Appendix S3
Solutions to Chapter 3 exercises

Solution to Exercise 3.1.

a) We calculate the right-hand side of Eq. (3.4) using decomposition (3.2):

$$\langle x|\ \psi\rangle = \int_{-\infty}^{+\infty} \psi(x')\left\langle x\middle|\ x'\right\rangle \mathrm{d}x' \overset{(3.1a)}{=} \int_{-\infty}^{+\infty} \psi(x')\,\delta(x-x')\mathrm{d}x' \overset{(D.5)}{=} \psi(x). \tag{S3.1}$$

b) Let us act with operator $\hat{I} = \int_{-\infty}^{+\infty} |x\rangle\langle x|\,\mathrm{d}x$ upon an arbitrary state $|\psi\rangle$. We have, according to the properties of the outer product,

$$\hat{I}\,|\psi\rangle = \int_{-\infty}^{+\infty} |x\rangle\,\langle x|\ \psi\rangle\,\mathrm{d}x \overset{(3.4)}{=} \int_{-\infty}^{+\infty} |x\rangle\ \psi(x)\mathrm{d}x \overset{(3.2)}{=} |\psi\rangle\,.$$

The operator $\hat{I}$ acting on any state returns the same state, i.e. it is the identity operator.

c) We insert the identity operator (3.5) into $\langle\psi_1|\ \psi_2\rangle$:

$$\begin{aligned}\langle\psi_1|\ \psi_2\rangle &= \left\langle \psi_1\middle| \left(\int_{-\infty}^{+\infty}|x\rangle\langle x|\,\mathrm{d}x\right)\middle|\ \psi_2\right\rangle \\ &= \int_{-\infty}^{+\infty}\langle\psi_1|\ x\rangle\,\langle x|\ \psi_2\rangle\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}\psi_1^*(x)\psi_2(x)\mathrm{d}x.\end{aligned}$$

Solution to Exercise 3.2. Applying Eq. (3.6), we find that

$$\langle \psi | \psi \rangle = \int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x.$$

The left-hand side of this equation is 1 because $|\psi\rangle$ is a physical state.

Solution to Exercise 3.3.

a) Integrating the squared absolute value of the wavefunction over the real axis we have

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x = A^2 \int_a^b \mathrm{d}x = A^2(b-a),$$

and thus

$$A = \frac{1}{\sqrt{b-a}}.$$

b) using Eq. (B.17) and assuming A real we find

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x = A^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{d^2}} \mathrm{d}x = A^2 \sqrt{\pi} d,$$

so

$$A = \frac{1}{\pi^{1/4}\sqrt{d}}.$$

Solution to Exercise 3.4. According to Eq. (3.4), the wavefunction of the state $|x_0\rangle$ is

$$\langle x | x_0 \rangle = \delta(x - x_0).$$

Solution to Exercise 3.5. In accordance with the continuous-variable observable definition (3.11),

$$\hat{x}|x\rangle = \left(\int_{-\infty}^{+\infty} x' |x'\rangle \langle x'| \mathrm{d}x' \right) |x\rangle = \int_{-\infty}^{+\infty} x' |x'\rangle \langle x' | x \rangle \mathrm{d}x' = \int_{-\infty}^{+\infty} x' |x'\rangle \delta(x'-x) \mathrm{d}x' = x|x\rangle.$$

Solution to Exercise 3.6.

a) We insert the identity operator (3.5) at both sides of $\hat{A}$:

$$
\begin{aligned}
\hat{A} &= \hat{\mathbf{1}}\hat{A}\hat{\mathbf{1}} \\
&= \left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\,\mathrm{d}x\right)\hat{A}\left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\,\mathrm{d}x'\right) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x\rangle\left\langle x\right|\hat{A}\left|x'\right\rangle\left\langle x'\right|\mathrm{d}x\mathrm{d}x' \\
&\overset{(3.13)}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(x,x')\,|x\rangle\left\langle x'\right|\mathrm{d}x\mathrm{d}x'.
\end{aligned}
$$

b) Using the continuous-basis operator function definition (3.12), we find

$$
\begin{aligned}
\langle\psi|\,f(\hat{x})\,|\psi\rangle &= \int_{-\infty}^{+\infty} f(x)\,\langle\psi|\,x\rangle\,\langle x|\,\psi\rangle\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \psi^*(x)f(x)\,\psi(x)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} |\psi(x)|^2 f(x)\mathrm{d}x.
\end{aligned}
$$

c) Using Eq. (3.14), we find

$$
\begin{aligned}
\langle\phi|\,\hat{A}\,|\psi\rangle &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle\phi|\,x\rangle\,A(x,x')\left\langle x'\right|\,\psi\rangle\,\mathrm{d}x\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi^*(x)A(x,x')\,\psi(x')\mathrm{d}x\,\mathrm{d}x'.
\end{aligned}
$$

d)

$$
\begin{aligned}
\langle x|\,\hat{A}\,|\psi\rangle &= \langle x|\,\hat{A}\left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\,\mathrm{d}x'\right)|\psi\rangle \\
&= \int_{-\infty}^{+\infty} \left\langle x\right|\hat{A}\left|x'\right\rangle\left\langle x'\right|\,\psi\rangle\,\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty} A(x,x')\,\psi(x')\,\mathrm{d}x'.
\end{aligned}
$$

e) Similarly,

$$
\begin{aligned}
\langle\psi|\hat{A}\,|x\rangle &= \langle\psi|\left(\int_{-\infty}^{+\infty}|x'\rangle\langle x'|\,\mathrm{d}x'\right)\hat{A}\,|x\rangle \\
&= \int_{-\infty}^{+\infty}\langle\psi|\,x'\rangle\,\langle x'|\,\hat{A}|\,x\rangle\,\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty}\psi^*(x')A(x',x)\,\mathrm{d}x'.
\end{aligned}
$$

f) According to the properties of adjoint operators (see Ex. A.59),

$$(A^\dagger)(x,x') = \langle x|\,\hat{A}^\dagger|\,x'\rangle = \langle x'|\,\hat{A}|\,x\rangle^* = A^*(x',x).$$

g) Inserting the identity operator between $\hat{A}$ and $\hat{B}$, we find

$$
\begin{aligned}
\langle x|\hat{A}\hat{B}\,|x'\rangle &= \langle x|\hat{A}\left(\int_{-\infty}^{+\infty}|x''\rangle\langle x''|\,\mathrm{d}x''\right)\hat{B}\,|x'\rangle \\
&= \int_{-\infty}^{+\infty}\langle x|\,\hat{A}|\,x''\rangle\,\langle x''|\,\hat{B}|\,x'\rangle\,\mathrm{d}x'' \\
&= \int_{-\infty}^{+\infty}A(x,x'')B(x'',x')\mathrm{d}x''.
\end{aligned}
$$

Solution to Exercise 3.7. Using Eq. (3.15) for $f(x)\equiv x$, we find

$$
\begin{aligned}
\langle\psi|\hat{x}\,|\psi\rangle &= \int_{-\infty}^{+\infty}x\,|\psi(x)|^2\,\mathrm{d}x \\
&\overset{(3.22)}{=} \int_{-\infty}^{+\infty}x\,\mathrm{pr}(x)\,\mathrm{d}x,
\end{aligned}
$$

where $\mathrm{pr}(x)$ is the probability density. The latter expression gives the mean of a continuous observable according to Eq. (B.13).

Solution to Exercise 3.8. We need to show that function (3.25) is periodic with period λ_{dB}. This is indeed so, because

$$\langle x+\lambda_{dB}|\,p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p(x+2\pi\hbar/p)}{\hbar}} = \frac{1}{\sqrt{2\pi\hbar}}e^{i\left(\frac{px}{\hbar}+2\pi\right)} = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{px}{\hbar}} = \langle x|\,p\rangle\,.$$

Solution to Exercise 3.9.

a) If a 1000-kg car is moving at a velocity of 20 m/s (72 km/h), its momentum is $p = 2\times10^4$ kg·m/s. Using the tabulated value of $2\pi\hbar = 6.6\times10^{-34}$ m^2·kg/s, we find the de Broglie wavelength to equal $\lambda = 2\pi\hbar/p = 3.3\times10^{-38}$ m.
b) The average translational velocity of molecules of a gas is $v = \sqrt{3k_BT/M}$, and the momentum is $p = \sqrt{3k_BTm}$, where $k_B = 1.38\times10^{-23}$ J/K is the Boltzmann constant, $T = 300$ K is the room temperature and $M = M_{\rm air}/N_A = 4.7\times10^{-26}$ kg is the average molecular mass (where $M_{\rm air} = 0.028$ kg/mol is the molar mass of air and $N_A = 6\times10^{23}$ is the Avogadro number). We find $p = 2.4\times10^{-23}$ kg·m/s and hence $\lambda = 2.7\times10^{-11}$ m.
c) The kinetic energy of the electron is $p^2/2M = eU$, where $M = 9.1\times10^{-31}$ kg is the electron mass, $e = 1.6\times10^{-19}$ Col is the electron charge and $U = 10^5$ V is the accelerating voltage. We find $p = 1.9\times10^{-22}$ kg·m/s and $\lambda = 3.5\times10^{-12}$ m. Because the de Broglie wavelength of the electron is much smaller than the wavelength of light, the electron microscope achieves much higher resolution than optical.
d) By analogy with part (b), we find the mass of rubidium atoms as $m = 0.087/(6\times10^{23})$ kg=1.5×10^{-25} kg and their momentum $p = \sqrt{3k_BTm} = 7.9\times10^{-28}$ kg·m/s. The de Broglie wavelength is 8.3×10^{-7} m=0.83 μm. This wavelength is comparable to the distance between atoms in the condensate, which leads to quantum effects in interaction between atoms.

Solution to Exercise 3.10. Using the resolution of the identity, we write

$$|p\rangle = \int_{-\infty}^{+\infty} |x\rangle \langle x|\, p\rangle \,\mathrm{d}x \overset{(3.25)}{=} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{i\frac{px}{\hbar}}\, |x\rangle \,\mathrm{d}x.$$

Equation (3.27b) is proven similarly.

Solution to Exercise 3.11. According to Eq. (3.6),

$$\langle p|\, p'\rangle \overset{(3.27a)}{=} \frac{1}{2\pi\hbar} \iint_{-\infty}^{+\infty} e^{i\frac{p'x'-px}{\hbar}} \underbrace{\langle x|\, x'\rangle}_{\delta(x-x')} \mathrm{d}x\mathrm{d}x' = \int_{-\infty}^{+\infty} e^{i\frac{(p'-p)x}{\hbar}}\,\mathrm{d}x \overset{(D.19)}{=} \frac{1}{2\pi\hbar} 2\pi\delta\left(\frac{p'-p}{\hbar}\right) \overset{(D.6)}{=} \delta(p'-p).$$

Solution to Exercise 3.13. To convert between the position and momentum bases, we apply our usual trick of inserting the resolution of the identity:

$$\begin{aligned} \psi(x) &= \langle x|\, \psi\rangle \\ &= \langle x| \left(\int_{-\infty}^{+\infty} |p\rangle\langle p|\,\mathrm{d}p \right) |\psi\rangle \\ &= \int_{-\infty}^{+\infty} \langle x|\, p\rangle \langle p|\, \psi\rangle \,\mathrm{d}p \\ &\overset{(3.25)}{=} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{i\frac{px}{\hbar}}\, \psi(p)\,\mathrm{d}p; \end{aligned}$$

$$\begin{aligned}
\psi(p) &= \langle p|\ \psi\rangle \\
&= \langle p| \left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\, \mathrm{d}x \right) |\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle p|\ x\rangle \langle x|\ \psi\rangle\, \mathrm{d}x \\
&\overset{(3.25)}{=} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-i\frac{px}{\hbar}}\, \psi(x)\, \mathrm{d}x.
\end{aligned}$$

Solution to Exercise 3.15.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \tilde{\psi}^*(p)\tilde{\varphi}(p)\mathrm{d}p &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\frac{px}{\hbar}}\, \psi^*(x) e^{-i\frac{px'}{\hbar}}\, \varphi(x')\mathrm{d}x\mathrm{d}x'\mathrm{d}p \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} e^{i\frac{p(x-x')}{\hbar}}\, \mathrm{d}p \right] \psi^*(x)\varphi(x')\mathrm{d}x\mathrm{d}x' \\
&\overset{(D.19)}{=} \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[2\pi\delta\left(\frac{x-x'}{\hbar}\right) \right] \psi^*(x)\varphi(x')\mathrm{d}x\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty} \psi^*(x)\varphi(x)\mathrm{d}x.
\end{aligned}$$

Solution to Exercise 3.16. We recall that the probability to detect a certain value of momentum is

$$\mathrm{pr}(p) = \langle p|\ \psi\rangle = |\tilde{\psi}(p)|^2,$$

where the wavefunction $\tilde{\psi}(p)$ in the momentum basis is the Fourier transform of the wavefunction $\psi(x)$ in the position basis. Because the latter is real, $\tilde{\psi}(p) = \tilde{\psi}^*(-p)$ [Ex. D.5(b)] and thus $\mathrm{pr}(p) = \mathrm{pr}(-p)$.

The expectation value of the momentum observable, given by

$$\langle p\rangle = \int_{-\infty}^{+\infty} p\, \mathrm{pr}(p)\mathrm{d}p = 0,$$

vanishes because $p\,\mathrm{pr}(p)$ is an odd function.

Solution to Exercise 3.17. Using the definition (3.25) of the de Broglie wave, we find

$$\begin{aligned}A(p,p') &= \langle p|\hat{A}|p'\rangle \\ &\stackrel{(3.5)}{=} \iint\limits_{-\infty}^{+\infty} \langle p|x\rangle\langle x|\hat{A}|x'\rangle\langle x'|p'\rangle \mathrm{d}x\mathrm{d}x' \\ &= \frac{1}{2\pi\hbar}\iint\limits_{-\infty}^{+\infty} A(x,x')e^{-\frac{ipx}{\hbar}}e^{\frac{ip'x'}{\hbar}}\mathrm{d}x\mathrm{d}x'.\end{aligned}$$

Solution to Exercise 3.18.

a) Since the potential is a function of the position observable, we have

$$V(\hat{x}) = \int\limits_{-\infty}^{+\infty} V(y)\,|y\rangle\langle y|\,\mathrm{d}y \tag{S3.2}$$

(where y is the integration variable). Hence

$$\begin{aligned}V(x,x') &= \langle x|\,V(\hat{x})\,|x'\rangle \\ &= \int\limits_{-\infty}^{+\infty} V(y)\,\langle x|y\rangle\,\langle y|x'\rangle\,\mathrm{d}y \\ &= \int\limits_{-\infty}^{+\infty} V(y)\delta(y-x)\delta(y-x')\mathrm{d}y \\ &= V(x)\delta(x-x').\end{aligned}$$

In the latter equality above, we used identity (D.5), with $a=x$ and $f(y)=V(y)\delta(y-x')$. This is somewhat frivolous because Eq. (D.5) requires a smooth function $f(\cdot)$. To make this argument rigorous, we could, for example, replace $\delta(y-x')$ by a Gaussian function $G_b(y-x')$, as defined in Eq. (D.1), and take the limit $b\to 0$.

b) Using Eq. (S3.2), as well as the definition (3.25) of the de Broglie wave, we find

$$\begin{aligned}V(p,p') &= \langle p|\,V(\hat{x})\,|p'\rangle \\ &= \int\limits_{-\infty}^{+\infty} V(x)\,\langle p|x\rangle\,\langle x|p'\rangle\,\mathrm{d}x \\ &= \frac{1}{2\pi\hbar}\int\limits_{-\infty}^{+\infty} V(x)e^{-\frac{ipx}{\hbar}}e^{\frac{ip'x}{\hbar}}\mathrm{d}x,\end{aligned}$$

which is the same as Eq. (3.41).

Solution to Exercise 3.19. Writing the momentum observable as $\hat{p} = \int_{-\infty}^{+\infty} p\,|p\rangle\langle p|\,\mathrm{d}p$, we find

$$\left\langle x\right|\,\hat{p}\left|\,x'\right\rangle = \int_{-\infty}^{+\infty} p\,\langle x|\,p\rangle\left\langle p\right|\,x'\right\rangle \mathrm{d}p = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} p\,e^{i\frac{p}{\hbar}(x-x')}\mathrm{d}p.$$

To calculate this integral, we express $p\,e^{i\frac{p}{\hbar}(x-x')} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}e^{i\frac{p}{\hbar}(x-x')}$. Hence

$$\begin{aligned}
\left\langle x\right|\,\hat{p}\left|\,x'\right\rangle &= \frac{1}{2\pi\hbar}(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{+\infty} e^{i\frac{p}{\hbar}(x-x')}\mathrm{d}p \\
&\overset{(D.19)}{=} \frac{1}{2\pi\hbar}(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}(2\pi\hbar)\delta(x-x') \\
&= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x').
\end{aligned}$$

Solution to Exercise 3.20. Inserting the identity operator after the momentum and using the result of the Ex. 3.19, we find

$$\begin{aligned}
\langle x|\,\hat{p}|\,\psi\rangle &= \langle x|\,\hat{p}\left(\int_{-\infty}^{+\infty}\left|x'\right\rangle\left\langle x'\right|\,\mathrm{d}x'\right)|\psi\rangle \\
&= \int_{-\infty}^{+\infty}\left\langle x\right|\,\hat{p}\left|\,x'\right\rangle\left\langle x'\right|\,\psi\rangle\,\mathrm{d}x' \\
&= -i\hbar\int_{-\infty}^{+\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\right]\psi(x')\,\mathrm{d}x' \\
&= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-\infty}^{+\infty}\delta(x-x')\psi(x')\,\mathrm{d}x'\right] \\
&= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)
\end{aligned}$$

Solution to Exercise 3.22. Using the results of Ex. 3.19 and 3.20,

$$\begin{aligned}
\left\langle x\right|\hat{p}^2\left|\psi\right\rangle &= \langle x|\,\hat{p}\left(\int_{-\infty}^{+\infty}\left|x'\right\rangle\left\langle x'\right|\mathrm{d}x'\right)\hat{p}\,|\psi\rangle \\
&= \int_{-\infty}^{+\infty}\left\langle x\right|\hat{p}\left|x'\right\rangle\left\langle x'\right|\hat{p}\left|\psi\right\rangle \mathrm{d}x' \\
&= -i\hbar\int_{-\infty}^{+\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\right]\left[-i\hbar\frac{\mathrm{d}}{\mathrm{d}x'}\psi(x')\right]\mathrm{d}x' \\
&= (-i\hbar)^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-\infty}^{+\infty}\delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x'}\psi(x')\,\mathrm{d}x'\right] \\
&= -\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\right] \\
&= -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x).
\end{aligned}$$

Solution to Exercise 3.23.

a) Since $\langle x|\,\hat{x} = x\,\langle x|$,

$$\langle x|\,\hat{x}\hat{p}|\,\psi\rangle = x\,\langle x|\,\hat{p}|\,\psi\rangle \overset{(3.44)}{=} -i\hbar x\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).$$

b) Let us denote $\hat{x}\,|\psi\rangle = |\phi\rangle$; then the wavefunction of this state is $\phi(x) = \langle x|\,\hat{x}|\,\psi\rangle = x\psi(x)$. Therefore

$$\langle x|\,\hat{p}\hat{x}|\,\psi\rangle = \langle x|\,\hat{p}|\,\phi\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}[x\psi(x)] = -i\hbar\psi(x) - i\hbar x\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).$$

Note that the above relation can also be found using the resolution of the identity. The reader can try this independently.

c) Using the two results above, we find

$$\langle x|\,[\hat{x},\hat{p}]|\,\psi\rangle = \langle x|\,\hat{x}\hat{p}|\,\psi\rangle - \langle x|\,\hat{p}\hat{x}|\,\psi\rangle = i\hbar\psi(x).$$

Therefore, applying the operator $[\hat{x},\hat{p}]$ to any $|\psi\rangle$ is equivalent to multiplying this state by $i\hbar$. We conclude that $[\hat{x},\hat{p}] = i\hbar\hat{\mathbf{1}}$.

Solution to Exercise 3.24. Writing the uncertainty principle (1.21) for any normalized state $|\psi\rangle$, we find:

$$\begin{aligned}
\left\langle\psi\right|\Delta\hat{x}^2\left|\psi\right\rangle\left\langle\psi\right|\Delta\hat{p}^2\left|\psi\right\rangle &\geq \frac{1}{4}\left|\left\langle\psi\right|i\hbar\hat{\mathbf{1}}\left|\psi\right\rangle\right|^2 \\
&= \frac{1}{4}\hbar^2|\,\langle\psi|\,\psi\rangle\,|^2 \\
&= \frac{1}{4}\hbar^2.
\end{aligned}$$

Solution to Exercise 3.25.

a) The probability density corresponding to the wavefunction (3.51) is

$$|\psi(x)|^2 = \frac{1}{d\sqrt{\pi}} e^{-\frac{(x-a)^2}{d^2}}. \tag{S3.3}$$

This is identical to the Gaussian probability density (B.15), whose normalization we verified in Ex. B.18.

b) To reduce the bookkeeping, let us first convert from the position basis into the wavenumber (rather than momentum) basis. We apply the direct Fourier transform as per Eq. (3.38).

$$\begin{aligned}
\tilde{\psi}(k) &= \frac{1}{(\pi d^2)^{1/4}} \mathscr{F}[e^{ik_0 x} e^{-\frac{(x-a)^2}{2d^2}}](k) \qquad (\text{where } k_0 = \frac{p_0}{\hbar}) \\
&\stackrel{(D.14)}{=} \frac{1}{(\pi d^2)^{1/4}} \mathscr{F}[e^{-\frac{(x-a)^2}{2d^2}}](k-k_0) \\
&\stackrel{(D.13)}{=} \frac{1}{(\pi d^2)^{1/4}} e^{-i(k-k_0)a} \mathscr{F}[e^{-\frac{x^2}{2d^2}}](k-k_0) \\
&\stackrel{(D.16)}{=} \frac{1}{(\pi d^2)^{1/4}} e^{-i(k-k_0)a} d e^{-(k-k_0)^2 d^2/2} \\
&= \frac{\sqrt{d}}{\pi^{1/4}} e^{-i(k-k_0)a} e^{-(k-k_0)^2 d^2/2}.
\end{aligned} \tag{S3.4}$$

Now we can rewrite this result in the momentum basis using Eq. (3.39):

$$\tilde{\psi}(p) = \frac{\sqrt{d}}{\pi^{1/4}\sqrt{\hbar}} e^{-i(p-p_0)a/\hbar} e^{-(p-p_0)^2 d^2/2\hbar^2}. \tag{S3.5}$$

c) In the position basis, the probability density

$$\mathrm{pr}(x) = |\psi(x)|^2 = \frac{1}{\sqrt{\pi}d} e^{-\frac{(x-a)^2}{d^2}}$$

is a Gaussian curve centered around $x = a$ with width d. Using the results of Ex. B.18, we find that $\langle x \rangle = a$ and $\left\langle \Delta x^2 \right\rangle = d^2/2$.

