Appendix S2
Solutions to Chapter 2 exercises

Solution to Exercise 2.1. Let us choose an arbitrary $|a\rangle \in \mathbb{V}_A$ and consider the sum $|\Psi\rangle = |a\rangle \otimes |b\rangle + |\text{zero}\rangle_{\mathbb{V}_A} \otimes |b\rangle$. According to Eq. (2.3a), we find $|\Psi\rangle = \left(|a\rangle + |\text{zero}\rangle_{\mathbb{V}_A}\right) \otimes |b\rangle = |a\rangle \otimes |b\rangle$. In other words, adding $|\text{zero}\rangle_{\mathbb{V}_A} \otimes |b\rangle$ to an element of $\mathbb{V}_A \otimes \mathbb{V}_B$ did not change this element. According to Ex. A.2(b), we find that $|\text{zero}\rangle_{\mathbb{V}_A} \otimes |b\rangle$ must be the zero element $\mathbb{V}_A \otimes \mathbb{V}_B$.

The second identity is proven in a similar fashion.

Solution to Exercise 2.2. For simplicity, let us consider the polarization Hilbert space of two photons and show that $B = \{|H\rangle \otimes |H\rangle, |H\rangle \otimes |V\rangle, |V\rangle \otimes |H\rangle, |V\rangle \otimes |V\rangle\}$ is a basis.

First, we prove that B is a spanning set. Consider an arbitrary separable vector $|a\rangle \otimes |b\rangle$ of $\mathbb{V}_A \otimes \mathbb{V}_B$. Decomposing $|a\rangle$ and $|b\rangle$ into the canonical bases of their home Hilbert spaces,

$$|a\rangle = a_H |H\rangle + a_V |V\rangle;$$

$$|b\rangle = b_H |H\rangle + b_V |V\rangle,$$

we use Eqs. (2.2) and (2.3) to write

$$|a\rangle \otimes |b\rangle = a_H b_H |H\rangle \otimes |H\rangle + a_H b_V |H\rangle \otimes |V\rangle + a_V b_H |V\rangle \otimes |H\rangle + a_V b_V |V\rangle \otimes |V\rangle. \tag{S2.1}$$

In other words, any separable element of $\mathbb{V}_A \otimes \mathbb{V}_B$ can be written as a linear combination of elements of B. This property is readily generalized to entangled vectors because any entangled vector is a linear combination of separable vectors.

Second, we need to prove that B is linearly independent. This follows from the fact that all elements of B are orthogonal to each other [see Eq. (2.4)] and the fact that any set of mutually orthogonal vectors is linearly independent (Ex. A.17).

Solution to Exercise 2.3. Because $|30^\circ\rangle = \sqrt{3}/2\,|H\rangle + 1/2\,|V\rangle$; $|R\rangle = 1/\sqrt{2}\,|H\rangle + i/\sqrt{2}\,|V\rangle$, we have

$$|30^\circ\rangle \otimes |R\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|HH\rangle + \frac{\sqrt{3}i}{2\sqrt{2}}|HV\rangle + \frac{1}{2\sqrt{2}}|VH\rangle + \frac{i}{2\sqrt{2}}|VV\rangle \simeq \begin{pmatrix} \sqrt{3}/2\sqrt{2} \\ \sqrt{3}i/2\sqrt{2} \\ 1/2\sqrt{2} \\ i/2\sqrt{2} \end{pmatrix}.$$

State $|30^\circ\rangle \otimes |R\rangle$ is separable.

Solution to Exercise 2.4.

a) First let us present both states in the canonical basis

$$\begin{aligned}
|\Pi\rangle =& 5|HH\rangle + 6i\frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \\
=& 5|HH\rangle + 3i(|HH\rangle - |HV\rangle + i|VH\rangle - i|VV\rangle) \\
=& (5+3i)|HH\rangle - 3i|HV\rangle - 3|VH\rangle + 3|VV\rangle; \\
|\Omega\rangle =& 2\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle) + 3\frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) \\
=& |HH\rangle - i|HV\rangle + |VH\rangle - i|VV\rangle + \frac{3}{2}(|HH\rangle + i|HV\rangle + i|VH\rangle - |VV\rangle) \\
=& \frac{5}{2}|HH\rangle + \frac{i}{2}|HV\rangle + \left(1 + \frac{3i}{2}\right)|VH\rangle + \left(-\frac{3}{2} - i\right)|VV\rangle.
\end{aligned}$$

Hence

$$\langle\Pi|\,\Omega\rangle = (5-3i)\frac{5}{2} + 3i\frac{i}{2} - 3\left(1 + \frac{3i}{2}\right) + 3\left(-\frac{3}{2} - i\right) = \frac{7}{2} - 15i.$$

b) Because $|\Pi\rangle$ and $|\Omega\rangle$ are both separable, we have

$$\begin{aligned}
\langle\Pi|\,\Omega\rangle =& -i(2\langle H| - i\langle V|)(2i|H\rangle - 3i|V\rangle) \times (\langle H| - i\langle V|)(|H\rangle + |V\rangle)/2 \\
=& -i[2\times(2i) + (-i)\times(-3i)][1\times 1 + (-i)\times 1]/2 \\
=& -i(-3+4i)(1-i)/2 = (7-i)/2.
\end{aligned}$$

Solution to Exercise 2.6. Consider, for example, $|\Phi^+\rangle$. Suppose this state can be written as a product

$$\left|\Phi^+\right\rangle = |a\rangle_A \otimes |b\rangle_B, \tag{S2.2}$$

where $|a\rangle$ and $|b\rangle$ are some states in $\mathbb{V}_A$ and $\mathbb{V}_B$, respectively. These states can be decomposed into the canonical bases of their respective spaces:

$$|a\rangle = a_H|H\rangle + a_V|V\rangle;$$
$$|b\rangle = b_H|H\rangle + b_V|V\rangle.$$

Substituting these decompositions into Eq. (S2.2), comparing the result with the definition (2.5c) of $|\Phi^+\rangle$, and using the uniqueness of the decomposition of a vector into a basis, we find:

$$\begin{cases} a_H b_H = 1/\sqrt{2} \\ a_H b_V = 0 \\ a_V b_H = 0 \\ a_V b_V = 1/\sqrt{2}. \end{cases} \tag{S2.3}$$

From the second equation in the above system, we find that either $a_H = 0$ or $b_V = 0$. Therefore, either $a_H b_H$ or $a_V b_V$ must vanish, which contradicts either the first or the fourth equations of system (S2.3).

The proof for other Bell states runs similarly.

