on two-mode Fock states $|0,0\rangle$, $|0,1\rangle$, $|2,1\rangle$. Express your answer in the Fock basis, including terms up to 2 photons. Do not use the explicit form of the two-mode squeezing Hamiltonian; instead, use the technique analogous to that implemented in class to find the matrix of the beam splitter operator [see Eq. (4.82) in the notes].

Problem 5.2. The beam splitter enacts the following transformation of mode operators.

$$\hat{a}_1' = \hat{U}^\dagger\hat{a}_1\hat{U} = \tau\hat{a}_1 - \rho\hat{a}_2; \tag{3}$$
$$\hat{a}_2' = \hat{U}^\dagger\hat{a}_2\hat{U} = \rho\hat{a}_1 + \tau\hat{a}_2. \tag{4}$$

Find the Hamiltonian $\hat{H}$ such that $\hat{U} = e^{-i\hat{H}t}$ for some t. Quantities τ and ρ are real.

Problem 5.3. Show that the Fock decomposition of the two-mode squeezed vacuum state

$$\hat{S}_2(\zeta)|0,0\rangle = \exp[\zeta(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger)]|0,0\rangle. \tag{5}$$

is given by

$$\hat{S}_2(\zeta)|0,0\rangle = \sum_{n=0}^{\infty}\frac{1}{\cosh\zeta}(-\tanh\zeta)^n|n,n\rangle \tag{6}$$

using the following approach.

a) Show that the decomposition contains only terms with equal photon numbers.

b) Knowing how the two-mode squeezing transforms quadrature observables, determine the wavefunction of $\hat{S}_2(\zeta)|0,0\rangle$ in the position basis.

c) Calculate $\langle\alpha,\alpha|\hat{S}_2(\zeta)|0,0\rangle$ for an arbitrary coherent state $|\alpha\rangle$ with a real α by integrating the wavefunctions.

d) You have obtained an expression of the form $\langle\alpha,\alpha|\hat{S}_2(\zeta)|0,0\rangle = f(\alpha,\zeta)$. Decompose the left-hand side of that expression into the Fock basis and the right-hand side into the Taylor series with respect to α. Apply the result from part (a) to simplify the result. Use it to find $|n,n\rangle\hat{S}_2(\zeta)|0,0\rangle$.

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Problem 5.4. Find the interaction picture Hamiltonian $\hat{V}(t) = e^{i(\hat{H}_0/\hbar)t}\hat{H}_I e^{-i(\hat{H}_0/\hbar)t}$ if (unlike what was done in the class) $\hat{H}_0$ is the Hamiltonian of the unperturbed atom. Apply the rotating wave approximation to simplify your result.

Problem 5.5. In class, we have written, but not derived, the general expression for the evolution of the two-level atom:

$$\psi_a(t) = \left(-i\frac{\Omega}{W}\sin Wt\right)e^{i\Delta t/2}; \tag{7}$$

$$\psi_b(t) = \left(\cos Wt - i\frac{\Delta}{2W}\sin Wt\right)e^{i\Delta t/2}. \tag{8}$$

with $W = \sqrt{\Delta^2/4 + |\Omega|^2}$. We also found that the evolution of the Bloch vector under the Hamiltonian $\hat{H} = \hbar\vec{v}\cdot\hat{\vec{\sigma}}$ is precession according to $\dot{\hat{\vec{\sigma}}} = \vec{v}\times\hat{\vec{\sigma}}$. Verify that these two results are consistent with each other for $Wt = 0, \pi/2, \pi$.