For the momentum basis, $\mathrm{pr}(p) = \frac{d}{\sqrt{\pi}\hbar} e^{-\frac{d^2(p-p_0)^2}{\hbar^2}}$. Hence $\langle p \rangle = p_0$ and $\left\langle \Delta p^2 \right\rangle = \hbar^2/2d^2$. The product of the uncertainties is

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4},$$

which is the minimum allowed by the uncertainty principle.

Solution to Exercise 3.27.

a) The wavefunction in the momentum representation (for convenience, we use the physically identical wavenumber representation) can be found using the standard conversion formula (3.38). The Fourier transformation has to be applied to both x_A and x_B.

$$
\begin{aligned}
\tilde{\Psi}(k_A,k_b) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\Psi(x_A,x_B)e^{-ik_Ax_A}e^{-ik_Bx_B}\mathrm{d}x_A\mathrm{d}x_B \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\delta(x_A-x_B)e^{-ik_Ax_A-ik_Bx_B}\mathrm{d}x_A\mathrm{d}x_B \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}e^{-i(k_A+k_B)x_A}\mathrm{d}x_A \\
&\overset{(D.19)}{=} \frac{1}{2\pi}2\pi\delta(k_A+k_B) \\
&= \delta(k_A+k_B).
\end{aligned}
$$

b) The wavefunction $\Psi(x_A,x_B)=\delta(x_A-x_B)$ of the system in the position basis implies that the positions of Alice's and Bob's particles must be identical. If Alice detects her particle at a position x_0, Bob's particle will be remotely prepared in a state with the same position, i.e. $|x_0\rangle$.
c) Similarly, because $\tilde{\Psi}(k_A,k_b)=\delta(k_A+k_B)$, Alice's detection of wavenumber k_0 (or momentum $p_0=\hbar k_0$) will project Bob's state onto $|-k_0\rangle$ (or $|-p_0\rangle$).

Solution to Exercise 3.28. In the absence of potential, the Hamiltonian is a function of the momentum: $\hat{H}=\hat{p}^2/2M$. An eigenstate $|p\rangle$ of the momentum is therefore automatically an energy eigenstate with the eigenvalue $E=p^2/2M$. According to the general solution (1.29) of the Schrödinger equation, this state evolves as follows:

$$
|p\rangle \to e^{-\frac{i}{\hbar}Et}|p\rangle = e^{-i\frac{p^2}{2M\hbar}t}|p\rangle .
$$

Assuming that the wavefunction of the momentum eigenstate at the moment $t=0$ is given by the de Broglie wave (3.25), its evolution can be written in the position basis as

$$
\psi_{|p\rangle}(x,t)=\left\langle x\right| e^{-\frac{i}{\hbar}Et}\left| p\right\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{px}{\hbar}-i\frac{p^2}{2M\hbar}t}.
$$

Solution to Exercise 3.29.

a) We found the wavenumber basis decomposition (S3.4) of the initial wavepacket in Ex. 3.25. Let us rewrite it as follows:

$$
|\psi(0)\rangle = \frac{\sqrt{d}}{\pi^{1/4}}\int\limits_{-\infty}^{+\infty}e^{-i\kappa a}e^{-\frac{\kappa^2d^2}{2}}|k_0+\kappa\rangle\,\mathrm{d}\kappa, \tag{S3.6}
$$

where we defined $\kappa=k-k_0$. Since each wavenumber eigenstate is also an eigenstate of the Hamiltonian with the eigenvalue $\hbar^2(k_0+\kappa)^2/2M$, we have for the evolution of state $|\psi\rangle$

$$|\psi(t)\rangle = \frac{\sqrt{d}}{\pi^{1/4}} \int_{-\infty}^{+\infty} e^{-i\kappa a} e^{-\frac{\kappa^2 d^2}{2}} e^{-i\frac{\hbar(k_0+\kappa)^2 t}{2M}} |k_0+\kappa\rangle \, \mathrm{d}\kappa. \tag{S3.7}$$

b) We rewrite the above equation as

$$|\psi(t)\rangle = \frac{\sqrt{d}}{\pi^{1/4}} e^{-i\frac{\hbar k_0^2 t}{2M}} \int_{-\infty}^{+\infty} e^{-i\kappa\left(a+\frac{\hbar k_0 t}{M}\right)} e^{-\kappa^2\left(\frac{d^2}{2}+i\frac{\hbar t}{2M}\right)} |k_0+\kappa\rangle \, \mathrm{d}\kappa. \tag{S3.8}$$

Let us now write this result back in the position basis. We have

$$\begin{aligned}\psi(x,t) = \langle x|\ \psi(t)\rangle &= \frac{\sqrt{d}}{\pi^{1/4}} e^{-i\frac{\hbar k_0^2 t}{2M}} \int_{-\infty}^{+\infty} e^{-i\kappa\left(a+\frac{\hbar k_0 t}{M}\right)} e^{-\kappa^2\left(\frac{d^2}{2}+i\frac{\hbar t}{2M}\right)} \langle x|\ k_0+\kappa\rangle \, \mathrm{d}\kappa \\ &= \frac{\sqrt{d}}{\pi^{1/4}} e^{-i\frac{\hbar k_0^2 t}{2M}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\kappa\left(a+\frac{\hbar k_0 t}{M}\right)} e^{-\kappa^2\left(\frac{d^2}{2}+i\frac{\hbar t}{2M}\right)} e^{i(k_0+\kappa)x} \mathrm{d}\kappa\right].\end{aligned}$$

The expression in brackets is an inverse Fourier transform, which is not surprising because we are transitioning from the wavenumber to the position basis. The first exponential in the above integral is a linear phase factor, which after the Fourier transform translates, according to Eq. (D.14), into the shift of the position by $a+\hbar k_0 t/M$ — the movement of the wavepacket. The second one is a Gaussian function, whose Fourier image is also a Gaussian. The resulting wavefunction is therefore

$$\psi(x,t) = \frac{\sqrt{d}}{\pi^{1/4}} e^{-i\frac{\hbar k_0^2 t}{2M}} \left(d^2 + i\frac{\hbar t}{M}\right)^{-1/2} e^{ik_0 x} e^{-\left(x-a-\frac{\hbar k_0 t}{M}\right)^2 \Big/ 2\left(d^2+i\frac{\hbar t}{M}\right)}, \tag{S3.9}$$

c) We first calculate the probability density taking into account the complexity of the Gaussian exponent in Eq. (S3.9). We find

$$\begin{aligned}\mathrm{pr}(x) &= \psi^*(x,t)\psi(x,t) \\ &= \frac{d}{\sqrt{\pi}} \left(d^4 + \frac{\hbar^2 t^2}{M^2}\right)^{-1/2} e^{-d^2\left(x-a-\frac{\hbar k_0 t}{M}\right)^2 \Big/ \left(d^4+\frac{\hbar^2 t^2}{M^2}\right)} \\ &= \frac{1}{d\sqrt{\pi}} \left(1 + \frac{\hbar^2 t^2}{M^2 d^4}\right)^{-1/2} e^{-\left(x-a-\frac{\hbar k_0 t}{M}\right)^2 \Big/ d^2\left(1+\frac{\hbar^2 t^2}{M^2 d^4}\right)}\end{aligned}$$

This is a Gaussian distribution centered at $\langle x\rangle = a + \frac{\hbar k_0 t}{M} = a + \frac{p_0 t}{M}$ with the width $b = d\sqrt{1+\frac{\hbar^2 t^2}{M^2 d^4}}$. To determine the position variance, we use Ex. B.18 and find

$$\langle \Delta x^2\rangle = \frac{b^2}{2} = \frac{d^2}{2}\left(1 + \frac{\hbar^2 t^2}{M^2 d^4}\right). \tag{S3.10}$$

Solution to Exercise 3.30.

a) According to Eq. (S3.10), the width of the Gaussian wavepacket behaves for large t according to

$$\sqrt{\langle \Delta x^2 \rangle} \sim \frac{\hbar}{Md} t. \tag{S3.11}$$

We can rewrite this as $t \sim \sqrt{\langle \Delta x^2 \rangle} Md/\hbar$. Substituting $\sqrt{\langle \Delta x^2 \rangle} = 10^{-3}$ m, $d = 10^{-10}$ m and $M \approx 10^{-30}$ kg we find $t \approx 1$ ns.

b) For $M = 10^{-3}$ kg, $t \approx 10^{18}$ s, i.e. of the same magnitude as the age of the universe.

c) According to Eq. (S3.10), the required time satisfies $\hbar t/Md^2 \approx 1$, so $t \sim 1$ s.

Solution to Exercise 3.31. The condition that p_0 greatly exceeds the momentum uncertainty of the initial wavepacket means, in accordance with Ex. 3.25, that $p_0 \gg \hbar/d$. This means that the traveled distance, $p_0 t/M$, is much greater than $\hbar t/Md$, i.e. it is much greater than $\sqrt{\langle \Delta x^2 \rangle}$ in accordance with Eq. (S3.11).

Solution to Exercise 3.32. We rewrite the time-independent Schrödinger equation

$$\left[V(\hat{x}) + \frac{\hat{p}^2}{2M} \right] |\psi\rangle = E\, |\psi\rangle$$

in the position basis

$$\langle x|\, V(\hat{x})|\, \psi\rangle - \left\langle x \middle| \frac{\hat{p}^2}{2M} \middle| \psi \right\rangle = E\, \langle x|\, \psi\rangle\, ,$$

and use the result of Ex. 3.22:

$$V(x)\psi(x) - \frac{\hbar^2}{2M} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \psi(x) = E\psi(x).$$

Solution to Exercise 3.33. We can rewrite the time-independent Schrödinger equation (3.60) as

$$\frac{\hbar^2}{2M} \frac{d^2}{dx^2} \psi(x) = (V_0 - E)\psi(x), \tag{S3.12}$$

which can be simplified to

$$\frac{d^2}{dx^2} \psi(x) = \kappa^2 \psi(x),$$

where $\kappa = \sqrt{2M(V_0 - E)}/\hbar$ does not depend on x. This second-order differential equation has two linearly-independent solutions:

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}. \tag{S3.13}$$

Factor κ is real only if $E < V_0$, i.e. the total energy is below the potential energy level. Otherwise, κ becomes imaginary and the solution (S3.13) takes the form of the de Broglie wave:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{S3.14}$$

where $k = i\kappa = \sqrt{2M(E - V_0)}/\hbar$ is a real wavenumber.

Solution to Exercise 3.34. Consider the operator $\hat{H} - V_{\min}$, where $V_{\min}$ is the minimum value of $V(x)$. This operator the energy operator (3.55) is a sum of two nonnegative functions $\hat{p}^2/2M$ and $V(\hat{x}) - V_{\min}$ of the momentum and position, respectively, and hence a non-negative operator (Ex. A.73, A.87). Such an operator cannot have negative eigenvalues (Ex. A.72). Hence the operator $\hat{H}$ cannot have eigenvalues below $V_{\min}$.

Solution to Exercise 3.35. Let us refer again to Eq. (S3.12). If both $V(x)$ and $\psi(x)$ are finite for all x, so is the right-hand side of that equation. This means that $\mathrm{d}^2\psi(x)/\mathrm{d}x^2$ is finite for all x as well. This implies in turn that the first derivative of the wavefunction is continuous for all x. Because $\mathrm{d}\psi(x)/\mathrm{d}x$ is continuous, it must be finite for all x. Therefore, $\psi(x)$ must be finite and continuous for all x, too.

Solution to Exercise 3.36. Suppose there exists a Hamiltonian eigenstate $|\psi\rangle$ with eigenvalue E which cannot be expressed as a linear combination of eigenstates with real wavefunctions. Let us write its wavefunction as a sum of real and imaginary parts: $\psi(x) = \psi_1(x) + i\psi_2(x)$, where $\psi_{1,2}(x) \in \mathbb{R}$. The time-independent Schrödinger equation (3.60) then takes the form

$$\frac{\hbar^2}{2M}\frac{d^2}{dx^2}[\psi_1(x) + i\psi_2(x)] = [V(x) - E][\psi_1(x) + i\psi_2(x)].$$

This equation is satisfied because $|\psi\rangle$ is a Hamiltonian eigenstate with eigenvalue E. Taking the real and imaginary parts of both sides of this equation, we find that both $\psi_1(x)$ and $\psi_2(x)$ satisfy this equation, so the corresponding states $|\psi_1\rangle$ and $|\psi_2\rangle$ are also eigenstates of $\hat{H}$ with eigenvalue E. Further, state $|\psi\rangle$ can be expressed as a linear combination $|\psi\rangle = |\psi_1\rangle + i|\psi_2\rangle$ of energy eigenstates with real eigenvalues. We have arrived at a contradiction.

Solution to Exercise 3.37. By the same logic, consider an energy eigenstate $|\psi\rangle$ with eigenvalue E and wavefunction $\psi(x)$. If $\psi(x)$ satisfies the time-independent Schrödinger equation with an even potential, so does $\psi(-x)$. To see this, we replace $x \to -x$ in the time-independent Schrödinger equation (3.60):

$$\frac{\hbar^2}{2M}\frac{d^2}{d(-x)^2}\psi(-x) = [V(-x) - E]\psi(-x).$$

Because the potential is even, $V(-x) = V(x)$. Also the second derivative has the property $\frac{d^2}{d(-x)^2} = \frac{d^2}{dx^2}$. Therefore the above equation can be rewritten as

$$\frac{\hbar^2}{2M}\frac{d^2}{dx^2}\psi(-x) = [V(x) - E]\psi(-x),$$

so the state $|\psi^-\rangle$ with the wavefunction $\psi(-x)$ is also an eigenstate of the Hamiltonian.

This means that states $|\psi_{1,2}\rangle = |\psi\rangle \pm |\psi^-\rangle$ are also eigenstates of the Hamiltonian with the same energy. Further, $|\psi_1\rangle$ has an even wavefunction and $|\psi_2\rangle$ has an odd wavefunction. State $|\psi\rangle$ can then be expressed as a linear combination of these:

$$|\psi\rangle = \frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle).$$

We have arrived at a contradiction.

Solution to Exercise 3.38. As discussed in Ex. 3.33, energies E below a constant potential level V_0 are associated with eigenwavefunctions $\psi(x) = Ae^{\pm\kappa x}$, with $\kappa = \sqrt{2M(V_0 - E)}/\hbar$. Because of the normalization condition, the wavefunctions cannot have components that exponentially grow at infinity, and thus we must have

$$\psi(x) \to \begin{cases} Ae^{-\kappa x} \text{ at } x \to +\infty \\ A'e^{\kappa x} \text{ at } x \to -\infty \end{cases}$$

In other words, $\psi(x) \to 0$ for $|x| \to \pm\infty$, so we have a bound state.

Conversely, if the energy exceeds the potential at infinity, the eigenwavefunctions tend to $\psi(x) \to Ae^{ikx} + A'e^{-ikx}$, with $k = \sqrt{2M(E - V_0)}/\hbar$. If at least one of the factors A or A' does not vanish, the state is not bound.

Solution to Exercise 3.39. We can write the generic solution to the time-independent Schrödinger equation in this potential using the result of Ex. 3.33:

$$\psi(x) = \begin{cases} B_1 e^{\kappa x} + B_2 e^{-\kappa x}, & x < -a/2 \\ A_1 \cos kx + A_2 \sin kx, & -a/2 \le x \le a/2 \\ B_3 e^{\kappa x} + B_4 e^{-\kappa x}, & x > a/2 \end{cases} \tag{S3.15}$$

We can immediately eliminate the terms $B_2 e^{-\kappa x}$ and $B_3 e^{\kappa x}$ that are exponentially growing at $x \to \pm\infty$ and therefore unphysical.

Next, because the potential is an even function of x, it suffices to look for even and odd solutions of the time-independent Schrödinger equation, as we found in Ex. 3.37. Let us consider these two cases separately.

We write a general *odd* solution as

$$\psi(x) = \begin{cases} -Be^{\kappa x}, & x < -a/2 \\ A \sin kx, & -a/2 \le x \le a/2 \\ Be^{-\kappa x}, & x > a/2 \end{cases} \tag{S3.16}$$

with real A and B and

$$k = \frac{\sqrt{2ME}}{\hbar}, \tag{S3.17a}$$

$$\kappa = \frac{\sqrt{2M(V_0 - E)}}{\hbar}. \tag{S3.17b}$$

Because the potential is finite, both the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ must be continuous. Writing these conditions for the boundary of the box $x = a/2$, we find

$$A \sin kx\big|_{x=a/2} = Be^{-\kappa x}|_{x=a/2};$$
$$Ak \cos kx\big|_{x=a/2} = -\kappa Be^{-\kappa x}|_{x=a/2}$$

or

$$A \sin \frac{ka}{2} = Be^{-\kappa a/2}; \tag{S3.18}$$

$$Ak \cos \frac{ka}{2} = -\kappa Be^{-\kappa a/2}. \tag{S3.19}$$

The continuity condition for $x = -a/2$ yields the same set of equations.

These equations restrict the set of energy values at which the time-independent Schrödinger equation has a solution. To see this, let us divide Eqs. (S3.18) and (S3.19) by each other. We obtain

$$\cot\frac{ka}{2} = -\frac{\kappa}{k}. \tag{S3.20}$$

This equation relates k and κ. Another relation between these quantities is due to Eqs. (S3.17), which can be incorporated into our calculations as follows. Let us denote $ka/2 = \theta$ and $\kappa a/2 = \theta_1$. From Eq. (S3.17) we then have

$$\theta^2 + \theta_1^2 = \theta_0^2,$$

where

$$\theta_0 = \frac{\sqrt{2MV_0}}{\hbar}\frac{a}{2}. \tag{S3.21}$$

Equation (S3.20) now takes the form

$$\cot\theta = -\frac{\theta_1}{\theta} \tag{S3.22}$$

or

$$-\cot\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{S3.23}$$

This equation contains only one unknown variable, θ, which is related to the energy eigenvalue. Unfortunately, this equation is transcendental and cannot be solved in elementary functions.

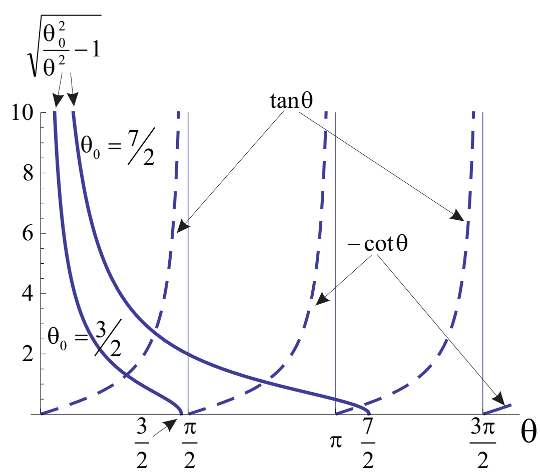

Fig. S3.1 Graphic solution to transcendental equations (S3.23) and (S3.27). The left-hand sides are plotted with dashed lines, right-hand sides with solid lines. Values $\theta_0 = 3/2$ and $\theta_0 = 7/2$ correspond to $V_0 = \frac{9}{2}\frac{\hbar^2}{Ma^2}$ and $V_0 = \frac{49}{2}\frac{\hbar^2}{Ma^2}$, respectively.

A generic *even* solution is given by

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x < -a/2 \\ A\cos kx, & -a/2 \le x \le a/2 \\ Be^{-\kappa x}, & x > a/2 \end{cases} \tag{S3.24}$$

Proceeding in a fashion similar to the odd case, we find the continuity conditions for the boundary of the well

$$A\cos\frac{ka}{2} = Be^{-\kappa a/2}; \tag{S3.25}$$

$$-Ak\sin\frac{ka}{2} = -\kappa Be^{-\kappa a/2}, \tag{S3.26}$$

and the transcendental equation for θ

$$\tan\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{S3.27}$$

By plotting the left and right-hand sides of transcendental equations (S3.23) and (S3.27) as functions of θ, we obtain a graphic solution as shown in Fig. S3.1. The corresponding energies and example wavefunctions are plotted in Fig. 3.2.