Solution to Exercise 2.7. The Bell states comprise a spanning set because the four canonical basis elements can be expressed through these states:

$$|HH\rangle = \left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)/\sqrt{2}; \tag{S2.4a}$$

$$|VV\rangle = \left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)/\sqrt{2}; \tag{S2.4b}$$

$$|HV\rangle = \left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)/\sqrt{2}; \tag{S2.4c}$$

$$|VH\rangle = \left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right)/\sqrt{2}. \tag{S2.4d}$$

Because the dimension of this tensor product space is 4, and according to Ex. A.7(b), the four Bell states form a basis. The orthonormality of this basis can be verified by direct calculation, i.e.:

$$\left\langle \Phi^+ \middle| \Phi^+ \right\rangle = \left(\langle HH|HH\rangle + \langle HH|VV\rangle + \langle VV|HH\rangle + \langle VV|VV\rangle\right)/2 = (1+0+0+1)/2 = 1;$$
$$\left\langle \Phi^+ \middle| \Phi^- \right\rangle = \left(\langle HH|HH\rangle - \langle HH|VV\rangle + \langle VV|HH\rangle - \langle VV|VV\rangle\right)/2 = (1-0+0-1)/2 = 0$$

and so on.

Solution to Exercise 2.8. Using $|\theta\rangle = \cos\theta\,|H\rangle + \sin\theta\,|V\rangle$ and $\left|\frac{\pi}{2}+\theta\right\rangle = -\sin\theta\,|H\rangle + \cos\theta\,|V\rangle$ (see Table 1.1), we find

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left[|\theta\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle - \left|\frac{\pi}{2}+\theta\right\rangle \otimes |\theta\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[(\cos\theta\,|H\rangle + \sin\theta\,|V\rangle)\otimes(-\sin\theta\,|H\rangle + \cos\theta\,|V\rangle) - (-\sin\theta\,|H\rangle + \cos\theta\,|V\rangle)\otimes(\cos\theta\,|H\rangle + \sin\theta\,|V\rangle)\right] \\ &= \frac{1}{\sqrt{2}}\left[(\cos^2\theta + \sin^2\theta)\,|HV\rangle - (\cos^2\theta + \sin^2\theta)\,|VH\rangle\right] = \left|\Psi^-\right\rangle. \end{aligned}$$

Solution to Exercise 2.9.

a) The probability of detecting the state $|\Psi\rangle = |R\rangle\,|-30^\circ\rangle$ is the square absolute value of the overlap

$$\begin{aligned} \mathrm{pr}_{|\Psi\rangle} &= |\left\langle \Psi^- \middle| \Psi \right\rangle|^2 = \frac{1}{2}\,|\langle HV|\,R,-30^\circ\rangle - \langle VH|\,R,-30^\circ\rangle|^2 \\ &= \frac{1}{2}\,|\langle H|R\rangle\,\langle V|-30^\circ\rangle - \langle V|R\rangle\,\langle H|-30^\circ\rangle|^2 = \frac{1}{2}\left|\frac{1}{\sqrt{2}}\frac{-1}{2} - \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2}\right|^2 = \frac{1}{4}. \end{aligned}$$

b) Similarly,

$$\mathrm{pr}_{|\Psi\rangle} = \left|\frac{1}{3\sqrt{2}}(\langle HV| - \langle VH|)(|HV\rangle + 2\,|VH\rangle + 2\,|VV\rangle)\right|^2 = \frac{1}{18}|1-2|^2 = \frac{1}{18}.$$

Solution to Exercise 2.10.

a) For the canonical basis, we write

$$\begin{aligned}\mathrm{pr}_{HH} &= |\langle HH|\,\Psi\rangle|^2 = 0;\\ \mathrm{pr}_{HV} &= |\langle HV|\,\Psi\rangle|^2 = \frac{1}{2};\\ \mathrm{pr}_{VH} &= |\langle VH|\,\Psi\rangle|^2 = \frac{1}{2};\\ \mathrm{pr}_{VV} &= |\langle VV|\,\Psi\rangle|^2 = 0.\end{aligned} \tag{S2.5}$$

To find the probabilities for the diagonal basis measurement, let us decompose $|\Psi\rangle$ in that basis. Knowing that $|H\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$, $|V\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, we write

$$\begin{aligned}|\Psi\rangle =& \frac{|{+}{+}\rangle - |{+}{-}\rangle + |{-}{+}\rangle - |{-}{-}\rangle}{2\sqrt{2}} + e^{i\phi}\,\frac{|{+}{+}\rangle + |{+}{-}\rangle - |{-}{+}\rangle - |{-}{-}\rangle}{2\sqrt{2}}\\ =& \frac{(1+e^{i\phi})\,|{+}{+}\rangle + (-1+e^{i\phi})\,|{+}{-}\rangle + (1-e^{i\phi})\,|{-}{+}\rangle + (-1-e^{i\phi})\,|{-}{-}\rangle}{2\sqrt{2}}\\ =& e^{i\phi/2}\,\frac{\cos(\phi/2)\,|{+}{+}\rangle + i\sin(\phi/2)\,|{+}{-}\rangle - i\sin(\phi/2)\,|{-}{+}\rangle - \cos(\phi/2)\,|{-}{-}\rangle}{\sqrt{2}}.\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{pr}_{++} = \mathrm{pr}_{--} &= \cos^2(\phi/2)/2;\\ \mathrm{pr}_{+-} = \mathrm{pr}_{-+} &= \sin^2(\phi/2)/2.\end{aligned}$$

b) The state $|\Psi^+\rangle$ corresponds to the case $\phi = 0$, state $|\Psi^-\rangle$ to $\phi = \pi$. They cannot be distinguished in the canonical basis because both of them give the same probabilities (S2.5). But in the diagonal basis, the states behave differently: for the state $|\Psi^+\rangle$, projections onto $|{+}{+}\rangle$ and $|{-}{-}\rangle$ occur with probabilities $1/2$ each, while projections onto $|{+}{-}\rangle$ and $|{-}{+}\rangle$ do not occur, while state $|\Psi^-\rangle$ only projects onto $|{+}{-}\rangle$ and $|{-}{+}\rangle$ but not onto $|{+}{+}\rangle$ and $|{-}{-}\rangle$. Hence a measurement in the diagonal basis will immediately distinguish these two states.

Solution to Exercise 2.11. The measurement procedure is complicated because the measurement basis $\{|H-\rangle, |H+\rangle, |VR\rangle, |VL\rangle\}$ cannot be written as a set of tensor products of elements of Alice's and Bob's local bases. One way to address this complication would be as follows.

- First, Alice measures her photon in the canonical basis and communicates the result to Bob by a classical channel.

- Once Bob hears from Alice, he sets his measurement basis to diagonal if Alice observed $|H\rangle$ and circular if Alice observed $|V\rangle$. Then he measures his photon in this chosen basis.

