Phys 615

Final Exam

Solutions

1 In Heisenberg picture,

$$A(t) = e^{iHt} A(0) e^{-iHt}$$
$$B(t) = e^{iHt} B(0) e^{-iHt}$$

$$[A(t)\ B(t)] = e^{iHt} A(0) B(0) e^{-iHt} - e^{iHt} B(0) A(0) e^{-iHt}$$
$$= e^{iHt} [A(0)\ B(0)] e^{-iHt} = C(t)$$

Interaction picture similarly.

2 a) $U|0\ 0\rangle = |0\ 0\rangle$

$U|1\ 0\rangle = t|1\ 0\rangle + r|0\ 1\rangle$ with $t = \sqrt{T}$, $r = \sqrt{1-T}$

$|\Psi\rangle = U(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|0\ 0\rangle + \beta t|1\ 0\rangle + \beta r|0\ 1\rangle$

b) $\mathrm{Tr}_2 |\Psi\rangle\langle\Psi| = (\alpha|0\rangle + \beta t|1\rangle)(\alpha^*\langle 0| + \beta^* t\langle 1|) + |\beta r|^2|0\rangle\langle 0|$

$$= \begin{pmatrix} |\alpha|^2 + |\beta|^2 r & \beta^* t \\ \beta t & |\beta|^2 t^2 \end{pmatrix}$$

c) $\Pi_{click} = \eta|1\rangle\langle 1|$

$\mathrm{Tr}_2'(\Pi_{click}|\Psi\rangle\langle\Psi|) = \eta|\beta r|^2|0\rangle\langle 0|$ $\quad pr = \eta|\beta r|^2$

$\Pi_{no\text{-}click}\ (1-\eta)|1\rangle\langle 1| + |0\rangle\langle 0|$

$\mathrm{Tr}_2(\Pi_{no\text{-}click}|\Psi\rangle\langle\Psi|) = (1-\eta)|\beta r|^2|0\rangle\langle 0|$

$$+ (\alpha|0\rangle + \beta t|1\rangle)(\alpha^*\langle 0| + \beta^* t\langle 1|)$$

$$= \begin{pmatrix} |\alpha|^2 + (1-\eta)|\beta r|^2 & \beta^* t \\ \beta t & |\beta|^2 t^2 \end{pmatrix}$$

$pr = |\alpha|^2 + (1-\eta)|\beta r|^2 + |\beta t|^2 = 1 - \eta|\beta r|^2$

$\boxed{3}$ a) $\vec{B} = B\cos\theta\,\hat{k} + B\sin\theta\,\hat{i}$

$$\hat{H} = \mu B\cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \mu B\sin\theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mu B\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

b) $|\dot{\psi}\rangle = -i\hat{H}|\psi\rangle$

$$\begin{cases}\dot{\psi}_\uparrow = -i\Omega(\cos\theta\,\psi_\uparrow + \sin\theta\,\psi_\downarrow)\\ \dot{\psi}_\downarrow = -i\Omega(\sin\theta\,\psi_\uparrow - \cos\theta\,\psi_\downarrow)\end{cases}$$

where $\Omega = \frac{\mu B}{\hbar}$

Look for solution in form $\psi_\uparrow = Ae^{\omega t}$, $\psi_\downarrow = Be^{\omega t}$

$$\begin{cases}A\omega = -i\Omega(\cos\theta\,A + \sin\theta\,B)\\ B\omega = -i\Omega(\sin\theta\,A - \cos\theta\,B)\end{cases}$$

$$A = B\,\frac{-i\Omega\sin\theta}{\omega + i\Omega\cos\theta}$$

$$B\omega = B\,\frac{-\Omega^2\sin^2\theta}{\omega + i\Omega\cos\theta} + i\Omega B\cos\theta$$

$$(\omega - i\Omega\cos\theta)(\omega + i\Omega\cos\theta) = -\Omega^2\sin^2\theta$$

$$\omega^2 + \Omega^2\cos^2\theta = -\Omega^2\sin^2\theta$$

$$\omega = \pm i\Omega$$

$$\begin{cases}\psi_\uparrow(t) = A_1e^{i\Omega t} + A_2e^{-i\Omega t}\\ \psi_\downarrow(t) = B_1e^{i\Omega t} + B_2e^{-i\Omega t}\end{cases}$$