It remains to answer the question regarding the dependence of the number of bound states on V_0. As evident from Fig. S3.1, there are N solutions to both transcendental equations when $(N-1)\pi/2 < \theta_0 < N\pi/2$. This corresponds to

$$\frac{[\pi\hbar(N-1)]^2}{2Ma^2} < V_0 < \frac{(\pi\hbar N)^2}{2Ma^2}.$$

Solution to Exercise 3.40. When V_0 is infinite, so is the right-hand side of Eqs. (S3.23) and (S3.27). The tangent in the left-hand side of Eq. (S3.27) takes on a positive infinite value when $\theta = (2j+1)\pi/2$, and the negative cotangent in (S3.23) when $\theta = \pi j$, where j is an arbitrary natural number. The general solution in the limit $V_0 \to \infty$ can then be written as $\theta = n\pi/2$, with n being an arbitrary natural number: an even $n = 2j$ produces an odd solution, and an odd $n = 2j+1$ an even solution. Using $\theta = ka/2$, we find wavenumber values $k_n = n\pi/a$, which correspond to energy eigenvalues

$$E_n = \frac{\hbar^2 k^2}{2M} = \frac{\hbar^2\pi^2 n^2}{2Ma^2}.$$

Substituting this result into Eqs. (S3.18) and (S3.25) we determine that the oscillating parts of the wavefunctions, inside the box,

$$\psi_n(x) = \begin{cases} A\sin\left(\frac{n\pi x}{a}\right), & \text{even } n \\ A\cos\left(\frac{n\pi x}{a}\right), & \text{odd } n. \end{cases}; \tag{S3.28}$$

vanish at $x = \pm a/2$. This implies that $B = 0$ for both odd and even cases, and that the wavefunction vanishes outside the box.

We can now find the normalization constant A. To this end, we integrate the square absolute value of the wavefunction over the real axis. We find, for both even and odd solutions,

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x = \int_{-a/2}^{a/2} |\psi(x)|^2 \mathrm{d}x = |A|^2 \frac{a}{2},$$

so $A = \sqrt{2/a}$.

Solution to Exercise 3.41. Because the potential is an even function of x, we can restrict to even and odd wavefunctions. At $x \neq 0$, the potential is zero. The energy of a bound state must then be negative, so a generic odd solution must be of the form

$$\psi(x) = \begin{cases} -Be^{\kappa x}, & x < 0 \\ Be^{-\kappa x}, & x > 0 \end{cases} \tag{S3.29}$$

with $\kappa = \sqrt{2M(-E)}/\hbar$. Unless $B = 0$ (i.e. $\psi(x) \equiv 0$), the odd wavefunction has a discontinuity at $x = 0$, i.e. is unphysical.

The even solution is given by

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x < 0 \\ Be^{-\kappa x}, & x > 0 \end{cases} \tag{S3.30}$$

The solution (S3.30) is valid for an arbitrary κ at all values of x except $x = 0$. At $x = 0$, its derivative has a discontinuity:

$$\Delta \psi'(x)\big|_{x=0} = -2B\kappa. \tag{S3.31}$$

There is no contradiction with the wavefunction continuity condition (Ex. 3.35), because the potential is singular at $x = 0$. However, the amplitude of the potential imposes a specific condition on the wavefunctions' derivative discontinuity, that is only satisfied for certain values of κ, as we see next.

Let us integrate both sides of the time-independent Schrödinger equation (3.60) over an infinitesimal interval around $x = 0$:

$$-\int_{-0}^{+0} \frac{\hbar^2}{2M} \frac{d^2}{dx^2} \psi(x) = \int_{-0}^{+0} [E - V(x)] \psi(x). \tag{S3.32}$$

Using the fundamental theorem of calculus as well as Eq. (D.9), we find

$$-\frac{\hbar^2}{2M} \Delta \psi'(x)\big|_{x=0} = W_0 \psi(0). \tag{S3.33}$$

Substituting $\psi(0) = B$ as well as Eq. (S3.31) into the above, we find that $\kappa = W_0 m/\hbar^2$ and thus

$$E = -\frac{(\hbar\kappa)^2}{2M} = -\frac{mW_0^2}{2\hbar^2}.$$

Let us now find the normalization coefficient. Because the well is infinitely narrow, we need to only take into account the part of the wavefunction that is localized outside the well. Using Eq. (S3.30), we have

$$\int_{-\infty}^{+\infty} |\psi(x)^2| \mathrm{d}x = 2B^2 \int_{0}^{\infty} e^{-2\kappa x} \mathrm{d}x = B^2, \tag{S3.34}$$

so $B=\sqrt{\kappa}$.

Solution to Exercise 3.42. Since $V_0 a = W_0$ we can rewrite Eq. (S3.21) as

$$\theta_0 = \frac{\sqrt{2MW_0}}{\hbar}\frac{\sqrt{a}}{2}. \tag{S3.35}$$

Because a tends to zero and W_0 is a constant, θ_0 also tends to zero. The solid curves in Fig. S3.1 shrink to a vertical line just next to the vertical axis. Therefore we have only one, even, energy eigenstate, and we rewrite Eq. (S3.27) using the fact that $\tan\theta \approx \theta$ for small θ:

$$\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{S3.36}$$

or

$$\theta^4 + \theta^2 - \theta_0^2 = 0. \tag{S3.37}$$

Therefore

$$\theta^2 = \frac{-1 \pm \sqrt{1+4\theta_0^2}}{2}.$$

We decompose the above solution into the Taylor series with respect to small parameter θ_0^2 to the *second* order (the reason why we need this will become clear shortly): $\sqrt{1+4\theta_0^2} \approx 1 + 2\theta_0^2 - 2\theta_0^4$. Then the two roots of Eq. (S3.37) can be rewritten as

$$\theta^2 \approx \begin{bmatrix} \theta_0^2 - \theta_0^4 \\ -1 - \theta_0^2 + \theta_0^4 \end{bmatrix}. \tag{S3.38}$$

Because we are looking for a bound solution, we expect θ to be real, so we choose the first root. Since $\theta_0 = \sqrt{2MV_0}a/2\hbar$ and $\theta = \sqrt{2ME}a/2\hbar$, we see that

$$E = \frac{2\hbar^2}{Ma^2}\theta^2 = V_0 - \frac{Ma^2V_0^2}{2\hbar^2} = V_0 - \frac{MW_0^2}{2\hbar^2}. \tag{S3.39}$$

We see that the second-order Taylor expansion was necessary to obtain the critical second term in the above equation.

Now, according to Eq. (S3.17b) we have

$$\kappa = \frac{\sqrt{2M(V_0 - E)}}{\hbar} = \frac{MW_0}{\hbar^2}. \tag{S3.40}$$

As we see, this coefficient is independent of a in the limit $a \to 0$ as long as $V_0 a = W_0$ is kept constant, and is the same as found in the previous exercise.

Solution to Exercise 3.43. The particle is initially prepared in the bound state of the original potential (see Ex. 3.41):

$$\psi_0(x) = \sqrt{\kappa_0}\begin{cases} e^{\kappa_0 x}, & x<0 \\ e^{-\kappa_0 x}, & x>0 \end{cases}$$

with $\kappa_0 = W_0 M/\hbar^2$. After the sudden change of the potential, the bound state is given by another wavefunction,

$$\psi_1(x) = \sqrt{\kappa_1} \begin{cases} e^{\kappa_1 x}, & x < 0 \\ e^{-\kappa_1 x}, & x > 0 \end{cases}$$

with $\kappa_1 = 2W_0 M/\hbar^2$. The probability that the particle will remain in the bound state of the new potential is given, according to the Second Postulate, by the squared inner product

$$\begin{aligned} \text{pr} &= |\langle \psi_0|\ \psi_1\rangle|^2 \\ &= \left| \int_{-\infty}^{+\infty} \psi_0^*(x)\psi_1(x)\mathrm{d}x \right|^2 \\ &= \kappa_0\kappa_1 \left| 2\int_0^{+\infty} e^{-\kappa_0 x} e^{-\kappa_1 x}\mathrm{d}x \right|^2 \\ &= \kappa_0\kappa_1 \left| \frac{2}{\kappa_0+\kappa_1} \right|^2 \\ &= \frac{8}{9}. \end{aligned}$$

Solution to Exercise 3.44. We start by following the logic of the solution to Ex. 3.41. The potential outside the wells is zero, so general odd and even wavefunctions in these areas will be of the form

$$\psi(x) = \begin{cases} -B_o e^{\kappa_o x}, & x < -a \\ A_o(e^{\kappa_o x} - e^{-\kappa_o x}), & -a < x < a \\ B_o e^{-\kappa_o x}, & x > a \end{cases} \tag{S3.41}$$

and

$$\psi(x) = \begin{cases} B_e e^{\kappa_e x}, & x < -a \\ A_e(e^{\kappa_e x} + e^{-\kappa_e x}), & -a < x < a \\ B_e e^{-\kappa_e x}, & x > a \end{cases}, \tag{S3.42}$$

respectively, with $\kappa_{e,o} = \sqrt{2M(-E_{e,o})}/\hbar$ (subscripts e and o standing for "even" and "odd", respectively) and both A's and B's real and positive (Fig. S3.2). In contrast to the single well, we cannot exclude the odd solution *a priori*.

Let us look at the even solution in detail. The continuity condition at $x = \pm a$ gives $A_e(e^{\kappa_e a} + e^{-\kappa_e a}) = B_e e^{-\kappa_e a}$, or $B_e = A_e(e^{2\kappa_e a} + 1)$. The discontinuity of the derivative at this point is then

$$\begin{aligned} \Delta\psi'(x)\big|_{x=\pm a} &= -A_e\kappa_e(e^{\kappa_e a} - e^{-\kappa_e a}) - B_e\kappa_e e^{-\kappa_e a} \\ &= -A_e\kappa_e(e^{\kappa_e a} - e^{-\kappa_e a} + e^{\kappa_e a} + e^{-\kappa_e a}) \\ &= -2A_e\kappa_e e^{\kappa_e a} \end{aligned}$$

Equation (S3.43) for our case takes the form

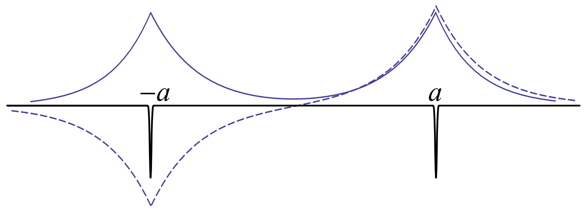

Fig. S3.2 The double delta potential (Ex. 3.44) and the even and odd energy eigenstate wavefunctions.

$$-\frac{\hbar^2}{2M}\left(\Delta\psi'(x)\big|_{x=\pm a}\right) = W_0\psi(\pm a), \tag{S3.43}$$

so using $\psi(\pm a) = A_e(e^{\kappa_e a} + e^{-\kappa_e a})$ we find

$$\kappa_e = \kappa_0(1 + e^{-2\kappa_e a}), \tag{S3.44}$$

where $\kappa_0 = \frac{W_0 M}{\hbar^2}$ is the wavefunction falloff coefficient in the case of the single delta potential (denoted as κ in Ex. 3.41). We see that in the limit of $a \to \infty$, this solution approaches that for a single well.

For a finite distance between the wells, Eq. (S3.44) is transcendental. Let us find the approximate solution for the case $\kappa_0 a \gg 1$. We write $\kappa_e = \kappa_0(1+\delta)$. Then Eq. (S3.44) becomes

$$\delta = e^{-2\kappa_0 a(1+\delta)} = e^{-2\kappa_0 a} e^{-2\delta\kappa_0 a},$$

from which we find that $\delta < e^{-2\kappa_0 a}$ and hence $2\delta\kappa_0 a \ll 1$. Therefore we can write in the first order $e^{-2\delta\kappa_0 a} = 1$, so $\delta = e^{-2\kappa_0 a}$ and

$$\kappa_e = \kappa_0(1 + e^{-2\kappa_0 a}). \tag{S3.45}$$

The corresponding energy is

$$E_e = -\frac{(\hbar\kappa_e)^2}{2M} = -\frac{(\hbar\kappa_0)^2}{2M}(1+\delta)^2 \approx -\frac{W_0^2 M}{2\hbar^2}(1+2\delta) = -\frac{W_0^2 M}{2\hbar^2}(1+2e^{-2\kappa_0 a}). \tag{S3.46}$$

The argument for the odd case is analogous, but in this case the energy shift is opposite:

$$E_o = -\frac{W_0^2 M}{2\hbar^2}(1 - 2e^{-2\kappa_0 a}). \tag{S3.47}$$

Solution to Exercise 3.45. Let $\psi_{\text{single}}(x)$ be the wavefunction (3.71) corresponding to a single delta-function well. Then, for $\kappa_0 a \gg 1$, the odd (S3.41) and even (S3.42) solutions to the double well problem can be approximated as

$$\psi_o(x) = \frac{\psi_{\text{single}}(x-a) - \psi_{\text{single}}(x+a)}{\sqrt{2}} \text{ and } \psi_e(x) = \frac{\psi_{\text{single}}(x-a) + \psi_{\text{single}}(x+a)}{\sqrt{2}}$$

(the factor of $\sqrt{2}$ due to normalization). We now express the localized states through the energy eigenstates as follows:

$$\psi_{\text{single}}(x-a)=\frac{\psi_e(x)+\psi_o(x)}{\sqrt{2}} \text{ and } \psi_{\text{single}}(x+a)=\frac{\psi_e(x)-\psi_o(x)}{\sqrt{2}}.$$

These states are orthogonal to each other to a good approximation.

The initial state's wavefunction is $\psi(x,0)=\psi_{\text{single}}(x-a)$. Knowing the energies $E_{e,o}=E_0\mp\Delta$ of the even and odd states, where

$$E_0=-\frac{W_0^2M}{2\hbar^2} \text{ and } \Delta=\frac{W_0^2M}{\hbar^2}e^{-2\kappa_0 a}$$

as found in Ex. 3.44, we write the evolution as

$$\begin{aligned}
\psi(x,t)&=\frac{1}{\sqrt{2}}\left[\psi_e(x)\exp\left(-i\frac{E_e}{\hbar}t\right)+\psi_o(x)\exp\left(-i\frac{E_o}{\hbar}t\right)\right]\\
&=\frac{1}{\sqrt{2}}\exp\left(-i\frac{E_0}{\hbar}t\right)\left[\psi_e(x)\exp\left(i\frac{\Delta}{\hbar}t\right)+\psi_o(x)\exp\left(-i\frac{\Delta}{\hbar}t\right)\right]\\
&=\frac{1}{2}\exp\left(-i\frac{E_0}{\hbar}t\right)\left[\left(\psi_{\text{single}}(x-a)+\psi_{\text{single}}(x+a)\right)\exp\left(i\frac{\Delta}{\hbar}t\right)\right.\\
&\qquad\left.+\left(\psi_{\text{single}}(x-a)-\psi_{\text{single}}(x+a)\right)\exp\left(-i\frac{\Delta}{\hbar}t\right)\right]\\
&=\exp\left(-i\frac{E_0}{\hbar}t\right)\left[\psi_{\text{single}}(x-a)\cos\left(\frac{\Delta}{\hbar}t\right)+i\psi_{\text{single}}(x+a)\sin\left(\frac{\Delta}{\hbar}t\right)\right].
\end{aligned}$$

Hence the probability to find the system in the state with wavefunction $\psi_{\text{single}}(x+a)$ equals $\sin^2\left(\frac{\Delta}{\hbar}t\right)$.

Solution to Exercise 3.46. Suppose there are two bound states, $|\psi_1\rangle$ and $|\psi_2\rangle$, corresponding to the same energy E. The time-independent Schrödinger equation (3.60) for both states is

$$\frac{\hbar^2}{2M}\psi_1''(x)=[V(x)-E]\psi_1(x)$$

and

$$\frac{\hbar^2}{2M}\psi_2''(x)=[V(x)-E]\psi_2(x).$$

Let us multiply the left-hand side of the first equation by the right-hand side of the second, and vice versa. At all points where $V(x)-E\neq 0$, we have

$$\psi_1''(x)\psi_2(x)=\psi_1(x)\psi_2''(x),$$

or

$$\psi_1''(x)\psi_2(x)-\psi_1(x)\psi_2''(x)=0.$$

This latter equality can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\psi_1'(x)\psi_2(x)-\psi_1(x)\psi_2'(x)\right]=0,$$

from which we conclude that

$$\left[\psi_1'(x)\psi_2(x)-\psi_1(x)\psi_2'(x)\right]=\text{const.}$$

The constant in the right-hand side of the above equation must be zero because it is known that the state is bound, i.e. both wavefunctions and their derivatives vanish at $x \to \pm\infty$. Dividing both sides of the above equation by $\psi_2^2(x)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\psi_1(x)}{\psi_2(x)}\right] = 0$$

at all points where $\psi_2(x) \neq 0$, or

$$\frac{\psi_1(x)}{\psi_2(x)} = \text{const},$$

so the two wavefunctions are proportional to each other.

Admittedly, the above proof does not apply to points at which $\psi_2(x) = 0$ or $V(x) = E$. I suggest to the reader to elaborate these cases as an independent exercise.

Solution to Exercise 3.49. Since the phase velocity of the de Broglie wave with momentum p and wavenumber k is $v_{\text{ph}} = p/2M = \hbar k/2M$, we have the following probability density currents:

$$j_A = \left(\frac{\hbar}{2M}\right) k_0 |A|^2$$

for the incident wave,

$$j_B = \left(\frac{\hbar}{2M}\right) k_0 |A|^2 \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2$$

for the reflected wave, and

$$j_C = \left(\frac{\hbar}{2M}\right) k_1 |A|^2 \left(\frac{2k_0}{k_0 + k_1}\right)^2$$

for the transmitted wave.

Accordingly, the reflection coefficient is

$$\frac{j_B}{j_A} = \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2$$

and the transmission coefficient is

$$\frac{j_C}{j_A} = 4\frac{k_0 k_1}{(k_0 + k_1)^2},$$

and their sum equals identity.

The reflection coefficient tends to 1 for $E \to V_0$ (so $k_1 \to 0$) and to 0 for $E \to \infty$ (so $k_0 - k_1 \to 0$). The behavior of the transmission coefficient is opposite.

Solution to Exercise 3.50. If the energy E is below the potential barrier level, the solution of the time-independent Schrödinger equation after the barrier is a falling exponential:

$$\psi(E, x) = \begin{cases} Ae^{ik_0 x} + Be^{-ik_0 x}, \; x < 0 \\ Ce^{-\kappa x}, \; x \geq 0 \end{cases}, \tag{S3.48}$$

where $k_0 = \sqrt{2ME}/\hbar$, $\kappa = \sqrt{2M(V_0 - E)}/\hbar$. Note that there is no D-wave in this case, because that wave would exhibit exponential growth at $x \to \infty$. The continuity condition now takes the form

$$A + B = C;$$

$$ik_0(A - B) = -\kappa C.$$

This system of two linear equations is readily solved, giving

$$B = A\frac{ik_0 + \kappa}{ik_0 - \kappa};$$

$$C = A\frac{2ik_0}{ik_0 - \kappa}.$$

Because $\left|\frac{ik_0+\kappa}{ik_0-\kappa}\right| = 1$, the amplitudes of the incident and reflected waves (A and B, respectively) have the same absolute value. Furthermore these waves propagate with the same phase and group velocities, and hence have the same probability density current. The reflection coefficient is therefore equal to unity.

Solution to Exercise 3.51. The initial wavepacket can be written in the wavenumber basis according to Eq. (3.52) as

$$|\psi(0)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa a} e^{-\kappa^2 d^2/2} |k_0 + \kappa\rangle \, d\kappa, \tag{S3.49}$$

where κ is small compared to k_0 and k_1. Our goal is to calculate the evolution of this state. In Ex. 3.29, we had the advantage that the momentum eigenstates in the right-hand side of Eq. (S3.49) were automatically the energy eigenstates. Here it is no longer the case. However, under the assumptions that we made it is safe to replace the momentum eigenstates in the above decomposition by corresponding energy eigenstates.

To see this, let us write the energy eigenstates (3.76) in the form

$$\langle x|\ \psi_{\text{bar}}(\kappa)\rangle = Ae^{i(k_0+\kappa)x}\theta(-x) + Be^{-i(k_0+\kappa)x}\theta(-x) + Ce^{i\sqrt{(k_0+\kappa)^2 - \frac{2MV_0}{\hbar^2}}x}\theta(x), \tag{S3.50}$$

where B and C are related to A by Eq. (3.78a). The first term in the above equation (the A-wave) is identical to the wavefunction $\langle x|\ k_0 = \kappa\rangle = \frac{1}{\sqrt{2\pi}}e^{i(k_0+\kappa)x}$ of the state $|k_0 + \kappa\rangle$ on the left side of the barrier for $A = \frac{1}{\sqrt{2\pi}}$. The second term (the B-wave) is also located to the left of the barrier, but has a negative wavenumber. The third term (the C-wave) is located to the right side of the barrier. The original wavepacket is located, almost completely, far to the left of the barrier and consists, almost entirely, of waves with positive wavenumbers. This means that its decomposition (S3.49) can be rewritten as

$$|\psi(0)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa a} e^{-\kappa^2 d^2/2} |\psi_{\text{bar}}(\kappa)\rangle \, d\kappa. \tag{S3.51}$$

Now since each $|\psi_{\text{bar}}(\kappa)\rangle$ is an eigenstate of the Hamiltonian, we can find the time evolution of the above state according to

$$|\psi(t)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}E_\kappa t} e^{-i\kappa a} e^{-\kappa^2 d^2/2} |\psi_{\text{bar}}(\kappa)\rangle \, d\kappa, \tag{S3.52}$$

where the energy of each $|\psi_{\text{bar}}(\kappa)\rangle$ is (neglecting quadratic terms in κ) $E_\kappa = \hbar^2(k_0+\kappa)^2/2M \approx (\hbar^2/2M)(k_0^2 + 2k_0\kappa)$. We find for the state vector

$$|\psi(t)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} e^{-i\hbar k_0^2 t/2M} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{M}t)} e^{-\kappa^2 d^2/2} |\psi_{\text{bar}}(\kappa)\rangle \, d\kappa. \tag{S3.53}$$

and for its wavefunction

$$\psi(x,t) = \left(\frac{d^2}{\pi}\right)^{1/4} e^{-i\hbar k_0^2 t/2M} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{M}t)} e^{-\kappa^2 d^2/2} \langle x|\ \psi_{\text{bar}}(\kappa)\rangle \, d\kappa. \tag{S3.54}$$

We can now calculate the integral (S3.54) for each wave in Eq. (S3.50) separately. The overall phase factor $e^{-i\hbar k_0^2 t/2M}$ can be neglected, as well as the the variation of the amplitudes B and C as a function of the small κ.