Solution to Exercise 2.12. For each matrix element of $\hat{A}\otimes\hat{B}$, we can write

$$\begin{aligned}(\hat{A}\otimes\hat{B})_{jji'j'} &= \langle v_i w_j|\hat{A}\otimes\hat{B}|v_{i'}w_{j'}\rangle \\ &= [\langle v_i|\otimes\langle w_j|]\left[(\hat{A}|v_{i'}\rangle)\otimes(\hat{B}|w_{j'}\rangle)\right] \\ &= (\langle v_i|\hat{A}|v_{i'}\rangle)\,(\langle w_j|\hat{B}|w_{j'}\rangle) \\ &= A_{ii'}B_{jj'}.\end{aligned}$$

In the second equality above, we used the definition of the tensor product operator; in the third one, Eq. (2.4).

Solution to Exercise 2.13. Let us write the operator $\hat{\sigma}_x\otimes\hat{\sigma}_y$ in the matrix form in the canonical basis $\{|HH\rangle,|HV\rangle,|VH\rangle,|VV\rangle\}$. Using Eq. (2.8), we have

$$\begin{aligned}\hat{\sigma}_x\otimes\hat{\sigma}_y &\simeq \begin{pmatrix} (\sigma_x)_{HH}(\sigma_y)_{HH} & (\sigma_x)_{HH}(\sigma_y)_{HV} & (\sigma_x)_{HV}(\sigma_y)_{HH} & (\sigma_x)_{HV}(\sigma_y)_{HV} \\ (\sigma_x)_{HH}(\sigma_y)_{VH} & (\sigma_x)_{HH}(\sigma_y)_{VV} & (\sigma_x)_{HV}(\sigma_y)_{VH} & (\sigma_x)_{HV}(\sigma_y)_{VV} \\ (\sigma_x)_{VH}(\sigma_y)_{HH} & (\sigma_x)_{VH}(\sigma_y)_{HV} & (\sigma_x)_{VV}(\sigma_y)_{HH} & (\sigma_x)_{VV}(\sigma_y)_{HV} \\ (\sigma_x)_{VH}(\sigma_y)_{VH} & (\sigma_x)_{VH}(\sigma_y)_{VV} & (\sigma_x)_{VV}(\sigma_y)_{VH} & (\sigma_x)_{VV}(\sigma_y)_{VV} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Further,

$$|\Psi^-\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

For the expectation value, we find

$$\langle\hat{\sigma}_x\otimes\hat{\sigma}_y\rangle = \langle\Psi^-|\hat{\sigma}_x\otimes\hat{\sigma}_y|\Psi^-\rangle \simeq \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

The uncertainty can be found via Eq. (B.3). One could perform a full matrix calculation again, but perhaps it is easier to notice that the square of any Pauli matrix is the identity operator, and thus

$$\langle\Psi^-|(\hat{\sigma}_x\otimes\hat{\sigma}_y)^2|\Psi^-\rangle = \langle\Psi^-|\hat{\mathbf{1}}|\Psi^-\rangle = \langle\Psi^-|\Psi^-\rangle = 1.$$

The mean square uncertainty hence equals

$$\langle\Delta(\hat{\sigma}_x\otimes\hat{\sigma}_y)^2\rangle = \langle(\hat{\sigma}_x\otimes\hat{\sigma}_y)^2\rangle - \langle\hat{\sigma}_x\otimes\hat{\sigma}_y\rangle^2 = 1-0 = 1.$$

Solution to Exercise 2.15. We choose a random separable state $|ab\rangle \in \mathbb{V}_A \otimes \mathbb{V}_B$ and apply the definition of the operator tensor product:

$$\begin{aligned}(\hat{A}_1\hat{A}_2)\otimes(\hat{B}_1\hat{B}_2)|a\rangle|b\rangle &= \hat{A}_1\hat{A}_2|a\rangle\otimes\hat{B}_1\hat{B}_2|b\rangle = \hat{A}_1\left(\hat{A}_2|a\rangle\right)\otimes\hat{B}_1\left(\hat{B}_2|b\rangle\right)\\ &= \left(\hat{A}_1\otimes\hat{B}_1\right)\left(\hat{A}_2|a\rangle\otimes\hat{B}_2|b\rangle\right) = \left(\hat{A}_1\otimes\hat{B}_1\right)\left(\hat{A}_2\otimes\hat{B}_2\right)|a\rangle|b\rangle\,.\end{aligned}$$

We see that the operators $\hat{A}_1\hat{A}_2\otimes\hat{B}_1\hat{B}_2$ and $\left(\hat{A}_1\otimes\hat{B}_1\right)\left(\hat{A}_2\otimes\hat{B}_2\right)$ act on each separable in $\mathbb{V}_A\otimes\mathbb{V}_B$ state in the same way. Because these are linear operators, the same applies to entangled states, which are linear combinations of separable states. This means that the two operators are identical.

Solution to Exercise 2.17. For arbitrary $|a\rangle \in \mathbb{V}_A$ and $|b\rangle \in \mathbb{V}_B$, we use again the definition of the tensor product operator to write

$$\begin{aligned}(\hat{A}\otimes\hat{B})|ab\rangle &= \hat{A}|a\rangle\otimes\hat{B}|b\rangle\\ &= \big(|a_1\rangle\langle a_2|\,a\rangle\big)\otimes\big(|b_1\rangle\langle b_2|\,b\rangle\big)\\ &= |a_1b_1\rangle\langle a_2|\,a\rangle\langle b_2|\,b\rangle\\ &\overset{(2.4)}{=} |a_1b_1\rangle\langle a_2b_2|\,ab\rangle\\ &= \big(|a_1b_1\rangle\langle a_2b_2|\big)\big(|ab\rangle\big).\end{aligned}$$

We see that the operators in the left- and right-hand sides of Eq. (2.9) map any separable state in the same way. It follows that the two operators are identical.

Solution to Exercise 2.18. Suppose cloning is possible. That is, there exists a linear operator $\hat{U}$ that performs cloning of any state $|a\rangle$ in accordance with Eq. (2.10). Applying that equation to two orthogonal states $|a_1\rangle$ and $|a_2\rangle$ and their linear superposition to cloning, we would obtain

$$\hat{U}|a_1\rangle\otimes|0\rangle = |a_1\rangle\otimes|a_1\rangle \tag{S2.6}$$

$$\hat{U}|a_2\rangle\otimes|0\rangle = |a_2\rangle\otimes|a_2\rangle \tag{S2.7}$$

$$\hat{U}\frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}\otimes|0\rangle = \frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}\otimes\frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}. \tag{S2.8}$$

On the other hand, adding Eqs. (S2.6) and (S2.7) together, and using the linearity of $\hat{U}$, we find

$$\hat{U}\frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}\otimes|0\rangle = \frac{1}{\sqrt{2}}(\hat{U}|a_1\rangle\otimes|0\rangle+\hat{U}|a_2\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(|a_1\rangle\otimes|a_1\rangle+|a_2\rangle\otimes|a_2\rangle),$$

which is inconsistent with Eq. (S2.8).