$\psi_\uparrow(0) = 1 \Rightarrow A_1 + A_2 = 1$

$\psi_\downarrow(0) = 0 \Rightarrow B_1 = -B_2$

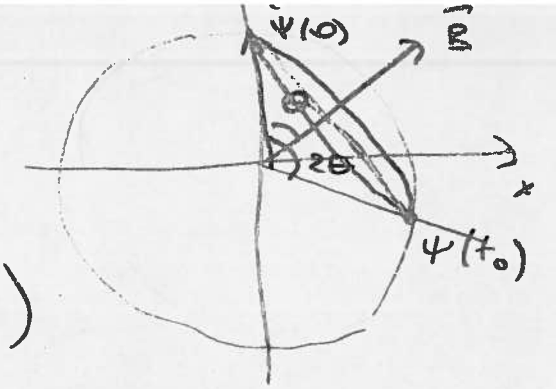
$$\begin{cases}\dot{\psi}_\uparrow(0) = -i\Omega\cos\theta = A_1(i\Omega) + A_2(-i\Omega)\\ \dot{\psi}_\downarrow(0) = -i\Omega\sin\theta = B_1(i\Omega) + B_2(-i\Omega)\end{cases} \Rightarrow \begin{cases}A_1 - A_2 = -\cos\theta\\ B_1 - B_2 = -\sin\theta\end{cases}$$

$$\begin{cases}A_1 = \frac{1-\cos\theta}{2}\\ A_2 = \frac{1+\cos\theta}{2}\\ B_1 = \frac{-\sin\theta}{2}\\ B_2 = \frac{\sin\theta}{2}\end{cases}$$

$$|\psi(t)\rangle = \begin{pmatrix}\cos\Omega t - i\cos\theta\sin\Omega t\\ -i\sin\theta\sin\Omega t\end{pmatrix} \qquad (1)$$

c) Bottom point: $(2\Theta, 0)$

$$\Rightarrow \Psi(2\Omega t = \pi) = \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}$$



Consistent with the result of (b):
if $\Omega t = \pi/2$, (1) becomes $\begin{pmatrix} -i\cos\Theta \\ -i\sin\Theta \end{pmatrix}$

$\boxed{4}$ a) Ramsey spectroscopy

b) Step 1: $\hat{U}|g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (|e\rangle - |g\rangle)/\sqrt{2}$

Optical state: $|\alpha\rangle = e^{-\alpha^2/2} \sum_n \alpha^n/\sqrt{n!}\, |n\rangle$

Step 2: $e^{-iHt/\hbar}\, \hat{U}|\alpha\rangle \otimes |g\rangle = e^{-\alpha^2/2} \sum_n \alpha^n/\sqrt{n!}\, |n\rangle \otimes \frac{1}{\sqrt{2}}(|e\rangle e^{-in6t} - |g\rangle)$

Step 3: $\hat{U}\frac{1}{\sqrt{2}}(|e\rangle e^{-in6} - |g\rangle) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-in6t} \\ -1 \end{pmatrix} =$

$$= \frac{1}{2}\begin{pmatrix} e^{-in6t} - 1 \\ e^{-in6t} + 1 \end{pmatrix} = e^{-in6t/2}\begin{pmatrix} -i\sin n6t/2 \\ \cos n6t/2 \end{pmatrix}$$

State of whole system:

$$|\Psi\rangle_{final} = e^{-\alpha^2/2} \sum_n \alpha^n/\sqrt{n!}\, |n\rangle \otimes \frac{1}{2}\begin{pmatrix} e^{-in6t} - 1 \\ e^{-in6t} + 1 \end{pmatrix}$$

c) If $n6t = n\pi$,

$$|\Psi_{final}\rangle = e^{-\alpha^2/2} \sum_n \alpha/\sqrt{n!}\, |n\rangle \otimes \frac{1}{2}\begin{pmatrix} (-1)^n - 1 \\ (-1)^n + 1 \end{pmatrix}$$

If n is odd, atomic state is $|e\rangle$
even $|g\rangle$

d) $\langle e|\Psi_{final}\rangle = e^{-\alpha^2/2} \sum_n \alpha/\sqrt{n!}\; n\; \frac{1}{2}[(-1)^n - 1] = \frac{|-\alpha\rangle - |\alpha\rangle}{2}$

$\langle g|\Psi_{final}\rangle = \frac{|-\alpha\rangle + |\alpha\rangle}{2}$

e) $\left.\begin{array}{l} |e\rangle \to (|e\rangle + |g\rangle)/\sqrt{2} \\ |g\rangle \to (|e\rangle - |g\rangle)/\sqrt{2} \end{array}\right\}$ Precession of Bloch vector around y axis with angular frequency $2v$ such that $2vt_1 = \pi/2$

$$\hat{H} = \hbar v \hat{\sigma}_y = \hbar \frac{\pi}{4t_1}\hat{\sigma}_y$$