A-wave. Applying standard Fourier transform rules (Ex. D.5), we obtain

$$\begin{aligned}
\psi_A(x,t) &= A\theta(-x)e^{ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{M}t)} e^{-\kappa^2 d^2/2} e^{i\kappa x} d\kappa \qquad \text{(S3.55)}\\
&= A\theta(-x)e^{ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \frac{\sqrt{2\pi}}{d} e^{-\frac{\left(x-a-\frac{\hbar k_0}{M}t\right)^2}{2d^2}}\\
&= \theta(-x)e^{ik_0x}\left(\frac{1}{\pi d^2}\right)^{1/4} e^{-\frac{\left(x-a-\frac{\hbar k_0}{M}t\right)^2}{2d^2}}.
\end{aligned}$$

This is a Gaussian wavepacket centered at the point $x = a + \frac{\hbar k_0}{M}t$ and propagating with the speed $\hbar k_0/M$ in the positive direction. When the wavepacket reaches the barrier (i.e. at $t_{\text{bar}} = \frac{-aM}{\hbar k_0}$), it disappears due to the factor $\theta(-x)$. Before this happens, the total probability associated with this wavepacket is $\text{pr}_A = \int_{-\infty}^{+\infty} |\psi_A(x,t)|^2 dx = 1$.

The *B-wave* is treated similarly, except that the integral corresponds to the inverse Fourier transform. We obtain

$$\begin{aligned}
\psi_B(x,t) &= B\theta(-x)e^{-ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \frac{\sqrt{2\pi}}{d} e^{-\frac{\left(x+a+\frac{\hbar k_0}{M}t\right)^2}{2d^2}}\\
&\overset{(3.78a)}{=} \frac{k_0-k_1}{k_0+k_1}\theta(-x)e^{-ik_0x}\left(\frac{1}{\pi d^2}\right)^{1/4} e^{-\frac{\left(x+a+\frac{\hbar k_0}{M}t\right)^2}{2d^2}}.
\end{aligned}$$

This wavepacket is a mirror image of the previous one. At $t = 0$, it is located at $x = -a$ but is "invisible" due to the factor $\theta(-x)$. It propagates in the negative direction. Once it reaches the barrier (simultaneously with the

A-packet), it becomes "visible". This wavepacket is associated with the reflection of the particle from the barrier. The total probability associated with this wavepacket is $\mathrm{pr}_B = \left(\frac{k_0-k_1}{k_0+k_1}\right)^2$.

C-wave. Using $k_1^2 = k_0^2 - 2MV_0/\hbar^2$ and again neglecting terms that are quadratic with respect to κ, we can replace in Eq. (S3.50)

$$\sqrt{(k_0+\kappa)^2 - \frac{2MV_0}{\hbar^2}} = \sqrt{k_0^2 + 2k_0\kappa - \frac{2MV_0}{\hbar^2}} = k_1\sqrt{1+2k_0\kappa/k_1^2} \approx k_1 + \kappa\frac{k_0}{k_1}.$$

As a result, we obtain for the C-wave

$$\begin{aligned}\psi_C(x,t) &= C\theta(x)e^{ik_1x}\left(\frac{d^2}{\pi}\right)^{1/4}\int\limits_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{M}t)}e^{-\kappa^2d^2/2}e^{i\kappa\frac{k_0}{k_1}x}d\kappa \\ &= C\theta(x)e^{ik_1x}\left(\frac{d^2}{\pi}\right)^{1/4}\frac{\sqrt{2\pi}}{d}e^{-\frac{\left(\frac{k_0}{k_1}x-a-\frac{\hbar k_0}{M}t\right)^2}{2d^2}} \\ &\overset{(3.78a)}{=} \frac{2k_0}{k_0+k_1}\theta(x)e^{ik_1x}\left(\frac{1}{\pi d^2}\right)^{1/4}e^{-\frac{\left(\frac{k_0}{k_1}x-a-\frac{\hbar k_0}{M}t\right)^2}{2d^2}}.\end{aligned}$$

This packet is narrower than the other two by a factor k_0/k_1. It begins to exist at $t = t_{\mathrm{bar}}$ and propagates in the positive direction at a speed $\hbar k_1/M$. This wavepacket is associated with the particle transmitted through the barrier and has the probability $\mathrm{pr}_C = \frac{k_1}{k_0}\left(\frac{2k_0}{k_0+k_1}\right)^2$. A direct calculation shows that $\mathrm{pr}_B + \mathrm{pr}_C = 1$.

Solution to Exercise 3.52. Proceeding similarly to Ex. 3.47 we find that the solution is a linear combination of six wavefunctions as shown in Fig. 3.6 and is thus a function of six parameters. For each of the two interfaces, there are two continuity conditions (for the wavefunction and its derivative):

$$A + B = C + D;$$

$$ik_0(A-B) = k_1(C-D);$$
$$Ce^{k_1L} + De^{-k_1L} = E + F;$$
$$k_1(Ce^{k_1L} - De^{-k_1L}) = ik_0(E-F),$$

where $k_0 = \sqrt{2ME}/\hbar$, $k_1 = \sqrt{2M(V_0-E)}/\hbar$. Again, each energy value is doubly degenerate: the linearly independent solutions correspond to the matter waves approaching from the left ($F = 0$) and from the right ($A = 0$). We are interested in the first option and solve the above equations assuming an arbitrary E and working our way to the left. We then find the relation between the incident, transmitted and reflected amplitudes:

$$A = E\left[\cosh k_1L + \frac{i}{2}\left(\frac{k_1}{k_0} - \frac{k_0}{k_1}\right)\sinh(k_1L)\right]; \tag{S3.56}$$

$$B = E\left[-\frac{i}{2}\left(\frac{k_1}{k_0} + \frac{k_0}{k_1}\right)\sinh(k_1L)\right]. \tag{S3.57}$$

The transmission and reflection coefficients are then given by Eqs. (3.81).

Solution to Exercise 3.53. We proceed similarly to Ex. 3.51, writing energy eigenstates in the form

$$\begin{aligned}\langle x|\ \psi_{\rm bar}(\kappa)\rangle = {} & [Ae^{i(k_0+\kappa)x}+Be^{-i(k_0+\kappa)x}]\theta(-x)\\ & +[Ce^{k_1x}+De^{-k_1x}]\theta(x)\theta(L-x)+Ee^{i(k_0+\kappa)(x-L)}\theta(x-L),\end{aligned} \tag{S3.58}$$

where the amplitude factors are related to each other as found in the previous exercise. Equations (S3.51)–(S3.54) apply to our case unchanged, as well as the solution (S3.55) for the A-wave. For the E-wave, on the other hand, we have (approximating E to be independent of κ)

$$\begin{aligned}\psi_E(x,t) &= E\theta(x-L)e^{ik_0(x-L)}\left(\frac{d^2}{\pi}\right)^{1/4}\int\limits_{-\infty}^{+\infty} e^{i\kappa(x-L)}e^{-i\kappa(a+\frac{\hbar k_0}{M}t)}e^{-\kappa^2d^2/2}d\kappa\\ &= E\theta(x-L)e^{ik_0(x-L)}\left(\frac{d^2}{\pi}\right)^{1/4}\frac{\sqrt{2\pi}}{d}e^{-\frac{\left(x-L-a-\frac{\hbar k_0}{M}t\right)^2}{2d^2}}.\end{aligned} \tag{S3.59}$$

The center of the Gaussian wavepacket in the above equation is located at point $x=L+a+\frac{\hbar k_0}{M}t$. Because of factor $\theta(x-L)$, it will emerge from the barrier at the moment when its center position exceeds L, i.e. at the same time $t_{\rm bar}=\frac{-aM}{\hbar k_0}$ when the center of the A-wave enters the barrier at $x=0$.

Solution to Exercise 3.54. Taking a time derivative of both sides of Eq. (3.83a) and substituting $\frac{dp}{dt}$ from Eq. (3.83b) we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{\kappa}{M}x(t).$$

The solution to this differential equation is

$$x(t)=A\cos\omega t+B\sin\omega t \tag{S3.60}$$

where $\omega=\sqrt{\kappa/M}$ and A and B are constants determined ftom the initial conditions. Setting $t=0$ into Eq. (3.54) gives $A=x(0)$. Taking the time derivatives of both sides of that equation we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-A\omega\sin\omega t+B\omega\cos\omega t.$$

Using $p(t)=M\frac{\mathrm{d}x}{\mathrm{d}t}$, we find

$$p(t)=-Am\omega\sin\omega t+Bm\omega\cos\omega t. \tag{S3.61}$$

Setting $t=0$ in the above equation yields $B=\frac{p(0)}{M\omega}$. Substituting A and B into Eqs. (S3.60) and (S3.61) and recalling recall again that $\omega=\sqrt{\kappa/M}$ gives Eqs. (3.84).

Solution to Exercise 3.55. Substituting $x=X/A$, $p=P/B$ into Eqs. (3.84), we obtain

$$X(t) = X(0)\cos\omega t + \frac{1}{M\omega}\frac{A}{B}P(0)\sin\omega t; \tag{S3.62a}$$

$$P(t) = P(0)\cos\omega t - M\omega\frac{B}{A}X(0)\sin\omega t. \tag{S3.62b}$$

In order for these equations to be of form (3.85), we must have

$$\frac{A}{B} = M\omega.$$

On the other hand, the commutator of the rescaled observables satisfies $[\hat{X},\hat{P}] = AB[\hat{x},\hat{p}] = i\hbar AB$. Since we need this commutator to equal i, we obtain the second equation:

$$AB = \frac{1}{\hbar}.$$

Solving these two equations for A and B, we find that

$$A = \sqrt{\frac{M\omega}{\hbar}}; \quad B = \frac{1}{\sqrt{M\omega\hbar}}.$$

Since $\hbar$ has the same dimension as the product of the position and momentum, i.e. kg·m^2/s, the dimension of A is m^{-1} (i.e. the same as x^{-1}) and that of B is s/(kg·m) (i.e. the same as p^{-1}).

Solution to Exercise 3.56.

a) Following the same logic as in Sec. 3.2, we have

$$\langle X|\, X'\rangle = \delta(X-X') = \delta\left[(x-x')\sqrt{\frac{M\omega}{\hbar}}\right] = \sqrt{\frac{\hbar}{M\omega}}\delta(x-x') = \langle x|\, x'\rangle\sqrt{\frac{\hbar}{M\omega}},$$

which means that $|X\rangle = \left(\frac{\hbar}{M\omega}\right)^{1/4}|x\rangle$. Similarly, $|P\rangle = (M\hbar\omega)^{1/4}|p\rangle$.

b) For the de Broglie wave we have

$$\langle X|\, P\rangle = \left(\frac{\hbar}{M\omega}\right)^{1/4}(M\hbar\omega)^{1/4}\langle x|\, p\rangle = \sqrt{\hbar}\frac{1}{\sqrt{2\pi\hbar}}e^{ixp/\hbar} = \frac{1}{\sqrt{2\pi}}e^{iXP}.$$

c)

$$\psi(X) = \langle X|\, \psi\rangle = \left(\frac{\hbar}{M\omega}\right)^{1/4}\langle x|\, \psi\rangle = \left(\frac{\hbar}{M\omega}\right)^{1/4}\psi(x);$$

$$\tilde{\psi}(P) = \langle P|\, \psi\rangle = (M\hbar\omega)^{1/4}\langle p|\, \psi\rangle = (M\hbar\omega)^{1/4}\tilde{\psi}(p).$$

d) Using the resolution of the identity, as well as the result from part (b), we find

$$\langle X|\ \psi\rangle = \int_{-\infty}^{+\infty} \langle X|\ P\rangle \langle P|\ \psi\rangle \,\mathrm{d}P$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\psi}(P) e^{iPX} \mathrm{d}P$$

and

$$\tilde{\psi}(P) = \int_{-\infty}^{+\infty} \langle P|\ X\rangle \langle X|\ \psi\rangle \,\mathrm{d}X$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(X) e^{-iPX} \mathrm{d}X.$$

e) Using the relations from part (d), we proceed in a fashion similar to Ex. 3.20:

$$\begin{aligned}
\left\langle X \right| \hat{P} \left| \psi \right\rangle &= \int_{-\infty}^{+\infty} P \langle X|\ P\rangle \langle P|\ \psi\rangle \,\mathrm{d}P \\
&\stackrel{(3.90)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P e^{iPX} \psi(P) \mathrm{d}P \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-i) \frac{\mathrm{d}}{\mathrm{d}X} e^{iPX} \psi(P) \mathrm{d}P \\
&\stackrel{(3.91)}{=} -i \frac{\mathrm{d}}{\mathrm{d}X} \psi(X)
\end{aligned}$$

The expression for the position operator in the momentum basis is obtained similarly.

f) From (3.87) we find

$$\left\langle \Delta X^2 \right\rangle \left\langle \Delta P^2 \right\rangle = \frac{M\omega}{\hbar} \frac{1}{M\omega\hbar} \left\langle \Delta x^2 \right\rangle \left\langle \Delta p^2 \right\rangle = \frac{1}{\hbar^2} \left\langle \Delta x^2 \right\rangle \left\langle \Delta p^2 \right\rangle. \tag{S3.63}$$

Now using the uncertainty principle (3.50) for the unrescaled position and momentum, we see that the right-hand side of the above equation is greater than of equal to $\frac{1}{4}$.

Solution to Exercise 3.57.

$$H = \frac{\hat{p}^2}{2M} + \frac{M\omega^2 \hat{x}^2}{2} = M\omega\hbar \frac{\hat{P}^2}{2M} + \frac{\hbar}{M\omega} \frac{M\omega^2 \hat{X}^2}{2} = \frac{1}{2}\hbar\omega \left(\hat{X}^2 + \hat{P}^2\right). \tag{S3.64}$$

Solution to Exercise 3.58.

a) Because the position and momentum operators are Hermitian, $\hat{X}^\dagger = \hat{X}$ and $(i\hat{P})^\dagger = -i\hat{P}$. Therefore, $\hat{a}^\dagger = (\hat{X}+i\hat{P})^\dagger/\sqrt{2} = (\hat{X}-i\hat{P})/\sqrt{2}$.
b) From part (a), $\hat{a} \neq \hat{a}^\dagger$.
c) Since $[\hat{X},\hat{P}] = i$,

$$[\hat{a},\hat{a}^\dagger] = \frac{1}{2}[\hat{X}+i\hat{P},\hat{X}-i\hat{P}] = \frac{1}{2}\big([\hat{X},\hat{X}] - i[\hat{X},\hat{P}] + i[\hat{P},\hat{X}] + [\hat{P},\hat{P}]\big) = 1.$$

d) The position and momentum operators are expressed through $\hat{a}$ and $\hat{a}^\dagger$ by solving Eqs. (3.96) and (3.97).
e)

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\hbar\omega(\hat{X}^2+\hat{P}^2)\\
&= \frac{1}{4}\hbar\omega\left[(\hat{a}+\hat{a}^\dagger)^2 + \frac{1}{i^2}(\hat{a}-\hat{a}^\dagger)^2\right]\\
&= \frac{1}{4}\hbar\omega\left[(\hat{a}^2+(\hat{a}^\dagger)^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}) - (\hat{a}^2+(\hat{a}^\dagger)^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a})\right]\\
&= \frac{1}{4}\hbar\omega[2\hat{a}\hat{a}^\dagger+2\hat{a}^\dagger\hat{a}]\\
&\overset{(3.98)}{=} \frac{1}{4}\hbar\omega[2\hat{a}^\dagger\hat{a}+2+2\hat{a}^\dagger\hat{a}]\\
&= \hbar\omega\left[\hat{a}^\dagger\hat{a}+\frac{1}{2}\right].
\end{aligned}$$

f) Using Eq. (A.44b):

$$[\hat{a},\hat{a}^\dagger\hat{a}] = \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a},\hat{a}^\dagger]\hat{a} = \hat{a};$$
$$[\hat{a}^\dagger,\hat{a}^\dagger\hat{a}] = \hat{a}^\dagger[\hat{a}^\dagger,\hat{a}] + [\hat{a}^\dagger,\hat{a}^\dagger]\hat{a} = -\hat{a}^\dagger.$$

Solution to Exercise 3.59.

a) In order to verify if state $\hat{a}|n\rangle$ is an eigenstate of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$, let us subject this state to the action of this operator and employ the result (3.100), rewritten in the form $\hat{n}\hat{a} = \hat{a}\hat{n} - \hat{a}$:

$$\hat{n}\hat{a}|n\rangle = [\hat{a}\hat{n}-\hat{a}]|n\rangle = [\hat{a}n-\hat{a}]|n\rangle = (n-1)\hat{a}|n\rangle,$$

as was required.
b) Similarly, from Eq. (3.100) we find $\hat{n}\hat{a}^\dagger = \hat{a}^\dagger\hat{n} + \hat{a}^\dagger$ and thus

$$\hat{n}\hat{a}^\dagger|n\rangle = [\hat{a}^\dagger\hat{n}+\hat{a}^\dagger]|n\rangle = [\hat{a}^\dagger n+\hat{a}^\dagger]|n\rangle = (n+1)\hat{a}^\dagger|n\rangle.$$

Solution to Exercise 3.60.

a) Let $|\psi\rangle = \hat{a}|n\rangle$. From the previous exercise, we know that $|\psi\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n-1$, i.e. $|\psi\rangle = A|n-1\rangle$, where A is some constant. We need to find A. To this end, we notice that $\langle\psi| = \langle n|\hat{a}^\dagger$ and calculate

$$\langle\psi|\,\psi\rangle = \left\langle n\right|\hat{a}^\dagger\hat{a}\left|\,n\right\rangle = n.$$

But on the other hand,

$$\langle\psi|\ \psi\rangle = |A|^2 \langle n-1|\ n-1\rangle = |A|^2,$$

where in the last equality we have used the fact that the eigenstates of the number operator are normalized. From the last two equations, we find $|A| = \sqrt{n}$.

b) Similarly, if $|\phi\rangle = \hat{a}^\dagger |n\rangle = B\,|n+1\rangle$, then, on the one hand,

$$\langle\phi|\ \phi\rangle = \langle n|\ \hat{a}\hat{a}^\dagger\,|\ n\rangle = \langle n|\ \hat{a}^\dagger\hat{a}+1\,|\ n\rangle = n+1,$$

and on the other hand

$$\langle\phi|\ \phi\rangle = |B|^2 \langle n+1|\ n+1\rangle = |B|^2.$$

Therefore $|B| = \sqrt{n+1}$.

Solution to Exercise 3.61.

$$|n\rangle \overset{(3.103b)}{=} \frac{\hat{a}^\dagger}{\sqrt{n}}\,|n-1\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}\frac{\hat{a}^\dagger}{\sqrt{n-1}}\,|n-2\rangle = \ldots = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\,.$$

Solution to Exercise 3.62. The vacuum state obeys the equation $\hat{a}\,|0\rangle = 0$, or

$$(\hat{X} + i\hat{P})\,|0\rangle = 0. \tag{S3.65}$$

In order to find the wavefunction in the position basis, we use Eq. (3.93) to write the momentum operator in this basis. Equation (S3.65) then becomes

$$\left(X + \frac{\mathrm{d}}{\mathrm{d}X}\right)\psi(X) = 0.$$

This is a first order ordinary differential equation which has a single solution

$$\psi(x) = Ae^{-X^2/2},$$

where A is the normalization constant, calculated in the usual manner:

$$\langle\psi|\ \psi\rangle = \int_{-\infty}^{+\infty} |\psi(X)|^2 \mathrm{d}X = |A|^2 \int_{-\infty}^{+\infty} e^{-X^2}\mathrm{d}X = |A|^2\sqrt{\pi}.$$

Requiring the norm of $|\psi\rangle$ to equal 1, we find $A = \pi^{-1/4}$.

The wavefunction in the momentum basis is calculated similarly.