Solution to Exercise 2.19. By definition, if a tensor product operator $\hat{A}\otimes\hat{B}$ acts on a separable state $|ab\rangle$, it will generate the state $\hat{A}|a\rangle\otimes\hat{B}|b\rangle$. Its adjoint (see Defn. A.21) must therefore satisfy

$$\text{Adjoint}\left(\hat{A}|a\rangle\otimes\hat{B}|b\rangle\right) = \text{Adjoint}\left[(\hat{A}\otimes\hat{B})|ab\rangle\right] = \langle ab|\left(\hat{A}\otimes\hat{B}\right)^\dagger. \tag{S2.9}$$

But, according to the definition (2.11), the adjoint of the tensor product state is the state

$$\text{Adjoint}\left(\hat{A}\,|a\rangle\otimes\hat{B}\,|b\rangle\right)=\text{Adjoint}(\hat{A}\,|a\rangle)\otimes\text{Adjoint}(\hat{B}\,|b\rangle)=\langle a|\hat{A}^\dagger\otimes\langle b|\hat{B}^\dagger. \tag{S2.10}$$

Comparing the last two equations, we obtain the required identity.

Solution to Exercise 2.20.

a) If the operators $\hat{A}$ in $\mathbb{V}_A$ and $\hat{B}$ in $\mathbb{V}_B$ are Hermitian, their matrices satisfy $A_{ii'}=A^*_{i'i}$ and $B_{ii'}=B^*_{i'i}$. Then, according to the result of Ex. 2.12,

$$(\hat{A}\otimes\hat{B})^*_{i'j'ij}=A^*_{i'i}B^*_{j'j}=A_{ii'}B_{jj'}=(\hat{A}\otimes\hat{B})_{iji'j'}.$$

When one transposes and conjugates the matrix of $(\hat{A}\otimes\hat{B})$, one obtains the same matrix, which is the signature of a Hermitian operator (Ex. A.53).

b) If operator $\hat{A}$ in $\mathbb{V}_A$ is unitary, it maps an orthonormal basis $\{|v_i\rangle\}$ onto another orthonormal basis $\{|v'_i\rangle\}$ (see Ex. A.81). Similarly, a unitary operator $\hat{B}$ in $\mathbb{V}_B$ transforms between orthonormal bases $\{|w_i\rangle\}$ and $\{|w'_i\rangle\}$. The tensor product of $\hat{A}$ and $\hat{B}$ transforms between $\{\left|v_iw_j\right\rangle\}$ and $\{\left|v'_iw'_j\right\rangle\}$, which are both orthonormal bases. An operator with such a property must be unitary.

Solution to Exercise 2.21. A local operator is a particular case of a tensor product operator, which, according to Ex. 2.16, cannot transform a separable state into an entangled one.

Neither is a reverse operation possible. This is because any unitary operator is invertible. If there existed a unitary operator enacting such a transformation, an inverse of that operator would transform a separable state into an entangled one, and this is not possible.

Solution to Exercise 2.22.

$$\left(\hat{A}\otimes\hat{\mathbf{1}}\right)|ab\rangle\overset{(2.7)}{=}\left(\hat{A}\,|a\rangle\right)\otimes\left(\hat{\mathbf{1}}\,|b\rangle\right)=\left(a\,|a\rangle\right)\otimes|b\rangle\overset{(2.2)}{=}a\,|ab\rangle\,.$$

Solution to Exercise 2.23.

a) $\hat{\sigma}_z\otimes\hat{\sigma}_z\,|\Phi^+\rangle=|\Phi^+\rangle$, but $\hat{\sigma}_{z,\text{Alice}}\,|\Phi^+\rangle=\hat{\sigma}_{z,\text{Bob}}\,|\Phi^+\rangle=|\Phi^-\rangle$.

b) Since $\hat{A}$ and $\hat{B}$ are Hermitian, they have spectral decompositions $\hat{A}=\sum_i a_i\,|v_i\rangle\langle v_i|$ and $\hat{B}=\sum_j b_j\left|w_j\right\rangle\left\langle w_j\right|$, where $\{|v_i\rangle\}$ and $\{\left|w_j\right\rangle\}$ are orthonormal bases in Alice's and Bob's spaces, respectively. Accordingly,

$$\hat{A}\otimes\hat{B}=\sum_{ij}a_ib_j\left|v_iw_j\right\rangle\left\langle v_iw_j\right|,$$

with $\{\left|v_iw_j\right\rangle\}$ being an orthonormal basis in $\mathbb{V}\otimes\mathbb{W}$. As per Ex. A.66, $|\Psi\rangle$ being an eigenstate of $\hat{A}\otimes\hat{B}$ with eigenvalue x means that it can be written as a linear combination of only those basis elements $\left|v_iw_j\right\rangle$ for which

$$\hat{A}\otimes\hat{B}\left|v_iw_j\right\rangle=x\left|v_iw_j\right\rangle. \tag{S2.11}$$

This means that, if measured in basis $\{|v_i\rangle\otimes\left|w_j\right\rangle\}$, state $|\Psi\rangle$ will project onto one of these basis elements. Alice's measuring $\hat{A}$ and Bob's measuring $\hat{B}$ does constitute a joint measurement of $|\Psi\rangle$ in basis $\{\left|v_iw_j\right\rangle\}$. This measurement will therefore yield a pair of vectors $|v_i\rangle\otimes\left|w_j\right\rangle$ for which Eq. (S2.11) holds. But we also have

$$\hat{A}\otimes\hat{B}\left|v_i w_j\right\rangle = (\hat{A}\left|v_i\right\rangle)\otimes(\hat{B}\left|w_j\right\rangle) = a_i b_j\left|v_i w_j\right\rangle, \tag{S2.12}$$

where a_i and b_j are the observable values associated with $\left|v_i\right\rangle$ and $\left|w_j\right\rangle$. Comparing Eqs. (S2.11) and (S2.12), we find $a_i b_j = x$.

Solution to Exercise 2.24.

a)
$$(\hat{\sigma}_z)_A\left|\Psi^-\right\rangle = (|H\rangle\langle H| - |V\rangle\langle V|)_A\frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) = \left|\Psi^+\right\rangle;$$

b)
$$(\hat{\sigma}_x)_A\left|\Psi^-\right\rangle = (|H\rangle\langle V| + |V\rangle\langle H|)_A\frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|VV\rangle - |HH\rangle) = -\left|\Phi^-\right\rangle;$$

c)
$$(\hat{\sigma}_y)_A\left|\Psi^-\right\rangle = (-i|H\rangle\langle V| + i|V\rangle\langle H|)_A\frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(i|VV\rangle + i|HH\rangle) = i\left|\Phi^+\right\rangle.$$

Solution to Exercise 2.25.

a) If $|\psi_{A,B}(t)\rangle$ are solutions of the Schrödinger equation in their respective spaces:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{A,B}(t)\rangle = -\frac{i}{\hbar}\hat{H}_{A,B}|\psi_{A,B}(t)\rangle$$

then for their tensor product we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left[|\psi_A(t)\rangle\otimes|\psi_B(t)\rangle\right] \\
&= |\dot{\psi}_A(t)\rangle\otimes|\psi_B(t)\rangle + |\psi_A(t)\rangle\otimes|\dot{\psi}_B(t)\rangle \\
&= \left[-\frac{i}{\hbar}\hat{H}_A|\psi_A(t)\rangle\right]\otimes|\psi_B(t)\rangle + |\psi_A(t)\rangle\otimes\left[-\frac{i}{\hbar}\hat{H}_B|\psi_B(t)\rangle\right] \\
&= -\frac{i}{\hbar}(\hat{H}_A + \hat{H}_B)(|\psi_A(t)\rangle\otimes|\psi_B(t)\rangle) \\
&= -\frac{i}{\hbar}\hat{H}|\Psi(t)\rangle.
\end{aligned}$$

b)

$$\begin{aligned}
\hat{H}|\Psi\rangle &= (\hat{H}_A + \hat{H}_B)(|\psi_A\rangle\otimes|\psi_B\rangle) \\
&= \hat{H}_A|\psi_A\rangle\otimes|\psi_B\rangle + \hat{H}_B|\psi_A\rangle\otimes|\psi_B\rangle \\
&= (\hat{H}_A|\psi_A\rangle)\otimes|\psi_B\rangle + |\psi_A\rangle\otimes(\hat{H}_B|\psi_B\rangle) \\
&= (E_A|\psi_A\rangle)\otimes|\psi_B\rangle + |\psi_A\rangle\otimes(E_B|\psi_B\rangle) \\
&= (E_A + E_B)(|\psi_A\rangle\otimes|\psi_B\rangle) \\
&= E|\Psi\rangle.
\end{aligned}$$

c) Since the eigenstates of the local Hamiltonians $\hat{H}_{A,B}$ form orthonormal bases (Ex. A.60), tensor products of these eigenstates form an orthonormal basis in the tensor product Hilbert space $\mathbb{V}_A \otimes \mathbb{V}_B$ (Ex. 2.2). Any eigenstate $|\Psi_E\rangle$ of $\hat{H}$ with energy E can be decomposed into that basis.
Now suppose the decomposition contains a term $|\psi_A\rangle \otimes |\psi_B\rangle$ with the corresponding energy $E_A + E_B \neq E$. Then, as we found in part (b), this term is also an eigenstate of the full bipartite Hamiltoninan with an eigenvalue that is unequal to E. But it follows from the spectral theorem (Ex. A.60) that an observable's eigenstates corresponding to different eigenvalues are orthogonal to each other. This means that the term $|\psi_A\rangle \otimes |\psi_B\rangle$ is orthogonal to $|\Psi_E\rangle$. But the decomposition of a vector into a basis cannot contain terms that are orthogonal to that state. We have arrived at a contradiction.

Solution to Exercise 2.26. According to Ex. 2.8, state $|\Psi^-\rangle$ can be written as

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|\theta\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle - \left|\frac{\pi}{2}+\theta\right\rangle \otimes |\theta\rangle\right).$$

This expression implies that, whenever Alice has a photon in the state $|\theta\rangle$, Bob's photon is in the state $\left|\frac{\pi}{2}+\theta\right\rangle$. Since both terms have amplitude $1/\sqrt{2}$, the corresponding probabilities are $1/2$.

Solution to Exercise 2.27. Since $|H\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ and $|V\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, we have

$$\begin{aligned} |\Psi\rangle &= \frac{1}{3}(|HH\rangle - 2|HV\rangle + 2|VV\rangle) \\ &= \frac{1}{3\sqrt{2}}[(|+\rangle + |-\rangle)\otimes|H\rangle - 2(|+\rangle + |-\rangle)\otimes|V\rangle + 2(|+\rangle - |-\rangle)\otimes|V\rangle] \\ &= \frac{1}{3\sqrt{2}}[|+\rangle \otimes |H\rangle + |-\rangle \otimes (|H\rangle - 4|V\rangle)], \end{aligned}$$

which is the same as Eq. (2.13).

Solution to Exercise 2.28. According to Eq. (2.16),

$$\sum_i 1/\mathcal{N}_i^2 = \sum_{ij} |\Psi_{ij}|^2 = 1.$$

In the last equality, we used the fact that the state $|\Psi\rangle$ is also normalized.

Solution to Exercise 2.29.

a) We can rewrite the state in question as

$$\begin{aligned} |\Psi\rangle &= \mathcal{N}\frac{1}{\sqrt{2}}[(|H\rangle + i|V\rangle)\otimes|V\rangle + |H\rangle \otimes (|H\rangle + |V\rangle)] \\ &= \mathcal{N}\frac{1}{\sqrt{2}}[|HH\rangle + 2|HV\rangle + i|VV\rangle]. \end{aligned}$$

Accordingly, $\langle\Psi|\,\Psi\rangle = 3\mathcal{N}^2$, so $\mathcal{N} = 1/\sqrt{3}$.

b) In order to rewrite state $|\Psi\rangle$ in the form of Eq. (2.15), we group the terms associated with Alice's horizontal and vertical polarizations and re-normalize each term:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{6}}[|H\rangle \otimes (|H\rangle + 2|V\rangle) + i|V\rangle \otimes |V\rangle] \\ &= \sqrt{\frac{5}{6}}|H\rangle \otimes \frac{|H\rangle + 2|V\rangle}{\sqrt{5}} + i\sqrt{\frac{1}{6}}|V\rangle \otimes |V\rangle .\end{aligned}$$

c) It follows from the above result that Alice will detect $|H\rangle$ with the probability $\text{pr}_H = \frac{5}{6}$, in which case the state prepared with Bob will be $\frac{|H\rangle+2|V\rangle}{\sqrt{5}}$, and Alice will detect $|V\rangle$ with the probability $\text{pr}_V = \frac{1}{6}$, in which case the state prepared with Bob will be $|V\rangle$.