Solution to Exercise 3.63.

a) The single-photon Fock state is obtained from the vacuum state by applying a single creation operator. Using Eq. (3.93), we express the creation operator in the position basis as

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \simeq \frac{1}{\sqrt{2}}\left(X - \frac{\mathrm{d}}{\mathrm{d}X}\right) \tag{S3.66}$$

and thus the wavefunction of the state $|1\rangle = \hat{a}^\dagger |0\rangle$ is

$$\psi_1(X) = \frac{1}{\sqrt{2}\pi^{1/4}} \left(X - \frac{\mathrm{d}}{\mathrm{d}X} \right) e^{-X^2/2} = \frac{\sqrt{2}}{\pi^{1/4}} X e^{-X^2/2}. \tag{S3.67}$$

The two-photon Fock state is obtained by applying the creation operator to the single-photon state:

$$|2\rangle \overset{(3.103b)}{=} \frac{\hat{a}^\dagger}{\sqrt{2}} |1\rangle .$$

In the position basis,

$$\psi_2(X) = \frac{1}{2} \left(X - \frac{\mathrm{d}}{\mathrm{d}X} \right) \psi_1(x) = \frac{1}{\sqrt{2}\pi^{1/4}} \left(X - \frac{\mathrm{d}}{\mathrm{d}X} \right) X e^{-X^2/2} = \frac{1}{\sqrt{2}\pi^{1/4}} (2X^2 - 1) e^{-X^2/2}. \tag{S3.68}$$

b) We now show by induction that Eq. (3.109) describes the wavefunction of the Fock state $|n\rangle$. First, applying Eqs. (3.109) and (3.110) for $n = 0$, we obtain the vacuum state wavefunction (3.106a). Next, let us suppose Eq. (3.109) is valid for a given $n = k$ and prove that it must be valid for $n = k+1$. We can write the recursion relation $|k+1\rangle = \hat{a}^\dagger |k\rangle / \sqrt{k+1}$ in the position basis using Eq. (S3.66):

$$\psi_{k+1}(X) = \frac{1}{\sqrt{2(k+1)}} \left(X - \frac{\mathrm{d}}{\mathrm{d}X} \right) \psi_k(X).$$

Applying this to Eq. (3.109), we find

$$\begin{aligned}
\psi_{k+1}(X) &= \frac{1}{\sqrt{2(k+1)}} \left(X - \frac{\mathrm{d}}{\mathrm{d}X} \right) \left[\frac{H_k(X)}{\pi^{1/4}\sqrt{2^k k!}} e^{-X^2/2} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{k+1}(k+1)!}} \left[2X H_k(X) e^{-X^2/2} - \left(\frac{\mathrm{d}}{\mathrm{d}X} H_k(X) \right) e^{-X^2/2} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{k+1}(k+1)!}} H_{k+1}(X) e^{-X^2/2},
\end{aligned}$$

which is consistent with Eq. (3.109) for $n = k+1$. To write the last equality, we observed from Eq. (3.110) that

$$H_{k+1}(X) = \left(2X - \frac{\mathrm{d}}{\mathrm{d}X} \right) H_k(X).$$

Solution to Exercise 3.64. The matrices of the two observables can in principle be obtained by integrating the wavefunctions in the position and momentum bases. However, a more elegant way would be expressing these observables in terms of the creation and annihilation operators according to Eq. (3.99). Using Eq. (3.103), we find the matrices of the creation and annihilation operators in the Fock basis as

$$\hat{a} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \quad \hat{a}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{S3.69}$$

Hence

$$\hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \frac{1}{\sqrt{2}} & 0 & \dots \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} & \dots \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \quad \hat{P} = \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i} = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & \frac{1}{\sqrt{2}} & 0 & \dots \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} & \dots \\ 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{S3.70}$$

Solution to Exercise 3.65. For an arbitrary Fock state $|n\rangle$, we have

$$\begin{aligned} \langle n|\,\hat{X}\,|n\rangle &= \frac{1}{\sqrt{2}} \langle n|\,(\hat{a}+\hat{a}^\dagger)\,|n\rangle \\ &= \frac{1}{\sqrt{2}} \langle n| \left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right) \\ &= 0. \end{aligned} \tag{S3.71}$$

Similarly,

$$\langle n|\,\hat{P}\,|n\rangle = 0.$$

For the uncertainties, we have

$$\begin{aligned} \langle \Delta X^2 \rangle &= \langle n|\,\hat{X}^2\,|n\rangle \\ &= \frac{1}{2} \langle n|\,(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)\,|n\rangle \\ &= \frac{1}{2} \langle n| \left[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{n+1}^2\,|n\rangle + \sqrt{n}^2\,|n\rangle + \sqrt{(n+1(n+2)}\,|n+2\rangle\right] \\ &= \frac{1}{2}(2n+1). \end{aligned} \tag{S3.72}$$

The same answer holds for the momentum uncertainty:

$$\langle \Delta P^2 \rangle = \frac{1}{2}(2n+1).$$

Solution to Exercise 3.66.

a) For the evolution of a superposition of multiple Fock states we have

$$|\psi(t)\rangle = \sum_n \psi_n e^{-i\omega t(n+\frac{1}{2})}\,|n\rangle\,.$$

The expectation value of the annihilation operator is then

$$\begin{aligned}
\langle\hat{a}\rangle(t) &= \langle\psi(t)|\,\hat{a}\,|\psi(t)\rangle \\
&= \left(\sum_n \psi_n^* e^{i\omega t(n+\frac{1}{2})}\,\langle n|\right)\hat{a}\left(\sum_m \psi_m e^{-i\omega t(m+\frac{1}{2})}\,|m\rangle\right) \\
&= \sum_n \sqrt{n+1}\,\psi_n^*\psi_{n+1}e^{-i\omega t}.
\end{aligned}$$

Here we used the fact that the annihilation operator couples only consecutive Fock states with $\langle n|\,\hat{a}|\,n+1\rangle = \sqrt{n+1}$. The above result can be rewritten as $\langle\hat{a}\rangle(t) = \langle\hat{a}\rangle(0)e^{-i\omega t}$.
To derive the counterpart expression for the creation operator, we recall that it is the adjoint of the annihilation operator:

$$\left\langle\hat{a}^\dagger\right\rangle(t) = \langle\psi(t)|\,\hat{a}^\dagger\,|\psi(t)\rangle = \langle\psi(t)|\,\hat{a}\,|\psi(t)\rangle^* = \left[\langle\hat{a}\rangle(0)e^{-i\omega t}\right]^* = \langle\hat{a}^\dagger\rangle(0)e^{i\omega t}.$$

b) Writing the position observable as $\hat{X} = (\hat{a}+\hat{a}^\dagger)/\sqrt{2}$, we find

$$\begin{aligned}
\langle X\rangle(t) &= \frac{1}{\sqrt{2}}\left[\langle\hat{a}\rangle(t) + \left\langle\hat{a}^\dagger\right\rangle(t)\right] \\
&= \frac{1}{\sqrt{2}}\left[\langle\hat{a}\rangle(0)e^{-i\omega t} + \left\langle\hat{a}^\dagger\right\rangle(0)e^{i\omega t}\right] \\
&= \frac{1}{\sqrt{2}}\left[\frac{\langle\hat{X}\rangle(0) + i\langle\hat{P}\rangle(0)}{\sqrt{2}}e^{-i\omega t} + \frac{\langle\hat{X}\rangle(0) - i\langle\hat{P}\rangle(0)}{\sqrt{2}}e^{i\omega t}\right] \\
&= \langle X\rangle(0)\cos\omega t + \langle P\rangle(0)\sin\omega t.
\end{aligned}$$

Similarly, for the momentum observable we find

$$\begin{aligned}
\langle P\rangle(t) &= \frac{1}{\sqrt{2}i}\left[\langle\hat{a}\rangle(t) - \left\langle\hat{a}^\dagger\right\rangle(t)\right] \\
&= \frac{1}{\sqrt{2}i}\left[\langle\hat{a}\rangle(0)e^{-i\omega t} - \left\langle\hat{a}^\dagger\right\rangle(0)e^{i\omega t}\right] \\
&= \frac{1}{\sqrt{2}i}\left[\frac{\langle\hat{X}\rangle(0) + i\langle\hat{P}\rangle(0)}{\sqrt{2}}e^{-i\omega t} - \frac{\langle\hat{X}\rangle(0) - i\langle\hat{P}\rangle(0)}{\sqrt{2}}e^{i\omega t}\right] \\
&= \langle P\rangle(0)\cos\omega t - \langle X\rangle(0)\sin\omega t.
\end{aligned}$$

Solution to Exercise 3.67. We will perform the calculation in the position basis. Similarly to Ex. 3.62, we rewrite Eq. (3.115) as

$$\frac{1}{\sqrt{2}}\left(X + \frac{\mathrm{d}}{\mathrm{d}X}\right)\psi(X) = (\mathrm{Re}\,\alpha + i\mathrm{Im}\,\alpha)\,\psi(X).$$

Substituting Eq. (3.116a) in the left-hand side of the above, we find

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(X+\frac{\mathrm{d}}{\mathrm{d}X}\right)\psi_\alpha(X) &= \frac{1}{\sqrt{2}\pi^{1/4}}e^{i\frac{P_\alpha X_\alpha}{2}}\left(X+\frac{\mathrm{d}}{\mathrm{d}X}\right)e^{iP_\alpha X}e^{-\frac{(X-X_\alpha)^2}{2}} \\ &= \frac{1}{\sqrt{2}\pi^{1/4}}e^{i\frac{P_\alpha X_\alpha}{2}}\left[X+iP_\alpha-(X-X_\alpha)\right]e^{iP_\alpha X}e^{-\frac{(X-X_\alpha)^2}{2}} \\ &= \frac{1}{\sqrt{2}}\left(X_\alpha+iP_\alpha\right)\psi_\alpha(X), \end{aligned} \tag{S3.73}$$

so Eq. (3.115) holds provided that $X_\alpha = \sqrt{2}\mathrm{Re}\,\alpha$ and $P_\alpha = \sqrt{2}\mathrm{Im}\,\alpha$.

The wavefunction (3.116b) in the momentum basis is obtained from that in the position basis by means of Fourier transform, similarly to Ex. 3.25.

The means and variances of the position and momentum can be obtained by integrating the wavefunction as in Ex. 3.25. However, one can also employ the approach similar to that used for Fock states in Ex. 3.65. Taking the adjoint of both parts of Eq. (3.115), we find that $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$ and hence

$$\begin{aligned}\langle\alpha|X|\alpha\rangle &= \frac{1}{\sqrt{2}}\left\langle\alpha\right|(\hat{a}+\hat{a}^\dagger)\left|\alpha\right\rangle \\ &= \frac{1}{\sqrt{2}}[\langle\alpha|(\hat{a}|\alpha\rangle)+(\langle\alpha|\hat{a}^\dagger)|\alpha\rangle] \\ &= \frac{1}{\sqrt{2}}[\langle\alpha|(\alpha|\alpha\rangle)+(\langle\alpha|\alpha^*)|\alpha\rangle] \\ &= \frac{\alpha+\alpha^*}{\sqrt{2}} \\ &= X_\alpha. \end{aligned} \tag{S3.74}$$

Similarly,

$$\left\langle\alpha\right|\hat{P}\left|\alpha\right\rangle = \frac{\alpha-\alpha^*}{\sqrt{2}i} = P_\alpha.$$

For the uncertainties, we have

$$\begin{aligned}\left\langle\alpha\right|X^2\left|\alpha\right\rangle &= \frac{1}{2}\left\langle\alpha\right|(\hat{a}\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}^\dagger)\left|\alpha\right\rangle \\ &= \frac{1}{2}\left\langle\alpha\right|(\hat{a}\hat{a}+2\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}^\dagger+1)\left|\alpha\right\rangle \\ &= \frac{1}{2}(\alpha^2+2\alpha^*\alpha+(\alpha^*)^2+1) \\ &= \frac{1}{2}[(\alpha+\alpha^*)^2+1], \end{aligned} \tag{S3.75}$$

so

$$\left\langle\Delta X^2\right\rangle = \left\langle X^2\right\rangle - \langle X\rangle^2 = \frac{1}{2}.$$

The same answer holds for the momentum uncertainty.

Solution to Exercise 3.68. Let us assume some decomposition of the coherent state into the number basis,

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle\,, \tag{S3.76}$$

and apply the definition (3.115) of the coherent state to this decomposition. For the left-hand side of Eq. (3.115), we have in accordance with Eq. (3.103a),

$$\begin{aligned} \hat{a}|\alpha\rangle &= \sum_{n=0}^{\infty} \alpha_n \hat{a}|n\rangle \\ &= \sum_{n=1}^{\infty} \alpha_n \sqrt{n}\,|n-1\rangle \\ &\overset{n'=n-1}{=} \sum_{n'=0}^{\infty} \alpha_{n'+1} \sqrt{n'+1}\,|n'\rangle\,. \end{aligned} \tag{S3.77}$$

We changed the lower summation index from $n = 0$ to $n = 1$ in the second equality above because the term with $n = 0$ comes with the coefficient $\sqrt{0}$ and therefore vanishes.

At the same time, the right-hand side of (3.115) can be written as

$$\alpha|\alpha\rangle = \sum_{n'=0}^{\infty} \alpha\alpha_{n'}\,|n'\rangle\,. \tag{S3.78}$$

Equalizing both sides, we find a recursive relation

$$\alpha_{n'+1} = \frac{\alpha\alpha_{n'}}{\sqrt{n'+1}}, \tag{S3.79}$$

so

$$\begin{aligned} \alpha_1 &= \alpha\alpha_0; \\ \alpha_2 &= \frac{\alpha\alpha_1}{\sqrt{2}} = \frac{\alpha^2\alpha_0}{\sqrt{2}}; \\ \alpha_3 &= \frac{\alpha\alpha_2}{\sqrt{3}} = \frac{\alpha^3\alpha_0}{\sqrt{6}}; \\ &\ldots, \end{aligned} \tag{S3.80}$$

or in general

$$\alpha_n = \frac{\alpha^n\alpha_0}{\sqrt{n!}}. \tag{S3.81}$$

It remains to find such a value of α_0 that state (S3.76) is normalized to one. We find

$$\langle\alpha|\,\alpha\rangle = \sum_{n=0}^{\infty} |\alpha_n|^2 = |\alpha_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!}. \tag{S3.82}$$

Tthe sum in the above expression, is the Taylor decomposition of $e^{|\alpha|^2}$, so we have $\langle\alpha|\,\alpha\rangle = |\alpha_0|^2 e^{|\alpha|^2}$. Requiring $\langle\alpha|\,\alpha\rangle = 1$, we find

$$|\alpha_0|^2 = e^{-|\alpha|^2} \tag{S3.83}$$

or, up to a arbitrary phase factor,

$$\alpha_0 = e^{-|\alpha|^2/2}. \tag{S3.84}$$

Combining Eqs. (S3.81) and (S3.84) we obtain

$$\alpha_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}. \tag{S3.85}$$

Solution to Exercise 3.69. For the Fock decomposition of the coherent state (3.121), we immediately see

$$\langle 0|\,\alpha\rangle = e^{-|\alpha|^2/2}.$$

On the other hand, from the position basis decompositions (wavefunctions) of the vacuum and coherent states [Eqs. (3.106a) and (3.116a), respectively] we find

$$\begin{aligned}
\langle 0|\,\alpha\rangle &= \int\limits_{-\infty}^{+\infty} \psi_0^*(X)\psi_\alpha(X)\mathrm{d}X \\
&= \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty} e^{-\frac{X^2}{2}} e^{-i\frac{P_\alpha X_\alpha}{2}} e^{iP_\alpha X} e^{-\frac{(X-X_\alpha)^2}{2}}\mathrm{d}X \\
&= \frac{1}{\sqrt{\pi}} e^{-i\frac{P_\alpha X_\alpha}{2}} \int\limits_{-\infty}^{+\infty} e^{iP_\alpha X} e^{-\left(X^2 - XX_\alpha + \frac{X_\alpha^2}{2}\right)}\mathrm{d}X \\
&= \frac{1}{\sqrt{\pi}} e^{-i\frac{P_\alpha X_\alpha}{2}} e^{-\frac{X_\alpha^2}{4}} \int\limits_{-\infty}^{+\infty} e^{iP_\alpha X} e^{-\left(X-\frac{X_\alpha}{2}\right)^2}\mathrm{d}X \\
&= \sqrt{2} e^{-i\frac{P_\alpha X_\alpha}{2}} e^{-\frac{X_\alpha^2}{4}} \mathscr{F}^{-1}\left[e^{-\left(X-\frac{X_\alpha}{2}\right)^2}\right](P_\alpha) \\
&\overset{(D.13),(D.16)}{=} e^{-i\frac{P_\alpha X_\alpha}{2}} e^{-\frac{X_\alpha^2}{4}} e^{i\frac{P_\alpha X_\alpha}{2}} e^{-\frac{P_\alpha^2}{4}} \\
&= e^{-\frac{X_\alpha^2+P_\alpha^2}{4}} \\
&\overset{(3.117)}{=} e^{-|\alpha|^2/2}.
\end{aligned}$$

Solution to Exercise 3.70. For the mean energy, we have

$$\langle E\rangle = \left\langle\alpha\right|\hat{H}\left|\,\alpha\right\rangle \overset{(3.101)}{=} \hbar\omega\left\langle\alpha\left|\,\hat{a}^\dagger\hat{a}+\frac{1}{2}\right|\,\alpha\right\rangle = \hbar\omega\left(|\alpha|^2+\frac{1}{2}\right),$$

where we used the definition of the coherent state $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and the Hermitian adjoint to this relation, $\langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^*$.

For the energy variance we find

$$\begin{aligned}\left\langle E^2\right\rangle &= \left\langle\alpha\right|\hat{H}^2\left|\,\alpha\right\rangle \\ &= (\hbar\omega)^2\left\langle\alpha\left|\,\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\right|\,\alpha\right\rangle \\ &\overset{(3.98)}{=} (\hbar\omega)^2\left\langle\alpha\left|\,\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a}+1\right)\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\right|\,\alpha\right\rangle \\ &= (\hbar\omega)^2\left(|\alpha|^4+2|\alpha|^2+\frac{1}{4}\right)\end{aligned}$$

and hence

$$\left\langle\Delta E^2\right\rangle = \left\langle E^2\right\rangle - \langle E\rangle^2 = (\hbar\omega)^2|\alpha|^2.$$

Both these results are in agreement with Eq. (3.123) because $\hat{H} = \hbar\omega(\hat{n}+\frac{1}{2})$.

Solution to Exercise 3.71. Given that the coherent state is decomposed into the Fock basis according to Eq. (3.121) and that each Fock state is an eigenstate of the Hamiltonian with eigenvalue $\hbar\omega(n+1/2)$, we find

$$\begin{aligned}e^{-i\hat{H}t/\hbar}\,|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}e^{-i\hat{H}t/\hbar}\,|n\rangle \\ &= e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega(n+1/2)t}\,|n\rangle \\ &= e^{-i\omega t/2}e^{-|\alpha|^2/2}\sum_n\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\,|n\rangle \\ &= e^{-i\omega t/2}\left|e^{-i\omega t}\alpha\right\rangle. \end{aligned} \tag{S3.86}$$

Solution to Exercise 3.72.

a) According to Eq. (3.124), the coherent state, while evolving, remains a coherent state, i.e. an eigenstate of the annihilation operator. Hence

$$\langle\hat{a}\rangle(t) = \left\langle\alpha e^{-i\omega t}\right|\hat{a}\left|\,\alpha e^{-i\omega t}\right\rangle = \alpha e^{-i\omega t}$$

and

$$\left\langle\hat{a}^\dagger\right\rangle(t) = [\langle\hat{a}\rangle(t)]^* = \alpha e^{i\omega t}.$$

b) Using Eqs. (3.117) and (3.118), we find

$$\begin{aligned}\langle X\rangle(t) &= \sqrt{2}\text{Re}(\alpha e^{-i\omega t})\\ &= \sqrt{2}\text{Re}[(\text{Re}\alpha + i\text{Im}\alpha)(\cos\omega t - i\sin\omega t)]\\ &= \text{Re}[(X_\alpha + iP_\alpha)(\cos\omega t - i\sin\omega t)]\\ &= X_\alpha\cos\omega t + P_\alpha\sin\omega t.\end{aligned}$$

and

$$\begin{aligned}\langle P\rangle(t) &= \sqrt{2}\text{Im}(\alpha e^{-i\omega t})\\ &= \text{Im}[(X_\alpha + iP_\alpha)(\cos\omega t - i\sin\omega t)]\\ &= P_\alpha\cos\omega t - X_\alpha\sin\omega t.\end{aligned}$$

Solution to Exercise 3.73. Decomposing, according to Eq. (3.121),

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \text{ and } |\alpha'\rangle = e^{-|\alpha'|^2/2}\sum_n\frac{(\alpha')^n}{\sqrt{n!}}|n\rangle,$$

we find

$$\begin{aligned}\langle\alpha|\,\alpha'\rangle &= e^{-|\alpha|^2/2-|\alpha'|^2/2}\sum_{n,n'=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\frac{(\alpha')^{n'}}{\sqrt{n'!}}\langle n|\,n'\rangle\\ &= e^{-|\alpha|^2/2-|\alpha'|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha')^n}{n!}\\ &= e^{-|\alpha|^2/2-|\alpha'|^2/2}e^{\alpha^*\alpha'}.\end{aligned}$$

Solution to Exercise 3.74. Suppose there exists an eigenstate of the creation operator:

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle, \tag{S3.87}$$

where β is the eigenvalue. Let us assume some decomposition of this state into the number basis:

$$|\beta\rangle = \sum_{n=0}^{\infty}\beta_n|n\rangle. \tag{S3.88}$$

Substituting this decomposition into Eq. (S3.87), we find

$$\sum_{n=0}^{\infty}\beta_n\sqrt{n+1}\,|n+1\rangle = \beta\sum_{n=0}^{\infty}\beta_n|n\rangle. \tag{S3.89}$$

In the left-hand side of this equation, the vacuum state $|0\rangle$ is absent. This means it must also be absent in the right-hand side, so either $\beta = 0$ or $\beta_0 = 0$. If $\beta = 0$, the entire right-hand side of Eq. (S3.89) vanishes, and so does the left-hand side, hence all $\beta_i = 0$. On the other hand, if $\beta_0 = 0$, the left-hand side is also missing the term with the first Fock state $|1\rangle$, which in turn compels us to conclude that $\beta_1 = 0$. Continuing this chain of argument, we find that also in this case all β_i must vanish, so $|\beta\rangle = 0$.