Solution to Exercise 2.30.

$$\begin{aligned}\langle \psi_{\text{Bob}} |\, \Omega\rangle &= (2\langle H| - i\langle V|)_{\text{Bob}}(2|HH\rangle + 3|HV\rangle + 4|VH\rangle) \\ &= 2|H\rangle_{\text{Alice}}(2\langle H| - i\langle V|)_{\text{Bob}}|H\rangle_{\text{Bob}} \\ &+ 3|H\rangle_{\text{Alice}}(2\langle H| - i\langle V|)_{\text{Bob}}|V\rangle_{\text{Bob}} \\ &+ 4|V\rangle_{\text{Alice}}(2\langle H| - i\langle V|)_{\text{Bob}}|H\rangle_{\text{Bob}} \\ &= (4|H\rangle - 3i|H\rangle + 8|V\rangle)_{\text{Alice}} = [(4-3i)|H\rangle + 8|V\rangle]_{\text{Alice}};\end{aligned}$$

$$\begin{aligned}\langle \Pi |\, \psi_{\text{Alice}}\rangle &= (2\langle H| - i\langle V|)_{\text{Alice}} \otimes (2|H\rangle + i|V\rangle)_{\text{Alice}}(-i\langle H| - \langle V|)_{\text{Bob}} \\ &= [(2\langle H| - i\langle V|)(2|H\rangle + i|V\rangle)]_{\text{Alice}}(-i\langle H| - \langle V|)_{\text{Bob}} \\ &= 5(-i\langle H| - \langle V|)_{\text{Bob}}.\end{aligned}$$

Mind the complex conjugation when converting a ket to a bra.

Solution to Exercise 2.31. Let us decompose $|a\rangle$ and $|b\rangle$ in their respective bases:

$$\begin{aligned}|a\rangle &= \sum_i a_i |v_i\rangle; \\ |b\rangle &= \sum_j b_j |w_j\rangle .\end{aligned}$$

Then $|ab\rangle = \sum_{i,j} a_i b_j |v_i w_j\rangle$. Applying the definition (2.17a) of the partial inner product, we have

$$\langle a' |\, \Psi\rangle = \sum_{ij} a_i b_j \langle a' |\, v_i\rangle |w_j\rangle \tag{S2.13}$$

$$= \sum_i a_i \langle a' |\, v_i\rangle \times \sum_j b_j |w_j\rangle \tag{S2.14}$$

$$= \langle a' |\, a\rangle |b\rangle \tag{S2.15}$$

Solution to Exercise 2.32. Let $|\Psi\rangle = \sum_{ij} \Psi_{ij} |v_i\rangle |w_j\rangle$. Then, according to the definition (2.17a),

$$\langle b|\langle a|\Psi\rangle = \langle b|\left(\sum_{ij}\Psi_{ij}\langle a|v_i\rangle|w_j\rangle\right) = \sum_{ij}\Psi_{ij}\langle a|v_i\rangle\langle b|w_j\rangle;$$

$$\langle a|\langle b|\Psi\rangle = \langle a|\left(\sum_{ij}\Psi_{ij}|v_i\rangle\langle b|w_j\rangle\right) = \sum_{ij}\Psi_{ij}\langle a|v_i\rangle\langle b|w_j\rangle;$$

$$\langle ab|\Psi\rangle = \langle ab|\left(\sum_{ij}\Psi_{ij}|v_iw_j\rangle\right) = \sum_{ij}\Psi_{ij}\langle a|v_i\rangle\langle b|w_j\rangle,$$

where the last equation is obtained from the definition of the inner product in a tensor product space.

Solution to Exercise 2.33. We use $\lambda_{ij} = \langle v_iw_j|\Psi\rangle$ and $\mu_{kl} = \langle v'_kw'_l|\Psi\rangle$ as well as the resolution of the identity (Sec. A.6.3) to transform the left-hand side of Eq. (2.21). Specifically, we insert two identity operators, $\sum_k |v'_k\rangle\langle v'_k|$ and $\sum_l |w'_l\rangle\langle w'_l|$.

$$\begin{aligned}\sum_{ij}\Psi_{ij}\langle a|v_i\rangle|w_j\rangle &= \sum_{ij}\langle v_iw_j|\Psi\rangle\langle a|v_i\rangle|w_j\rangle\\ &= \sum_{ijkl}\langle v_iw_j|\Psi\rangle\langle a|v'_k\rangle\langle v'_k|v_i\rangle|w'_l\rangle\langle w'_l|w_j\rangle\\ &= \sum_{ijkl}\langle v_iw_j|\Psi\rangle\langle a|v'_k\rangle\langle v'_kw'_l|v_iw_j\rangle|w'_l\rangle\\ &= \sum_{kl}\langle v'_kw'_l|\left(\sum_{ij}|v_iw_j\rangle\langle v_iw_j|\right)|\Psi\rangle\langle a|v'_k\rangle|w'_l\rangle\\ &= \sum_{kl}\langle v'_kw'_l|\Psi\rangle\langle a|v'_k\rangle|w'_l\rangle = \sum_{kl}\Psi'_{kl}\langle a|v'_k\rangle|w'_l\rangle\end{aligned}$$

Solution to Exercise 2.34.

a) Taking the partial inner product of both sides of Eq. (2.15) with an arbitrary element $\langle v_j|$ of Alice's measurement basis, we find

$$_A\langle v_j|\Psi\rangle \overset{(2.18)}{=} \sum_i \frac{1}{\mathcal{N}_i}\langle v_j|v_i\rangle\otimes|b_i\rangle = \frac{1}{\mathcal{N}_j}|b_j\rangle$$

b) This follows from the previous result and the fact that $|b_j\rangle$ is normalized.

Solution to Exercise 2.35. Since $|\Psi\rangle = \frac{1}{\sqrt{3}}(|RV\rangle + |H+\rangle)$, we find

$$\begin{aligned}_A\langle H|\Psi\rangle &= \frac{1}{\sqrt{3}}(\langle H|R\rangle|V\rangle + \langle H|H\rangle|+\rangle) = \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}|V\rangle + |+\rangle) = \frac{1}{\sqrt{6}}(2|V\rangle + |H\rangle);\\ _A\langle V|\Psi\rangle &= \frac{1}{\sqrt{3}}(\langle V|R\rangle|V\rangle + \langle V|H\rangle|+\rangle) = \frac{1}{\sqrt{6}}|V\rangle.\end{aligned}$$

These are the (unnormalized) states in which Alice's measurement prepares Bob's photon. The probabilities are the squared norms of these states:

$$\mathrm{pr}_{A,H} = \left\| \frac{1}{\sqrt{6}}(2\,|V\rangle + |H\rangle) \right\| = \frac{5}{6};\ \mathrm{pr}_{A,V} = \left\| \frac{1}{\sqrt{6}}(|V\rangle + |H\rangle) \right\| = \frac{1}{6}.$$

Solution to Exercise 2.36. I will show the proof for the Bell state $|\Phi^+\rangle$. Let the first element in Alice's orthonormal basis be given by $|v_1\rangle = a\,|H\rangle + b\,|V\rangle$, where a and b are arbitrary complex numbers such that $|a|^2 + |b|^2 = 1$. Then

$$_A\langle v_1|\,\Phi^+\rangle = \frac{1}{\sqrt{2}}(a^*\,\langle H| + b^*\,\langle V|)(|HH\rangle + |VV\rangle) = \frac{1}{\sqrt{2}}\,(a^*\,|H\rangle + b^*\,|V\rangle)$$

and thus

$$\mathrm{pr}_1 = \frac{1}{2}\,(a\,\langle H| + b\,\langle V|)\,(a^*\,|H\rangle + b^*\,|V\rangle) = \frac{1}{2}(|a|^2 + |b|^2) = \frac{1}{2}.$$

Then the probability to observe the other element of Alice's basis must be $\mathrm{pr}_2 = 1 - \frac{1}{2} = \frac{1}{2}$. The argument for the other Bell states is similar.

Solution to Exercise 2.38. By analogy to Ex. 2.8 we notice that the state $|\Psi^-\rangle$ can be expressed as

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{H}\tilde{V}\rangle - |\tilde{V}\tilde{H}\rangle\right), \tag{S2.16}$$

where the states $|\tilde{H}\rangle = \alpha\,|H\rangle + \beta\,|V\rangle$ and $|\tilde{V}\rangle = -\beta^*\,|H\rangle + \alpha^*\,|V\rangle$ form an orthonormal basis and $|\tilde{H}\rangle$ is the state Alice desires to prepare at Bob's location. From Eq. (S2.16) we find that Alice should measure in the basis $\{|\tilde{H}\rangle, |\tilde{V}\rangle\}$. The remote state preparation of $|\tilde{H}\rangle$ occurs if Alice detects $|\tilde{V}\rangle$, which happens with a probability of 1/2 as per Ex. 2.36.

Solution to Exercise 2.39. As we know from Ex. 2.26, Alice, when measuring in the basis $\{|\theta\rangle, |\frac{\pi}{2}+\theta\rangle\}$, will observe either result with probability $\mathrm{pr}_{\text{Alice observes }|\theta\rangle} = \mathrm{pr}_{\text{Alice observes }|\frac{\pi}{2}+\theta\rangle} = 1/2$.

Suppose Alice observes $|\theta\rangle$. Bob's state will then project onto $|\frac{\pi}{2}+\theta\rangle$. Conditioned on this event, Bob, who measures in the canonical basis, will have the following probabilities:

$$\mathrm{pr}_{\text{Bob observes }|H\rangle\ |\ \text{Alice observes }|\theta\rangle} = \left|\left\langle H\middle|\,\frac{\pi}{2}+\theta\right\rangle\right|^2 = \sin^2\theta;$$
$$\mathrm{pr}_{\text{Bob observes }|V\rangle\ |\ \text{Alice observes }|\theta\rangle} = \left|\left\langle V\middle|\,\frac{\pi}{2}+\theta\right\rangle\right|^2 = \cos^2\theta.$$

Similarly, if Alice observes $|\frac{\pi}{2}+\theta\rangle$, Bob obtains $|\theta\rangle$ so the conditional probabilities are

$$\mathrm{pr}_{\text{Bob observes }|H\rangle\ |\ \text{Alice observes }|\frac{\pi}{2}+\theta\rangle} = |\langle H|\,\theta\rangle|^2 = \cos^2\theta;$$
$$\mathrm{pr}_{\text{Bob observes }|V\rangle\ |\ \text{Alice observes }|\frac{\pi}{2}+\theta\rangle} = |\langle V|\,\theta\rangle|^2 = \sin^2\theta.$$

To find the overall probability for Bob to observe H, we must use rule (B.6) for conditional probabilities:

$$\begin{aligned}
\mathrm{pr}_{\text{Bob observes }|H\rangle} =& \mathrm{pr}_{\text{Bob observes }|H\rangle\ |\ \text{Alice observes }|\theta\rangle}\mathrm{pr}_{\text{Alice observes }|\theta\rangle} \\
&+\mathrm{pr}_{\text{Bob observes }|H\rangle\ |\ \text{Alice observes }\left|\frac{\pi}{2}+\theta\right\rangle}\mathrm{pr}_{\text{Alice observes }\left|\frac{\pi}{2}+\theta\right\rangle} \\
&= \sin^2\theta/2+\cos^2\theta/2 = 1/2.
\end{aligned}$$

In the same fashion, we find

$$\mathrm{pr}_{\text{Bob observes }|H\rangle} = 1/2.$$

Solution to Exercise 2.40. For the first scenario, the result immediately follows from the original Measurement Postulate. Let us analyze the second scenario. In contrast with the previous solution, rather than using conditional probabilities, we will argue in terms of unnormalized states which incorporate probabilities as their norm. This difference is just a matter of bookkeeping, the physics is the same.

Alice's measurement will generate unnormalized state

$$\hat{\Pi}_{A,i}|\Psi\rangle = |v_i\rangle_A \otimes {}_A\langle v_i|\,\Psi\rangle_B\,,$$

where i is random. If Bob now performs his measurement on his portion of that state, the probability for him to observe $|w_j\rangle$ is

$$\mathrm{pr}_{\text{Bob observes }|w_j\rangle\text{ AND Alice observes }|v_i\rangle} = |\left\langle w_j\right|(\langle v_i|\,\Psi\rangle)|^2. \tag{S2.17}$$

As we found in Ex. 2.32, $\left\langle w_j\right|(\langle v_i|\,\Psi\rangle) = \left\langle v_i w_j\right|\Psi\rangle$. Accordingly,

$$\mathrm{pr}_{\text{Bob observes }|w_j\rangle\text{ AND Alice observes }|v_i\rangle} = |\left\langle v_i w_j\right|\Psi\rangle|^2, \tag{S2.18}$$

which is the same as what we had in the first scenario. The equivalence of the third scenario to the first one is proven in the same way.

Solution to Exercise 2.41.

To find the overall probability for Bob to detect $|w_j\rangle$, we must sum over all possible Alice's outcomes:

$$\begin{aligned}
\mathrm{pr}_{\text{Bob observes }|w_j\rangle} &= \sum_i \mathrm{pr}_{\text{Bob observes }|w_j\rangle\text{ AND Alice observes }|v_i\rangle} \\
&= \sum_i |\left\langle v_i, w_j\right|\Psi\rangle|^2 \\
&= \sum_i \left\langle\Psi\right| v_i, w_j\rangle\left\langle v_i, w_j\right|\Psi\rangle \\
&\stackrel{(2.19)}{=} \sum_i \left\langle\Psi\right| w_j\rangle_B |v_i\rangle_{A\ A}\langle v_i|_B\langle w_j|\,\Psi\rangle \\
&= \left\langle\Psi\right| w_j\rangle\,\hat{\mathbf{1}}\left\langle w_j\right|\Psi\rangle \\
&= \|\left\langle w_j\right|\Psi\rangle\|^2,
\end{aligned}$$

which is the same probability that Bob would have if he performed his measurement before Alice. Obviously, this probability does not depend on the sequence of Alice's and Bob's measurements nor on Alice's choice of basis $\{|v_i\rangle\}$.