Solution to Exercise 3.75. In the Schrödinger picture

$$|\psi(t)\rangle = e^{-i(\hat{H}/\hbar)t}|\psi_0\rangle \tag{S3.90}$$

from which the expectation value of $\hat{A}$ is

$$\left\langle \psi(t)\right| \hat{A} \left| \psi(t)\right\rangle = \left\langle \psi_0 \left| e^{i(\hat{H}/\hbar)t}\hat{A}e^{-i(\hat{H}/\hbar)t}\right| \psi_0 \right\rangle$$

which is the same as the expectation value of operator (3.126) evolving in accordance with the Heisenberg picture.

Solution to Exercise 3.76. Let us differentiate both sides of Eq. (3.126) with respect to time.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)e^{-\frac{i}{\hbar}\hat{H}t}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{i}{\hbar}\hat{H}t}\right)\hat{A}(0)e^{-\frac{i}{\hbar}\hat{H}t} + e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{i}{\hbar}\hat{H}t}\right) \\
&= \frac{i}{\hbar}\left(\hat{H}e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)e^{-\frac{i}{\hbar}\hat{H}t} - e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)\hat{H}e^{-\frac{i}{\hbar}\hat{H}t}\right) \\
&= \frac{i}{\hbar}\left(\hat{H}e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)e^{-\frac{i}{\hbar}\hat{H}t} - e^{\frac{i}{\hbar}\hat{H}t}\hat{A}(0)e^{-\frac{i}{\hbar}\hat{H}t}\hat{H}\right)
\end{aligned}$$

where the last line follows from the commutativity of $\hat{H}$ and $e^{i\hat{H}/\hbar}$. Hence we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) &= \frac{i}{\hbar}\left(\hat{H}\hat{A}(t) - \hat{A}(t)\hat{H}\right) \\
&= \frac{i}{\hbar}\left[\hat{H},\hat{A}(t)\right].
\end{aligned}$$

Solution to Exercise 3.77. Using the Heisenberg equation, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = \frac{i}{\hbar}[\hat{H},\hat{x}] = \frac{i}{\hbar}\frac{1}{2M}[\hat{p}^2,\hat{x}] \overset{(3.49),(A.46)}{=} \frac{i}{\hbar}\frac{1}{2M}(-2i\hbar\hat{p}) = \frac{\hat{p}}{M}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p} = \frac{i}{\hbar}[\hat{H},\hat{p}] = \frac{i}{\hbar}\frac{\kappa}{2}[\hat{x}^2,\hat{p}] = \frac{i}{\hbar}\frac{\kappa}{2}(2i\hbar\hat{x}) = -\kappa\hat{x}.$$

Solution to Exercise 3.78. The derivation of Eq. (3.132a) under Hamiltonian 3.55 is identical to that in the previous exercise. To derive Eq. (3.132b), we decompose the potential into a power series with respect to $\hat{x}$:

$$V(\hat{x}) = \sum_{n=0}^{\infty} V_n\hat{x}^n. \tag{S3.91}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p} = \frac{i}{\hbar}[V(\hat{x}),\hat{p}] \overset{(3.49),(A.46)}{=} \frac{i}{\hbar}\sum_{n=0}^{\infty} n\hat{x}^{n-1}(i\hbar) = -\sum_{n=0}^{\infty} n\hat{x}^{n-1}.$$

The resulting expression equals $-V'(\hat{x})$ according to Eq. (S3.91).

Solution to Exercise 3.79. The evolution operator is a function of the Hamiltonian and hence commutes with it. Therefore

$$\hat{H}(t) = e^{\frac{i}{\hbar}\hat{H}(0)t}\hat{H}(0)e^{-\frac{i}{\hbar}\hat{H}(0)t} = \hat{H}(0)e^{\frac{i}{\hbar}\hat{H}(0)t}e^{-\frac{i}{\hbar}\hat{H}(0)t} = \hat{H}(0).$$

Solution to Exercise 3.80. The position and momentum operators evolve in the Heisenberg picture according to

$$\hat{x}(t) = \hat{U}^\dagger(t)\hat{x}(0)\hat{U}(t);$$
$$\hat{p}(t) = \hat{U}^\dagger(t)\hat{p}(0)\hat{U}(t),$$

where $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ is the evolution operator. Substituting these into the right-hand side of Eq. (3.137) and using the power decomposition (S3.91) of the potential, we find

$$\begin{aligned}
V(\hat{x}(t)) + \frac{\hat{p}(t)^2}{2M} &= \sum_{n=0}^{\infty} V_n\left[\hat{U}^\dagger(t)\hat{x}(0)\hat{U}(t)\right]^n + \frac{1}{2M}\left[\hat{U}^\dagger(t)\hat{p}(0)\hat{U}(t)\right]^2 \\
&= \sum_{n=0}^{\infty} V_n\hat{U}^\dagger(t)\left[\hat{x}(0)\right]^n\hat{U}(t) + \frac{1}{2M}\hat{U}^\dagger(t)\left[\hat{p}(0)\right]^2\hat{U}(t) \\
&= \hat{U}^\dagger(t)\left[\sum_{n=0}^{\infty} V_n\hat{x}(0)^n\right]\hat{U}(t) + \hat{U}^\dagger(t)\left[\frac{\hat{p}(0)^2}{2M}\right]\hat{U}(t) \\
&= \hat{U}^\dagger(t)\hat{H}(0)\hat{U}(t) \\
&\overset{\text{Ex. 3.79}}{=} \hat{H}(0).
\end{aligned}$$

For the second equality in the chain above, we used the unitarity of the evolution operator $\hat{U}(t)\hat{U}^\dagger(t) = \hat{\mathbf{1}}$, e.g. as in

$$\left[\hat{U}^\dagger(t)\hat{p}(0)\hat{U}(t)\right]^2 = \hat{U}^\dagger(t)\hat{p}(0)\hat{U}(t)\hat{U}^\dagger(t)\hat{p}(0)\hat{U}(t) = \hat{U}^\dagger(t)\hat{p}(0)^2\hat{U}(t).$$

We thus find that the right-hand sides of Eqs. (3.136) and (3.137) are equal.

Solution to Exercise 3.81. The power decomposition of a multi-variate function is a sum of form

$$f\big((\hat{A}_1(t),\ldots,\hat{A}_m(t)\big) = \sum_{j=0}^{\infty} C_j\hat{A}^{(j,1)}(t)\hat{A}^{(j,2)}(t)\ldots A^{(j,N_j)}(t),$$

where C_j is a constant coefficient and each $A^{(j,1)}(t)$ is one of the operators $\hat{A}_1(t),\ldots,\hat{A}_m(t)$. Substituting the expression for the Heisenberg evolution of these operators, we find

$$
\begin{aligned}
f\big((\hat{A}_1(t),\dots,\hat{A}_m(t)\big) &= \sum_{j=0}^{\infty} C_j \left[\hat{U}^\dagger(t)\hat{A}^{(j,1)}(0)\hat{U}(t)\right]\left[\hat{U}^\dagger(t)\hat{A}^{(j,2)}(0)\hat{U}(t)\right]\dots\left[\hat{U}^\dagger(t)A^{(j,N_j)}(0)\hat{U}(t)\right]\\
&= \sum_{j=0}^{\infty} C_j \hat{U}^\dagger(t)\hat{A}^{(j,1)}(0)\hat{A}^{(j,2)}(0)\dots A^{(j,N_j)}(0)\hat{U}(0)\\
&= \hat{U}^\dagger(t)\left[\sum_{j=0}^{\infty} C_j \hat{A}^{(j,1)}(0)\hat{A}^{(j,2)}(0)\dots A^{(j,N_j)}(0)\right]\hat{U}(t)\\
&= \hat{U}^\dagger(t) f\big((\hat{A}_1(0),\dots,\hat{A}_m(0)\big)\hat{U}(t)\\
&= \hat{U}^\dagger(t)\hat{B}(0)\hat{U}(t)\\
&= \hat{B}(t).
\end{aligned}
$$

Solution to Exercise 3.82.

$$
\begin{aligned}
[\hat{x}(t),\hat{p}(t)] &= \hat{x}(t)\hat{p}(t)-\hat{p}(t)\hat{x}(t)\\
&= e^{\frac{i}{\hbar}\hat{H}t}\hat{x}(0)e^{-\frac{i}{\hbar}\hat{H}t}e^{\frac{i}{\hbar}\hat{H}t}\hat{p}(0)e^{-\frac{i}{\hbar}\hat{H}t} - e^{\frac{i}{\hbar}\hat{H}t}\hat{p}(0)e^{-\frac{i}{\hbar}\hat{H}t}e^{\frac{i}{\hbar}\hat{H}t}\hat{x}(0)e^{-\frac{i}{\hbar}\hat{H}t}\\
&= e^{\frac{i}{\hbar}\hat{H}t}\hat{x}(0)\hat{p}(0)e^{-\frac{i}{\hbar}\hat{H}t} - e^{\frac{i}{\hbar}\hat{H}t}\hat{p}(0)\hat{x}(0)e^{-\frac{i}{\hbar}\hat{H}t}\\
&= e^{\frac{i}{\hbar}\hat{H}t}[\hat{x}(0),\hat{p}(0)]e^{-\frac{i}{\hbar}\hat{H}t}\\
&= i\hbar.
\end{aligned}
$$

Solution to Exercise 3.83. Substituting solution (3.130) into Hamiltonian (3.82) and using $\omega=\sqrt{\frac{\kappa}{M}}$, we find

$$
\begin{aligned}
\frac{\hat{p}(t)^2}{2M}+\frac{\kappa\hat{x}(t)^2}{2} &= \frac{\kappa}{2}\left[\hat{x}(0)\cos\omega t+\frac{1}{M\omega}\hat{p}(0)\sin\omega t\right]^2+\frac{1}{2M}\left[\hat{p}(0)\cos\omega t-\frac{\kappa}{\omega}\hat{x}(0)\sin\omega t\right]^2\\
&= \frac{\hat{p}(0)^2}{2M}+\frac{\kappa\hat{x}(0)^2}{2}.
\end{aligned}
$$

Solution to Exercise 3.84. The Heisenberg equation for the position and momentum takes the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} &= \frac{i}{\hbar}[\hat{H},\hat{x}] = \frac{i}{\hbar}\beta(-i\hbar)=\beta;\\
\frac{\mathrm{d}}{\mathrm{d}t}\hat{p} &= \frac{i}{\hbar}[\hat{H},\hat{p}] = 0.
\end{aligned}
$$

Evolution for time $t_0=x_0/\beta$ will lead to displacement (3.142).

Solution to Exercise 3.85. The displacement operator is a complex exponential of a Hermitian operator, and hence unitary according to Ex. A.92. Hence $\hat{D}_x^\dagger(x)=\hat{D}_x^{-1}(x)$. Further, using Eq. (3.144) we find

$$
D_x(x)D_x(-x)=e^{-\frac{i}{\hbar}\hat{p}x_0}e^{\frac{i}{\hbar}\hat{p}x_0}=\hat{\mathbf{1}},
$$

and hence $\hat{D}_x(-x) = D_x^{-1}(x)$.

Solution to Exercise 3.86.

a) Let us first rewrite $|x\rangle$ in the momentum basis:

$$e^{-i\hat{p}x_0/\hbar}|x\rangle \stackrel{(3.27a)}{=} \frac{1}{\sqrt{2\pi\hbar}} e^{-i\hat{p}x_0/\hbar} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}|p\rangle\,\mathrm{d}p.$$

Each eigenstate $|p\rangle$ of the momentum observable is also an eigenstate of $e^{-i\hat{p}x_0/\hbar}$. Hence the above expression can be rewritten as

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-ip(x+x_0)/\hbar}|p\rangle\,\mathrm{d}p = |x+x_0\rangle\,.$$

b) Denoting the wavefunction of the displaced state as $\psi_d(x)$ we find

$$\begin{aligned}
\psi_d(x) &= \langle x|\, e^{-i\hat{p}x_0/\hbar}\,|\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle x|\, e^{-i\hat{p}x_0/\hbar}\,|x'\rangle \langle x'|\,\psi\rangle\,\mathrm{d}x' \\
&\stackrel{(3.145)}{=} \int_{-\infty}^{+\infty} \langle x|\, x'+x_0\rangle \langle x'|\,\psi\rangle\,\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty} \delta(x'+x_0-x)\psi(x')\mathrm{d}x' \\
&= \psi(x-x_0).
\end{aligned}$$

c) This follows directly from Ex. A.85.

d) If $|\psi\rangle = \int_{-\infty}^{+\infty} \tilde{\psi}(p)\,|p\rangle\,\mathrm{d}p$ then

$$e^{-i\hat{p}x_0/\hbar}|\psi\rangle = \int_{-\infty}^{+\infty} e^{-ipx_0/\hbar}\tilde{\psi}(p)\,|p\rangle\,\mathrm{d}p. \qquad \text{(S3.92)}$$

The wavefunction of this state in the momentum basis is $\tilde{\psi}_d(p) = e^{-ipx_0/\hbar}\tilde{\psi}(p)$.

Solution to Exercise 3.87.

a) In the Heisenberg picture, we have $\hat{x}(t) = \hat{x}(0) + x_0$ and $\hat{p}(t) = \hat{p}(0)$. Hence

$$\langle x(t)\rangle = \langle\psi|\,\hat{x}(0)|\,\psi\rangle + \langle\psi|\,x_0|\,\psi\rangle = \langle\psi|\,\hat{x}(0)|\,\psi\rangle + x_0\,\langle\psi|\,\psi\rangle = \langle x(0)\rangle + x_0$$

and

$$\langle p(t)\rangle = \langle\psi|\,\hat{p}(t)|\,\psi\rangle = \langle\psi|\,\hat{p}(0)|\,\psi\rangle = \langle x(0)\rangle\,.$$

In the Schrödinger picture we can argue that, since the operator displaces the whole wavefunction by x_0 (Fig. 3.11), it must also add x_0 to the mean position value. Formally this can be expressed as follows. For the mean position value in the state $|\psi_d\rangle = e^{-i\hat{p}x_0/\hbar}|\psi\rangle$, we have

$$\begin{aligned}\langle x\rangle_{|\psi_d\rangle} &= \int_{-\infty}^{+\infty} x|\psi(x-x_0)|^2\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}(x-x_0)|\psi(x-x_0)|^2\mathrm{d}x + x_0\int_{-\infty}^{+\infty}|\psi(x-x_0)|^2\mathrm{d}x\end{aligned} \tag{S3.93}$$

The first term in the expression above is equal to $\langle x\rangle_{|\psi\rangle}$ (we can see that by replacing the integration variable to $x' = x - x_0$). The second term is x_0 because the wavefunction is normalized.

For the momentum uncertainty, we obtain from Ex. 3.86(d) that $|\tilde{\psi}_d(p)|^2 = |\tilde{\psi}(p)|^2$ and hence

$$\begin{aligned}\langle p\rangle_{|\psi_d\rangle} &= \int_{-\infty}^{+\infty} p|\tilde{\psi}_d(p)|^2\mathrm{d}p \\ &= \int_{-\infty}^{+\infty} p|\tilde{\psi}(p)|^2\mathrm{d}p \\ &= \langle p\rangle_{|\psi\rangle}.\end{aligned} \tag{S3.94}$$

b) The fact that the uncertainties of the position and momentum of the displaced state are the same as those in the original state is, again, intuitive (Fig. 3.11). A rigorous proof can be done as follows.
In the Heisenberg picture, we have

$$\begin{aligned}\left\langle \Delta x(t)^2\right\rangle &= \left\langle x(t)^2\right\rangle - \langle x(t)\rangle^2 \\ &= \left\langle (x(0)+x_0)^2\right\rangle - (\langle x(0)\rangle + x_0)^2 \\ &= \left\langle (x(0)^2\right\rangle + 2x_0\langle x(0)\rangle + x_0^2 - \langle x(0)\rangle^2 - 2x_0\langle x(0)\rangle - x_0^2 \\ &= \left\langle (x(0)^2\right\rangle - \langle x(0)\rangle^2 \\ &= \left\langle \Delta x(0)^2\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle \Delta p(t)^2\right\rangle &= \left\langle p(t)^2\right\rangle - \langle p(t)\rangle^2 \\ &= \left\langle p(0)^2\right\rangle - \langle p(0)\rangle^2 \\ &= \left\langle \Delta p(0)^2\right\rangle\end{aligned}$$

In the Schrödinger picture, we have for the position:

$$\begin{aligned}
\left\langle \Delta x^2 \right\rangle_{|\psi_d\rangle} &= \int_{-\infty}^{+\infty} \left(x - \langle x\rangle_{|\psi_d\rangle}\right)^2 |\psi_d(x)|^2 \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \left(x - \langle x\rangle_{|\psi\rangle} - x_0\right)^2 |\psi(x-x_0)|^2 \mathrm{d}x \\
&\overset{x'=x-x_0}{=} \int_{-\infty}^{+\infty} \left(x' - \langle x\rangle_{|\psi\rangle}\right)^2 |\psi(x')|^2 \mathrm{d}x' \\
&= \left\langle \left(x - \langle x\rangle_{|\psi\rangle}\right)^2 \right\rangle_{|\psi\rangle} \\
&= \left\langle \Delta x^2 \right\rangle_{|\psi\rangle} .
\end{aligned} \tag{S3.95}$$

and for the momentum:

$$\begin{aligned}
\left\langle \Delta p^2 \right\rangle_{|\psi_d\rangle} &= \int_{-\infty}^{+\infty} \left(p - \langle p\rangle_{|\psi_d\rangle}\right)^2 |\tilde{\psi}_d(p)|^2 \mathrm{d}p \\
&= \int_{-\infty}^{+\infty} \left(p - \langle p\rangle_{|\psi\rangle}\right)^2 |\tilde{\psi}(p)|^2 \mathrm{d}p \\
&= \left\langle \Delta p^2 \right\rangle_{|\psi\rangle} .
\end{aligned} \tag{S3.96}$$

Solution to Exercise 3.88. The proof proceeds similarly to Ex. 3.86. For example:

$$\begin{aligned}
e^{i\hat{x}p_0/\hbar} |p\rangle &= \frac{1}{2\pi\hbar} e^{i\hat{x}p_0/\hbar} \int_{-\infty}^{+\infty} e^{ipx/\hbar} |x\rangle \, \mathrm{d}x \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{ix(p+p_0)/\hbar} |x\rangle \, \mathrm{d}x \\
&= |p+p_0\rangle .
\end{aligned}$$

Solution to Exercise 3.89.

a) Similarly to the case discussed in Ex. 3.86(c), the action of the momentum displacement operator in the position basis corresponds to multiplication by a complex exponential:

$$\left\langle x \middle| e^{ip_0\hat{x}/\hbar} \middle| \psi \right\rangle = e^{ip_0x/\hbar} \langle x | \psi \rangle = e^{ip_0x/\hbar} \psi(x).$$

Here we used the fact that vector $\langle x|$ is an eigenstate of the operator $e^{ip_0\hat{x}/\hbar}$.

b) We are acting with the position displacement operator on the state $e^{ip_0\hat{x}/\hbar}|\psi\rangle$, whose wavefunction we found in part (a). This results in a shift of the argument by x_0, i.e. a state with wavefunction

$$\left\langle x\middle| e^{-ix_0\hat{p}/\hbar}e^{ip_0\hat{x}/\hbar}\middle| \psi\right\rangle = e^{ip_0(x-x_0)/\hbar}\psi(x-x_0).$$

c) First applying the position displacement operator to state $|\psi\rangle$, we obtain the state with the wavefunction $\psi(x-x_0)$. A subsequent application of the momentum displacement multiplies this wavefunction by $e^{ip_0x/\hbar}$ [as we found in part (a)], so we obtain

$$\left\langle x\middle| e^{ip_0\hat{x}/\hbar}e^{-ix_0\hat{p}/\hbar}\middle| \psi\right\rangle = e^{ip_0x/\hbar}\psi(x-x_0).$$

Solution to Exercise 3.90. Equation (3.147) directly obtains from the Baker-Hausdorff-Campbell formula (A.54) if we set $\hat{A} = \frac{i}{\hbar}p_0\hat{x}$ and $\hat{B} = -\frac{i}{\hbar}x_0\hat{p}$. Then $c = \frac{1}{i}[\hat{A},\hat{B}] = \frac{1}{\hbar}x_0p_0$.

Solution to Exercise 3.91. The Hamiltonian leading to the phase-space displacement is $\hat{H} = \beta_x\hat{p} - \beta_p\hat{x}$, where $\beta_x = x_0/t_0$ and $\beta_p = p_0/t_0$, with t_0 being the amount of displacement. Indeed, in this case we have in the Heisenberg picture

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} &= \frac{i}{\hbar}[\hat{H},\hat{x}] = \frac{i}{\hbar}\beta_x(-i\hbar) = \beta_x;\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{p} &= \frac{i}{\hbar}[\hat{H},\hat{p}] = -\frac{i}{\hbar}\beta_p(i\hbar) = \beta_p.\end{aligned}$$

Solution to Exercise 3.92. In order to verify that the vector $e^{-\frac{i}{\hbar}\hat{H}t}|x\rangle$ is an eigenvector of operator $\hat{x}$, let us subject it to the action of that operator.

$$\begin{aligned}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}|x\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}|x\rangle\\ &\overset{(3.148)}{=} e^{-\frac{i}{\hbar}\hat{H}t}f(\hat{x},t)|x\rangle\\ &= e^{-\frac{i}{\hbar}\hat{H}t}f(x,t)|x\rangle\\ &= f(x,t)e^{-\frac{i}{\hbar}\hat{H}t}|x\rangle.\end{aligned}$$

In the third equality above, we used the fact that an eigenstate of an operator with eigenvalue x is also an eigenstate of a function of that operator with eigenvalue $f(x,t)$ (Sec. A.11). The result of this calculation shows that the action of the operator $\hat{x}$ on vector $e^{-\frac{i}{\hbar}\hat{H}t}|x\rangle$ is equivalent to multiplication by the scalar $f(x,t)$, which is what was required to prove.

Solution to Exercise 3.93. Because $f(x,t)$ is an invertible function, the overlap $\langle f(x,t)|\, f(x',t)\rangle$ takes nonzero values only in the ifinitesimal interval of $x \approx x'$. Decomposing $f(x',t)$ in the neighborhood of x as $f(x',t) \approx f(x,t) + f'(x,t)(x'-x)$, we find

$$\left\langle f(x,t)\middle|\, f(x',t)\right\rangle = \delta(f(x,t)-f(x',t)) \approx \delta(f'(x,t)(x-x')) \overset{(D.6)}{=} \frac{\delta(x-x')}{f'(x,t)}.$$

Solution to Exercise 3.94. Applying Eq. (3.149) to arbitrary x and x' and taking the inner product of both sides of the two resulting equations, we find

$$\left\langle x\right| x'\left.\right\rangle = |K(x,t)|^2 \left\langle f(x,t)\right| f(x',t)\left.\right\rangle . \tag{S3.97}$$

Now using $\langle x| x'\rangle = \delta(x-x')$ as well as Eq. (3.150), we obtain the desired result.

Solution to Exercise 3.95. We can write Eq. (3.149) for a negative time as follows:

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}|x\rangle = K(x,-t)\,|f(x,-t)\rangle .$$

Now taking the adjoint of both parts of the above and using the definition of the wavefunction, we write

$$\psi(x,t) = \langle x|\ \psi(t)\rangle = \left\langle x\right|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}\left|\ \psi(0)\right\rangle = K^*(x,-t)\,\langle f(x,-t)|\ \psi(0)\rangle = K^*(x,-t)\psi(f(x,-t),0).$$

Solution to Exercise 3.96. Let the evolution operator $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}$ corresponding to the negative time $-t$ act upon both sides of Eq. (3.149). We obtain

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|x\rangle = K(x,t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}|f(x,t)\rangle = K(x,t)K(x,-t)\,|f(f(x,t),-t)\rangle .$$

The left-hand site of this equation is also equal to $|x\rangle$ for any x. This means that $f(f(x,t),-t) = x$ [which means that $f(x,-t) = f^{-1}(x,t)$] and $K(x,t)K(x,-t) = 1$. Combining the latter result with Eq. (3.151), we obtain $|K(x,-t)|^2 = \frac{1}{f'(x,t)}$.

Solution to Exercise 3.97. Combining the results of Exercises 3.95 and 3.96, we find

$$|\psi(x,t)|^2 = |K(x,-t)|^2|\psi(f(x,-t),0)|^2 = \frac{1}{|f'(x,t)|}|\psi(f^{-1}(x,t),0)|^2.$$

Solution to Exercise 3.98. We notice that the displacement operator can be written as evolution

$$\hat{D}_{XP}(X_0,P_0) = e^{iP_0\hat{X}-iX_0\hat{P}} = e^{-\frac{i}{\hbar}\hat{H}t} \tag{S3.98}$$

under Hamiltonian

$$\hat{H} = \hbar\omega(-P_0\hat{X} + X_0\hat{P}),$$

where $\omega = 1/t$. Consequently,

$$\hat{D}^\dagger_{XP}(X_0,P_0) = e^{\frac{i}{\hbar}\hat{H}t}. \tag{S3.99}$$

The position and momentum operators evolve under this Hamiltonian under the Heisenberg picture according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X} = \frac{i}{\hbar}[\hat{H},\hat{X}] = i\omega X_0[\hat{P},\hat{X}] \overset{(3.86)}{=} \omega X_0;$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P} = \frac{i}{\hbar}[\hat{H},\hat{P}] = -i\omega P_0[\hat{X},\hat{P}] = \omega P_0.$$

Hence

$$\hat{X}(t) = \hat{X}(0) + X_0;$$
$$\hat{P}(t) = \hat{P}(0) + P_0.$$

As we know from Eq. (3.126),

$$\hat{X}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{X}(0)e^{-\frac{i}{\hbar}\hat{H}t} = \hat{D}^\dagger_{XP}(X_0,P_0)\hat{X}(0)\hat{D}_{XP}(X_0,P_0);$$
$$\hat{P}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{P}(0)e^{-\frac{i}{\hbar}\hat{H}t} = \hat{D}^\dagger_{XP}(X_0,P_0)\hat{P}(0)\hat{D}_{XP}(X_0,P_0).$$

Bringing these results together, we obtain Eqs. (3.154a,b).

For the annihilation operator, we use its definition (3.96) to write

$$\hat{a}(t) = \frac{\hat{X}(t) + i\hat{P}(t)}{\sqrt{2}} = \frac{\hat{X}(0) + i\hat{P}(0)}{\sqrt{2}} + \frac{X_0 + iP_0}{\sqrt{2}} = \hat{a}(0) + \frac{X_0 + iP_0}{\sqrt{2}}.$$

Solution to Exercise 3.99. We know from Eq. (3.154c) that in the Heisenberg picture the displacement operator transforms the annihilation operator $\hat{a}$ into its function $f(\hat{a}) = \hat{a} + \frac{X_\alpha + iP_\alpha}{\sqrt{2}}$. According to Ex. 3.92, it means that this evolution in the Schrödinger picture should transform the vacuum state — an eigenstate of $\hat{a}$ with eigenvalue 0 — into an eigenvector of the same operator with the eigenvalue $\frac{X_\alpha + iP_\alpha}{\sqrt{2}}$.

Solution to Exercise 3.100.

a) Using $\alpha = \frac{X_\alpha + iP_\alpha}{\sqrt{2}}$ and Eq. (3.99), we write

$$\begin{aligned}\hat{D}_{XP}(X_\alpha,P_\alpha) &= \exp(iP_\alpha\hat{X} - iX_\alpha\hat{P}) \\ &= \exp\left(i\frac{\alpha-\alpha^*}{\sqrt{2}i}\frac{a+a^\dagger}{\sqrt{2}} - i\frac{\alpha+\alpha^*}{\sqrt{2}}\frac{a-a^\dagger}{\sqrt{2}i}\right) \\ &= \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right). \end{aligned} \tag{S3.100}$$

b) Since the commutator
$$[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = -|\alpha|^2[\hat{a}^\dagger,\hat{a}] = |\alpha|^2,$$
is a number, we can use the Baker-Hausdorff-Campbell formula (A.54) and obtain Eq. (3.157).

c) We decompose the exponent into a Taylor series:

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_{n=0}^{\infty}\frac{(-\alpha^*)^n}{n!}\hat{a}^n|0\rangle = |0\rangle.$$

The last equality above holds because, since $\hat{a}$ is the annihilation operator, all the terms in the sum vanish except for $n=0$.

It follows that
$$\hat{D}_{XP}(X_\alpha,P_\alpha)|0\rangle \overset{(3.157)}{=} e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle.$$

Solution to Exercise 3.101. Decomposing Eq. (3.158) into the Taylor series, we find

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^\dagger)^n |0\rangle \\ &\stackrel{(3.105)}{=} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \end{aligned} \tag{S3.101}$$

Solution to Exercise 3.102.

a) This follows from the statement of Ex. A.85.
b) Using the previous result and the Fock decomposition (3.121) of the coherent state, we write

$$\begin{aligned} \hat{F}(\varphi)|\alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{F}(\varphi)|n\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\varphi n}|n\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{i\varphi})^n}{\sqrt{n!}} |n\rangle \\ &= \left|\alpha e^{-i\varphi}\right\rangle . \end{aligned}$$

Solution to Exercise 3.103. We follow the same logic as in Ex. 3.98. The fictitious Hamiltonian such that $F(\varphi) = e^{-i\varphi n} = e^{-\frac{i}{\hbar}\hat{H}t}$ is in this case $\hat{H} = \hbar\omega\hat{n}$ with $\omega = \varphi/t$. The annihilation operator evolves under this Hamiltonian as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = \frac{i}{\hbar}[\hat{H},\hat{a}] = i\omega[\hat{n},\hat{a}] \stackrel{(3.100)}{=} -i\omega a$$

and hence

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} = \hat{a}(0)e^{-i\varphi}.$$

Therefore

$$\hat{a}^\dagger(t) = [\hat{a}(0)e^{-i\omega t})]^\dagger = \hat{a}^\dagger(0)e^{i\varphi}.$$

Now using the results of Ex. 3.58 to express the position and momentum observables through the creation and annihilation operators and vice versa, we find

$$\begin{aligned} \hat{X}(t) &= \frac{\hat{a}(t) + \hat{a}^\dagger(t)}{\sqrt{2}} \\ &= \frac{\hat{a}(0)e^{-i\varphi} + \hat{a}^\dagger(0)e^{i\varphi}}{\sqrt{2}} \\ &= \frac{[\hat{X}(0) + i\hat{P}(0)]e^{-i\varphi} + [\hat{X}(0) - i\hat{P}(0)]e^{i\varphi}}{2} \\ &= \hat{X}(0)\cos\varphi + \hat{P}(0)\sin\varphi \end{aligned}$$

and

$$\begin{aligned}\hat{P}(t) &= \frac{\hat{a}(t)-\hat{a}^\dagger(t)}{\sqrt{2}i}\\ &= \frac{\hat{a}(0)e^{-i\varphi}-\hat{a}^\dagger(0)e^{i\varphi}}{\sqrt{2}i}\\ &= \frac{[\hat{X}(0)+i\hat{P}(0)]e^{-i\varphi}-[\hat{X}(0)-i\hat{P}(0)]e^{i\varphi}}{2i}\\ &= \hat{P}(0)\cos\varphi-\hat{X}(0)\sin\varphi.\end{aligned}$$

Solution to Exercise 3.105. Here we again follow the lines of the solution to Ex. 3.98. We can write

$$\hat{S}(r) = e^{r(\hat{a}^2-\hat{a}^{\dagger 2})/2} = e^{-\frac{i}{\hbar}\hat{H}t},$$

where the fictitious Hamiltonian is given by Eq. (3.169). This Hamiltonian can be transformed:

$$\hat{H} = \frac{i}{2}\hbar\gamma[\hat{a}^2-(\hat{a}^\dagger)^2] = \frac{i}{4}\hbar\gamma[(\hat{X}+i\hat{P})^2-(\hat{X}-i\hat{P})^2] = -\frac{1}{2}\hbar\gamma[\hat{X}\hat{P}+\hat{P}\hat{X}].$$

The position and momentum operators evolve under this Hamiltonian as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{X} &= \frac{i}{\hbar}[\hat{H},\hat{X}] = -i\frac{1}{2}\gamma\hat{X}[\hat{P},\hat{X}] = -\gamma\hat{X};\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{P} &= \frac{i}{\hbar}[\hat{H},\hat{P}] = -i\frac{1}{2}\gamma\hat{P}[\hat{X},\hat{P}] = \gamma\hat{P}.\end{aligned}$$

Hence

$$\begin{aligned}\hat{X}(t) &= \hat{X}(0)e^{-\gamma t} = \hat{X}(0)e^{-r};\\ \hat{P}(t) &= \hat{P}(0)e^{\gamma t} = \hat{P}(0)e^{r}.\end{aligned}$$

For the annihilation and creation operators, we find

$$\begin{aligned}\hat{a}(t) &= \frac{\hat{X}(t)+i\hat{P}(t)}{\sqrt{2}}\\ &= \frac{\hat{X}(0)e^{-r}+i\hat{P}(0)e^{r}}{\sqrt{2}}\\ &\overset{(3.99)}{=} \frac{[\hat{a}(0)+\hat{a}^\dagger(0)]e^{-r}+[\hat{a}(0)-\hat{a}^\dagger(0)]e^{r}}{2}\\ &= \hat{a}(0)\cosh r-\hat{a}^\dagger(0)\sinh r\end{aligned}$$

and

$$\hat{a}^\dagger(t) = [\hat{a}(0)\cosh r - \hat{a}^\dagger(0)\sinh r]^\dagger$$
$$= \hat{a}^\dagger(0)\cosh r - \hat{a}(0)\sinh r.$$

Solution to Exercise 3.106.

For the mean square position uncertainty of the state $\hat{S}(r)\left|\psi\right\rangle$ we can write

$$\begin{aligned}\left\langle\psi\right|\hat{S}^\dagger(r)\Delta X^2\hat{S}(r)\left|\psi\right\rangle &= \left\langle\psi\right|\hat{S}^\dagger(r)\hat{X}^2\hat{S}(r)\left|\psi\right\rangle - (\left\langle\psi\right|\hat{S}^\dagger(r)\hat{X}\hat{S}(r)\left|\psi\right\rangle)^2\\ &= \left\langle\psi\right|(\hat{X}e^{-r})^2\left|\psi\right\rangle - (\left\langle\psi\right|\hat{X}e^{-r}\left|\psi\right\rangle)^2\\ &= e^{-2r}\left[\left\langle\psi\right|\hat{X}^2\left|\psi\right\rangle - (\left\langle\psi\right|\hat{X}\left|\psi\right\rangle)^2\right]\\ &= e^{-2r}\left\langle\psi\right|\Delta X^2\left|\psi\right\rangle.\end{aligned}$$

The argument for the momentum uncertainty runs similarly.

Solution to Exercise 3.107.

a) We need to verify the integral $\int\limits_{-\infty}^{+\infty}|\psi_{sq}(X)|^2\mathrm{d}X = 1$. To calculate the integral, let us change the integration variable according to $X' = Xe^r$. Then $\mathrm{d}X = \mathrm{d}X'e^{-r}$ and

$$\begin{aligned}\int\limits_{-\infty}^{+\infty}|\psi_{sq}(X)|^2\mathrm{d}X &= \int\limits_{-\infty}^{+\infty}e^r|\psi_0(Xe^r)|^2\mathrm{d}X\\ &= \int\limits_{-\infty}^{+\infty}|\psi_0(X')|^2\mathrm{d}X'\\ &= 1,\end{aligned}$$

where we used the known fact that the vacuum state wavefunction is normalized.

b) From Eq. (3.170), we find $f(X,t) = Xe^{-r} = Xe^{-\gamma t}$, so $f'(X,t) = e^{-r}$ and $f^{-1}(X,t) = Xe^r$. Therefore Eq. (3.153) takes the form

$$|\psi(x,t)|^2 = e^r|\psi_0(e^rX)|^2.$$

This is consistent with Eq. (3.174a).

Solution to Exercise 3.108. Hamiltonian (3.169) can be written in the position basis

$$\hat{H} = -\frac{1}{2}\hbar\gamma\left[\hat{X}\hat{P}+\hat{P}\hat{X}\right] = -\hbar\gamma\left[\hat{X}\hat{P}-\frac{i}{2}\right] \stackrel{(3.93)}{\simeq} i\hbar\gamma\left[X\frac{\mathrm{d}}{\mathrm{d}X}+\frac{1}{2}\right]. \tag{S3.102}$$

The Schrödinger equation

$$\frac{d\left|\psi\right\rangle}{dt} = -\frac{i}{\hbar}\hat{H}\left|\psi\right\rangle$$

in the position basis takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{sq}(X,t) = \gamma\left[X\frac{\mathrm{d}}{\mathrm{d}X}+\frac{1}{2}\right]\psi_{sq}(X,t), \tag{S3.103}$$

where

$$\psi_{sq}(X,t) = \frac{e^{\gamma t/2}}{\pi^{1/4}}e^{-X^2e^{2\gamma t}/2}.$$

A direct calculation shows that the two sides of Eq. (S3.103) are equal. The proof for the wavefunction in the momentum basis is analogous.

Solution to Exercise 3.109.

a) The evolution operator under Hamiltonian (3.176) is

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{\gamma t(-\hat{a}_A\hat{a}_B+\hat{a}_A^\dagger\hat{a}_B^\dagger)} = \hat{S}_2(\gamma t).$$

Writing the creation and annihilation operators in terms of position and momentum, we transform the Hamiltonian as follows

$$\begin{aligned}\hat{H} &= \frac{i}{2}\hbar\gamma[-(\hat{X}_A+i\hat{P}_A)(\hat{X}_B+i\hat{P}_B)+(\hat{X}_A-i\hat{P}_A)(\hat{X}_B-i\hat{P}_B)]\\ &= \frac{i}{2}\hbar\gamma[-2i\hat{X}_A\hat{P}_B-2i\hat{P}_A\hat{X}_B]\\ &= \hbar\gamma[\hat{X}_A\hat{P}_B+\hat{P}_A\hat{X}_B].\end{aligned}$$

b) Applying Heisenberg's equation to the position and momentum observables and recalling that the operators associated with different oscillators commute, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X}_A = \frac{i}{\hbar}[\hat{H},\hat{X}_A] = i\gamma X_B[\hat{P}_A,\hat{X}_A] = \gamma X_B; \tag{S3.104}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X}_B = \frac{i}{\hbar}[\hat{H},\hat{X}_B] = i\gamma X_A[\hat{P}_B,\hat{X}_B] = \gamma X_A; \tag{S3.105}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_A = \frac{i}{\hbar}[\hat{H},\hat{P}_A] = i\gamma P_B[\hat{X}_A,\hat{P}_A] = -\gamma P_B; \tag{S3.106}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_B = \frac{i}{\hbar}[\hat{H},\hat{P}_B] = i\gamma P_A[\hat{X}_B,\hat{P}_B] = -\gamma P_A \tag{S3.107}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X}_\pm = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\hat{X}_A\pm\hat{X}_B}{\sqrt{2}} = \gamma\frac{\hat{X}_B\pm\hat{X}_A}{\sqrt{2}} = \pm\gamma X_\pm; \tag{S3.108}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_\pm = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\hat{P}_A\pm\hat{P}_B}{\sqrt{2}} = -\gamma\frac{\hat{P}_B\pm\hat{P}_A}{\sqrt{2}} = \mp\gamma P_\pm. \tag{S3.109}$$

These results lead to

$$\begin{aligned}\hat{X}_\pm(t) &= X_\pm(0)e^{\pm\gamma t};\\ \hat{P}_\pm(t) &= P_\pm(0)e^{\mp\gamma t},\end{aligned}$$

which is equivalent to Eqs. (3.177) and (3.178) because $r = \gamma t$.
To find the evolution of the annihilation operators, we define the operators

$$\hat{a}_\pm = \frac{\hat{a}_A \pm \hat{a}_B}{\sqrt{2}} = \frac{\hat{X}_\pm + i\hat{P}_\pm}{\sqrt{2}}.$$

The evolution of these operators can be found similarly to the single-mode case:

$$\begin{aligned}
\hat{a}_\pm(t) &= \frac{\hat{X}_\pm(t) + i\hat{P}_\pm(t)}{\sqrt{2}} \\
&= \frac{\hat{X}_\pm(0)e^{\pm r} + i\hat{P}_\pm(0)e^{\mp r}}{\sqrt{2}} \\
&= \frac{[\hat{a}_\pm(0) + \hat{a}_\pm^\dagger(0)]e^{\pm r} + [\hat{a}_\pm(0) - \hat{a}_\pm^\dagger(0)]e^{\mp r}}{2} \\
&= \frac{\hat{a}_\pm(0)[e^{\pm r} + e^{\mp r}] + \hat{a}_\pm^\dagger(0)[e^{\pm r} - e^{\mp r}]}{2} \\
&= \hat{a}_\pm(0)\cosh r \pm \hat{a}_\pm^\dagger(0)\sinh r,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\hat{a}_A(t) &= \frac{\hat{a}_+(t) + \hat{a}_-(t)}{\sqrt{2}} \\
&= \frac{\hat{a}_+(0)\cosh r + \hat{a}_+^\dagger(0)\sinh r + \hat{a}_-(0)\cosh r - \hat{a}_-^\dagger(0)\sinh r}{\sqrt{2}} \\
&= \frac{\hat{a}_+(0) + \hat{a}_-(0)}{\sqrt{2}}\cosh r + \frac{\hat{a}_+^\dagger(0) - \hat{a}_-^\dagger(0)}{\sqrt{2}}\sinh r \\
&= \hat{a}_A(0)\cosh r + \hat{a}_B^\dagger(0)\sinh r.
\end{aligned}$$

The calculation for $\hat{a}_B(t)$ is analogous.

c) Similarly to Ex. 3.106, we will use the fact proven when the Heisenberg picture was introduced: the expectation value of any observable $\hat{A} = \hat{A}(0)$ in the state $\hat{S}_2(r)\left|0,0\right\rangle$ is equal to the expectation value of the "squeezed" observable $\hat{S}_2^\dagger(r)\hat{A}\hat{S}_2(r) = \hat{A}(t)$ in the vacuum state $\left|0,0\right\rangle$. However, before we proceed with proving relations (3.182) and (3.183), it is convenient to first determine the moments of the "unsqueezed" observables $\hat{X}_\pm(0)$ with respect to the vacuum state. We find:

$$\left\langle \hat{X}_{\pm}(0)\right\rangle = \frac{\left\langle \hat{X}_A(0)\right\rangle \pm \left\langle \hat{X}_B(0)\right\rangle}{\sqrt{2}} = 0;$$
$$\left\langle \hat{X}_{\pm}^2(0)\right\rangle = \frac{\left\langle \hat{X}_A^2(0)\right\rangle \pm 2\left\langle \hat{X}_A(0)\hat{X}_B(0)\right\rangle + \left\langle \hat{X}_B^2(0)\right\rangle}{2} = \frac{\frac{1}{2}\pm 0+\frac{1}{2}}{2} = \frac{1}{2};$$
$$\left\langle \hat{X}_{+}(0)\hat{X}_{-}(0)\right\rangle = \frac{\left\langle \hat{X}_A^2(0)\right\rangle - \left\langle \hat{X}_A(0)\hat{X}_B(0)\right\rangle + \left\langle \hat{X}_B(0)\hat{X}_A(0)\right\rangle - \left\langle \hat{X}_B^2(0)\right\rangle}{2} = \frac{\frac{1}{2}-0+0-\frac{1}{2}}{2} = 0;$$
$$\left\langle \hat{X}_{+}(0)\hat{X}_{-}(0)\right\rangle = \frac{\left\langle \hat{X}_A^2(0)\right\rangle + \left\langle \hat{X}_A(0)\hat{X}_B(0)\right\rangle - \left\langle \hat{X}_B(0)\hat{X}_A(0)\right\rangle - \left\langle \hat{X}_B^2(0)\right\rangle}{2} = \frac{\frac{1}{2}+0-0-\frac{1}{2}}{2} = 0.$$

For the squeezed observables $\hat{X}_{\pm}(t)$, it then follows from Eqs. (3.177) and (3.178) that

$$\left\langle \hat{X}_{\pm}(t)\right\rangle = \left\langle \hat{X}_{\pm}(0)\right\rangle e^{\pm\gamma t} = 0;$$
$$\left\langle \hat{X}_{\pm}^2(t)\right\rangle = \left\langle \hat{X}_{\pm}^2(0)\right\rangle e^{\pm 2\gamma t} = \frac{1}{2}e^{\pm 2\gamma t},$$

where the averaging is still performed with respect to the vacuum state because we work in the Heisenberg picture. Hence for Alice's position we have

$$\left\langle \hat{X}_A(t)\right\rangle = \frac{\left\langle \hat{X}_{+}(t)\right\rangle + \left\langle \hat{X}_{-}(t)\right\rangle}{\sqrt{2}} = 0$$

and

$$\begin{aligned}\left\langle \hat{X}_A^2(t)\right\rangle &= \frac{\left\langle \hat{X}_{+}^2(t)\right\rangle + \left\langle \hat{X}_{+}(t)\hat{X}_{-}(t)\right\rangle + \left\langle \hat{X}_{-}(t)\hat{X}_{+}(t)\right\rangle + \left\langle \hat{X}_{-}^2(t)\right\rangle}{2} \\ &= \frac{\left\langle \hat{X}_{+}^2(0)e^{2\gamma t}\right\rangle + \left\langle \hat{X}_{+}(0)e^{\gamma t}\hat{X}_{-}(0)e^{-\gamma t}\right\rangle + \left\langle \hat{X}_{-}(0)e^{-\gamma t}\hat{X}_{+}(0)e^{\gamma t}\right\rangle + \left\langle \hat{X}_{-}^2(0)e^{-2\gamma t}\right\rangle}{2} \\ &= \frac{\frac{1}{2}e^{2\gamma t}+0+0+\frac{1}{2}e^{-2\gamma t}}{2} = \frac{\cosh 2\gamma t}{2}.\end{aligned}$$

For Bob's position, and for the momentum observable, the calculation is analogous.

Solution to Exercise 3.110. In the position basis, Hamiltonian (3.176) becomes

$$\hat{H} \simeq -i\hbar\gamma\left[X_A\frac{\mathrm{d}}{\mathrm{d}X_B} + X_B\frac{\mathrm{d}}{\mathrm{d}X_A}\right],$$

so the Schrödinger equation (1.31) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{sq2}(X_A,X_B,t) = -\gamma\left[X_A\frac{\mathrm{d}}{\mathrm{d}X_B} + X_B\frac{\mathrm{d}}{\mathrm{d}X_A}\right]\Psi_{sq2}(X_A,X_B,t). \tag{S3.110}$$

where $\Psi_{sq2}(X_A,X_B)$ is given by Eq. (3.185a) with $r=\gamma t$. The validity of Eq. (S3.110) is readily verified by way of a direct calculation.

The proof for the wavefunction in the momentum basis is analogous.

Solution to Exercise 3.111.

a) When Alice detects a particular position value X_A, state $|\Psi\rangle$ collapses onto $\langle X_A|\,\Psi\rangle$ in Bob's Hilbert space. The wavefunction of this state is

$$\psi_B(X_B) = \langle X_B|\left(\langle X_A|\,\Psi\rangle\right) = \langle X_A, X_B|\,\Psi\rangle = \Psi(X_A, X_B),$$

which the same as the wavefunction of the original two-mode squeezed vacuum state. This wavefunction should however be interpreted differently: now X_A is the specific value that has already been observed by Alice while X_B is the argument of Bob's wavefunction that has not yet been measured. Note that the above wavefunction is unnormalized in Bob's Hilbert space because it incorporates the probability for Alice to detect a specific X_A.

To find the position uncertainty, we rewrite this wavefunction as

$$\psi(X_B) = \mathcal{N} \exp\left[-u^2 X_A^2 + 2v^2 X_A X_B - u^2 X_B^2\right],$$

where

$$u^2 = \frac{e^{2r}}{4} + \frac{e^{-2r}}{4} = \frac{\cosh 2r}{2};\; v^2 = \frac{e^{2r}}{4} - \frac{e^{-2r}}{4} = \frac{\sinh 2r}{2}.$$

Transforming this expression further, we obtain

$$\begin{aligned}\psi(X_B) &= \mathcal{N} \exp\left[-u^2 X_A^2 + \frac{v^4}{u^2} X_A^2 - \frac{v^4}{u^2} X_A^2 + 2v^2 X_A X_B - u^2 X_B^2\right] \\ &= \mathcal{N} \exp\left[-u^2 X_A^2 + \frac{v^4}{u^2} X_A^2\right] \exp\left[-u^2\left(X_B - \frac{v^2}{u^2} X_A\right)^2\right].\end{aligned}$$

While the first of the above exponentials is a constant factor (because X_A is constant), the second one is a Gaussian function of X_B of width $1/u$. By comparing it with the Gaussian in Ex. 3.25, we find

$$\left\langle \Delta x_B^2 \right\rangle = 1/(4u^2) = \frac{1}{2\cosh 2r}.$$

b) The solution is analogous to part (a) and yields the same answer.

Solution to Exercise 3.112.

a) Let us set $\zeta = \frac{1}{\sqrt{dD}}$. Then the wavefunction will take the form

$$\Psi_{sq2}(X_A, X_B) = \mathcal{N} \mathrm{e}^{-\frac{s^2(X_A - X_B)^2}{4}} \mathrm{e}^{-\frac{(X_A + X_B)^2}{4s^2}},$$

where $s = \sqrt{\frac{D}{d}}$. It is equivalent to the two-mode squeezed state wavefunction (3.185a) with $s = e^r$.

b) Since $[\hat{x}_{A,B}, \hat{p}_{A,B}] = i\hbar$, we need $\hat{P}_{A,B} = \frac{1}{\hbar\zeta}\hat{p}_{A,B}$ in order to obtain $[\hat{X}_{A,B}, \hat{P}_{A,B}] = i$. Note that this transformation is equivalent to Eq. (3.87) with $\zeta = \sqrt{\frac{M\omega}{\hbar}}$.

Solution to Exercise 3.113.

a) Decomposing operator 3.168 into the power series up to the first term and applying it to the vacuum state, we find

$$\hat{S}(r)|0\rangle \approx \left[\hat{\mathbf{1}} + \frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]|0\rangle = |0\rangle - \frac{r}{\sqrt{2}}|2\rangle. \quad \text{(S3.111)}$$

The squared norm of this state is $\langle\psi|\,\psi\rangle = 1 + r^2/2$, which is approximated by 1 in the first order in r. The expectation values of the position and momentum in this state are

$$\begin{aligned}
\langle X\rangle &= \left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\left(|0\rangle - \frac{r}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\left(-r|1\rangle + |1\rangle - \frac{r}{\sqrt{6}}|3\rangle\right) \\
&= 0; \\
\langle P\rangle &= \left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i}\left(|0\rangle - \frac{r}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{\sqrt{2}i}\left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\left(-r|1\rangle - |1\rangle + \frac{r}{\sqrt{6}}|3\rangle\right) \\
&= 0.
\end{aligned}$$

The position uncertainty of this state is

$$\begin{aligned}
\langle X^2\rangle &= \left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\frac{\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2}{2}\left(|0\rangle - \frac{r}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{2}\left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\left(-r|0\rangle + |0\rangle - \frac{3r}{\sqrt{2}}|2\rangle - \frac{2r}{\sqrt{2}}|2\rangle + \sqrt{2}|2\rangle - \sqrt{6}r|4\rangle\right) \\
&\approx \frac{1}{2}(1-2r); \\
\langle P^2\rangle &= \left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\frac{-\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - (\hat{a}^\dagger)^2}{2}\left(|0\rangle - \frac{r}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{2}\left(\langle 0| - \frac{r}{\sqrt{2}}\langle 2|\right)\left(r|0\rangle + |0\rangle - \frac{3r}{\sqrt{2}}|2\rangle - \frac{2r}{\sqrt{2}}|2\rangle - \sqrt{2}|2\rangle + \sqrt{6}r|4\rangle\right) \\
&\approx \frac{1}{2}(1+2r),
\end{aligned}$$

where we removed terms of higher than the first order in r.
These results are consistent with those expected from the calculation in the Heisenberg picture (Ex. 3.106). Indeed, according to that calculation we expect in the first order of r:

$$\begin{aligned}
\langle X^2\rangle &= \frac{1}{2}e^{-2r} \approx \frac{1}{2}(1-2r); \\
\langle P^2\rangle &= \frac{1}{2}e^{2r} \approx \frac{1}{2}(1+2r),
\end{aligned}$$

where we used the fact that the position and momentum uncertainties of the vacuum state are equal to $\frac{1}{2}$.

b) Applying the two-oscillator squeezing operator (3.175) to the double vacuum sate, we find

$$\hat{S}_2(r)\,|0,0\rangle \approx \left[\hat{\mathbf{1}} + r(-\hat{a}_A\hat{a}_B + \hat{a}_A^\dagger\hat{a}_B^\dagger)\right]|0,0\rangle = |0,0\rangle + r\,|1,1\rangle\,. \tag{S3.112}$$

The squared norm if this state is $\langle\psi|\ \psi\rangle = 1 + r^2$, which, again, approximates to 1 in the first order in r. The expectation values of the observable $\hat{X}_\pm$ in this state are

$$\begin{aligned}\langle X_\pm\rangle &= (\langle 0,0| + r\,\langle 1,1|)\,\frac{\hat{a}_A + \hat{a}_A^\dagger \pm \hat{a}_B \pm \hat{a}_B^\dagger}{2}\,(|0,0\rangle + r\,|1,1\rangle) \\ &= \frac{1}{2}\,(\langle 0,0| + r\,\langle 1,1|)\left(r\,|0,1\rangle + |1,0\rangle + r\sqrt{2}\,|2,1\rangle \pm r\,|1,0\rangle \pm |0,1\rangle \pm r\sqrt{2}\,|1,2\rangle\right) \\ &= 0.\end{aligned}$$

A similar expression for $\left\langle X_\pm^2\right\rangle$ would contain 64 terms, which is unwieldy. To simplify it, we notice that we can obtain nonvanishing contributions only from those terms in $\hat{X}_\pm^2$ that keep number of photons in the two oscillators equal. These terms are $\hat{a}_A\hat{a}_A^\dagger$, $\hat{a}_A^\dagger\hat{a}_A$, $\hat{a}_B\hat{a}_B^\dagger$, $\hat{a}_B^\dagger\hat{a}_B$, $\hat{a}_A\hat{a}_B = \hat{a}_B\hat{a}_A$ and $\hat{a}_A^\dagger\hat{a}_B^\dagger = \hat{a}_B^\dagger\hat{a}_A^\dagger$. Hence

$$\begin{aligned}\left\langle X_\pm^2\right\rangle &= (\langle 0,0| + r\,\langle 1,1|)\,\frac{\hat{a}_A\hat{a}_A^\dagger + \hat{a}_A^\dagger\hat{a}_A + \hat{a}_B\hat{a}_B^\dagger + \hat{a}_B^\dagger\hat{a}_B \pm 2\hat{a}_A\hat{a}_B \pm 2\hat{a}_A^\dagger\hat{a}_B^\dagger}{4}\,(|0,0\rangle + r\,|1,1\rangle) \\ &= \frac{1}{4}\,(\langle 0,0| + r\,\langle 1,1|)\,(|0,0\rangle + 2r\,|1,1\rangle + r\,|1,1\rangle + |0,0\rangle + 2r\,|1,1\rangle + r\,|1,1\rangle \pm 2r\,|0,0\rangle \pm 2\,|1,1\rangle \pm 4r\,|2,2\rangle) \\ &\approx \frac{1}{2}(1 \pm 2r),\end{aligned}$$

where we again removed terms of higher than the first order in r.

Similarly to part (a), these results are consistent with those expected from the Heisenberg picture. The calculation for the momentum observable $\hat{P}_\pm$ is analogous to the above.

Solution to Exercise 3.114.

a) We calculate the required inner product using wavefunctions (3.116a) and (3.174a), remembering that α is real:

$$
\begin{aligned}
\left\langle\alpha\right|\hat{S}(r)\left|0\right\rangle &= \int_{-\infty}^{+\infty}\psi_\alpha(X)\psi_{sq}(X)\mathrm{d}X \qquad \text{(S3.113)}\\
&= \frac{e^{r/2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left[-\frac{(X-X_\alpha)^2+X^2e^{2r}}{2}\right]\mathrm{d}X\\
&= \frac{e^{r/2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left[-\frac{X^2(1+e^{2r})-2XX_\alpha+X_\alpha^2}{2}\right]\mathrm{d}X\\
&= \frac{e^{r/2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left[-\frac{X^2(1+e^{2r})-2XX_\alpha+\frac{X_\alpha^2}{1+e^{2r}}-\frac{X_\alpha^2}{1+e^{2r}}+X_\alpha^2}{2}\right]\mathrm{d}X\\
&= \frac{e^{r/2}}{\sqrt{\pi}}\exp\left[-\frac{-\frac{X_\alpha^2}{1+e^{2r}}+X_\alpha^2}{2}\right]\int_{-\infty}^{+\infty}\exp\left[-\frac{1+e^{2r}}{2}\left(X-\frac{X_\alpha}{1+e^{2r}}\right)^2\right]\mathrm{d}X\\
&\overset{(B.17)}{=} \frac{e^{r/2}}{\sqrt{\pi}}\exp\left[-\frac{X_\alpha^2}{2}\frac{e^{2r}}{1+e^{2r}}\right]\sqrt{\frac{2\pi}{1+e^{2r}}}\\
&\overset{X_\alpha=\alpha\sqrt{2}}{=} \exp\left[-\alpha^2\frac{e^{2r}}{1+e^{2r}}\right]\sqrt{\frac{1}{\cosh r}}
\end{aligned}
$$

b) Using the Fock decomposition (3.121) of the coherent state we transform the previous result (S3.113) into

$$\exp\left[-\frac{\alpha^2}{2}\right]\sum_{n=0}^{\infty}\left\langle n\right|\hat{S}(r)\left|0\right\rangle\frac{\alpha^n}{\sqrt{n!}}=\sqrt{\frac{1}{\cosh r}}\exp\left[-\alpha^2\frac{e^{2r}}{1+e^{2r}}\right].$$

or

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left\langle n\right|\hat{S}(r)\left|0\right\rangle\frac{\alpha^n}{\sqrt{n!}} &= \sqrt{\frac{1}{\cosh r}}\exp\left[-\frac{\alpha^2}{2}\frac{e^{2r}-1}{1+e^{2r}}\right]\\
&= \sqrt{\frac{1}{\cosh r}}\exp\left[-\alpha^2\frac{\tanh r}{2}\right].
\end{aligned}
$$

Now decomposing the right-hand side according to

$$\exp\left[-\alpha^2\frac{\tanh r}{2}\right]=\sum_{m=0}^{\infty}\frac{\alpha^{2m}}{m!}\left[-\frac{\tanh r}{2}\right]^m$$

we obtain

$$\sum_{n=0}^{\infty}\left\langle n\right|\hat{S}(r)\left|0\right\rangle\frac{\alpha^n}{\sqrt{n!}}=\sqrt{\frac{1}{\cosh r}}\sum_{m=0}^{\infty}\left[\frac{-\tanh r}{2}\right]^m\frac{\alpha^{2m}}{m!}. \qquad \text{(S3.114)}$$

We can now argue that, because the above equation is valid for all values of α, each pair of terms of the sums in the two sides corresponding to the same power $n=2M$ of α must be equal to each other. Therefore

$$\langle 2M|\,\hat{S}(r)|\,0\rangle \frac{\alpha^{2m}}{\sqrt{2m!}} = \sqrt{\frac{1}{\cosh r}}\left[\frac{-\tanh r}{2}\right]^m \frac{\alpha^{2m}}{m!}.$$

This result is equivalent to Eq. (3.190) because the single-mode squeezed state only contains terms with even numbers of photons.

Solution to Exercise 3.115.

a) Using the wavefunction (3.185aa) of the 2-mode squeezed vacuum state, we find

$$\begin{aligned}
&\langle \alpha,\alpha|\,\hat{S}_2(r)|\,0,0\rangle \qquad\qquad \text{(S3.115)}\\
&= \iint_{-\infty}^{+\infty} \psi_\alpha(X_A)\psi_\alpha(X_B)\Psi_{sq2}(X_A,X_B)\mathrm{d}X_A\mathrm{d}X_B\\
&= \frac{1}{\pi}\iint_{-\infty}^{+\infty} \exp\left[-\frac{(X_A-\alpha\sqrt{2})^2+(X_B-\alpha\sqrt{2})^2+X_-^2e^{2r}+X_+^2e^{-2r}}{2}\right]\mathrm{d}X_A\mathrm{d}X_B\\
&= \frac{1}{\pi}\iint_{-\infty}^{+\infty} \exp\left[-\frac{X_A^2+X_B^2-2\sqrt{2}\alpha(X_A+X_B)+4\alpha^2+X_-^2e^{2r}+X_+^2e^{-2r}}{2}\right]\mathrm{d}X_A\mathrm{d}X_B\\
&= \frac{1}{\pi}\iint_{-\infty}^{+\infty} \exp\left[-\frac{X_+^2+X_-^2-4\alpha X_++4\alpha^2+X_-^2e^{2r}+X_+^2e^{-2r}}{2}\right]\mathrm{d}X_-\mathrm{d}X_+\\
&= \frac{1}{\pi}\exp[-2\alpha^2]\int_{-\infty}^{+\infty}\exp\left[-\frac{X_-^2(1+e^{2r})}{2}\right]\mathrm{d}X_-\int_{-\infty}^{+\infty}\exp\left[-\frac{X_+^2(1+e^{-2r})-4\alpha X_+}{2}\right]\mathrm{d}X_+\\
&= \frac{1}{\pi}\exp[-2\alpha^2]\sqrt{\frac{2\pi}{1+e^{2r}}}\int_{-\infty}^{+\infty}\exp\left[-\frac{X_+^2(1+e^{-2r})-4\alpha X_++\frac{4\alpha^2}{1+e^{-2r}}-\frac{4\alpha^2}{1+e^{-2r}}}{2}\right]\mathrm{d}X_+\\
&= \frac{1}{\pi}\exp\left[-2\alpha^2+\frac{2\alpha^2}{1+e^{-2r}}\right]\sqrt{\frac{2\pi}{1+e^{2r}}}\int_{-\infty}^{+\infty}\exp\left[-\frac{1+e^{-2r}}{2}\left(X_+-\frac{2\alpha}{1+e^{-2r}}\right)^2\right]\mathrm{d}X_+\\
&= \frac{1}{\pi}\exp\left[\frac{-2e^{-2r}}{1+e^{-2r}}\alpha^2\right]\sqrt{\frac{2\pi}{1+e^{2r}}}\sqrt{\frac{2\pi}{1+e^{-2r}}}\\
&= \exp\left[\frac{-2}{1+e^{2r}}\alpha^2\right]\sqrt{\frac{4}{(e^r+e^{-r})^2}}\\
&= \exp\left[\frac{-2}{1+e^{2r}}\alpha^2\right]\frac{1}{\cosh r}.
\end{aligned}$$

At some point during this transformation, we have changed the integration variables from (X_A,X_B) to (X_+,X_-). The associated Jacobian is

$$J = \begin{vmatrix} \frac{\partial X_+}{\partial X_A} & \frac{\partial X_-}{\partial X_A} \\ \frac{\partial X_+}{\partial X_B} & \frac{\partial X_-}{\partial X_B} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = 1.$$

b) We now decompose the coherent states in the left-hand side into the Fock basis and recall that the only terms that contribute to $\hat{S}_2(r)|0,0\rangle$ are those with equal number of quanta. The above result then becomes

$$e^{-\alpha^2}\sum_{n=0}^{\infty}\langle n,n|\,\hat{S}_2(r)|\,0,0\rangle\frac{\alpha^{2n}}{n!} = \exp\left[\frac{-2}{1+e^{2r}}\alpha^2\right]\frac{1}{\cosh r}$$

or

$$\sum_{n=0}^{\infty}\langle n,n|\,\hat{S}_2(r)|\,0,0\rangle\frac{\alpha^{2n}}{n!} = \exp\left[\frac{-1+e^{2r}}{1+e^{2r}}\alpha^2\right]$$
$$= \exp\left[\alpha^2\tanh r\right]\frac{1}{\cosh r}$$

c) Decomposing the exponent in right-hand side of the above equation into the power series in α, we obtain

$$\sum_{n=0}^{\infty}\langle n,n|\,\hat{S}_2(r)|\,0,0\rangle\frac{\alpha^{2n}}{n!} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{n!}\tanh^n r.$$

We now set the terms with the same n equal to each other and obtain Eq. (3.192).