Solution to Exercise 2.42. If cloning were possible, Alice and Bob could implement the following protocol. They start with sharing an entangled state, e.g. $|\Psi^-\rangle$. When Alice needs to send a message to Bob, she encodes this message in the value of angle θ between 0 and $\frac{\pi}{2}$, and then performs a measurement of her photon in the basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$, thereby instantly remotely preparing one of these two states at Bob's station. Bob makes many copies of this state and performs quantum tomography (see Sec. 1.4.2) on them, thereby determining the polarization angle of his remotely prepared photon with an arbitrarily high precision. Even though this angle can be either θ or $\frac{\pi}{2}+\theta$, it is sufficient to determine θ, which is known to be between 0 and $\frac{\pi}{2}$. Then he decodes this value into Alice's original message.

Solution to Exercise 2.43. If Alice has measured her photon in the *canonical basis*, the resulting unnormalized states for Bob are as follows.

a)

$$
\begin{aligned}
{}_A\langle H|\,\Psi\rangle &= |H\rangle/\sqrt{5};\\
{}_A\langle V|\,\Psi\rangle &= 2\,|V\rangle/\sqrt{5}.
\end{aligned}
$$

Accordingly, the verbal description of Bob's photon is "either $|H\rangle$ with probability $1/5$ or $|V\rangle$ with probability $4/5$".

b)

$$
\begin{aligned}
{}_A\langle H|\,\Psi\rangle &= (|H\rangle+|V\rangle)/\sqrt{3} = \sqrt{2/3}\,|+\rangle;\\
{}_A\langle V|\,\Psi\rangle &= |V\rangle/\sqrt{3}.
\end{aligned}
$$

This state is verbally described as "either $|+\rangle$ with probability $2/3$ or $|V\rangle$ with probability $1/3$".
Note that, when projecting onto $|H\rangle$, Alice does not destroy the coherence between Bob's $|H\rangle$ and $|V\rangle$. This can also be seen by rewriting the initial state as

$$
|\Psi\rangle = \sqrt{2/3}\,|H+\rangle + \sqrt{1/3}\,|VV\rangle.
$$

For the *diagonal basis*,

a)

$$
\begin{aligned}
{}_A\langle +|\,\Psi\rangle &= \sqrt{1/10}\,|H\rangle + \sqrt{2/5}\,|V\rangle = \frac{1}{\sqrt{2}}(\sqrt{1/5}\,|H\rangle + \sqrt{4/5}\,|V\rangle);\\
{}_A\langle -|\,\Psi\rangle &= \sqrt{1/10}\,|H\rangle - \sqrt{2/5}\,|V\rangle = \frac{1}{\sqrt{2}}(\sqrt{1/5}\,|H\rangle - \sqrt{4/5}\,|V\rangle),
\end{aligned}
$$

where the state vectors in parentheses are normalized. Bob's photon is either $\sqrt{1/5}\,|H\rangle + \sqrt{4/5}\,|V\rangle$ or $\sqrt{1/5}\,|H\rangle - \sqrt{4/5}\,|V\rangle$ with probabilities $1/2$ each.

b)

$$
\begin{aligned}
{}_A\langle +|\,\Psi\rangle &= \sqrt{1/6}\,|H\rangle + \sqrt{2/3}\,|V\rangle = \sqrt{\frac{5}{6}}(\sqrt{1/5}\,|H\rangle + \sqrt{4/5}\,|V\rangle);\\
{}_A\langle -|\,\Psi\rangle &= \sqrt{1/6}\,|H\rangle.
\end{aligned}
$$

This state is either $\sqrt{1/5}\,|H\rangle+\sqrt{4/5}\,|V\rangle$ with probability 5/6 or $|H\rangle$ with probability 1/6.

Solution to Exercise 2.44. Let

$$\mathrm{pr}_{M_A,M_B,N_A,N_B}=\sum_{\lambda_A,\lambda_B}\mathrm{pr}_{\lambda_A,\lambda_B}\mathrm{pr}_{M_A|\lambda_A}\mathrm{pr}_{M_B|\lambda_B}\mathrm{pr}_{N_A|\lambda_A}\mathrm{pr}_{N_B|\lambda_B}. \tag{S2.19}$$

Given that

$$\sum_{M_A=-1}^{1}\mathrm{pr}_{M_A|\lambda_A}=1;\ \sum_{M_B=-1}^{1}\mathrm{pr}_{M_B|\lambda_B}=1;\ \sum_{N_A=-1}^{1}\mathrm{pr}_{N_A|\lambda_A}=1;\ \sum_{N_B=-1}^{1}\mathrm{pr}_{N_B|\lambda_B}=1 \tag{S2.20}$$

(because, e.g., for a given λ_A, the values that M_A can take are either $+1$ or -1), we find

$$\sum_{M_A,M_B,N_A,N_B=-1}^{+1}\mathrm{pr}_{M_A,M_B,N_A,N_B}=\sum_{\lambda_A,\lambda_B}\mathrm{pr}_{\lambda_A,\lambda_B}=1.$$

Let us now obtain the first term of Eq. (2.26) from the first term of Eq. (2.24); the remaining terms are calculated similarly. We have

$$\begin{aligned}\sum_{M_A,M_B=-1}^{+1}\mathrm{pr}_{M_A,M_B}M_AM_B &\overset{(2.25)}{=}\sum_{\lambda_A,\lambda_B}\left(\sum_{M_A,M_B=-1}^{+1}\mathrm{pr}_{\lambda_A,\lambda_B}\mathrm{pr}_{M_A|\lambda_A}\mathrm{pr}_{M_B|\lambda_B}M_AM_B\right)\\ &\overset{(S2.20)}{=}\sum_{\lambda_A,\lambda_B}\sum_{M_A,M_B,N_A,N_B=-1}^{+1}\mathrm{pr}_{\lambda_A,\lambda_B}\mathrm{pr}_{M_A|\lambda_A}\mathrm{pr}_{M_B|\lambda_B}\mathrm{pr}_{M_B|\lambda_B}\mathrm{pr}_{N_B|\lambda_B}M_AM_B\\ &\overset{(S2.19)}{=}\sum_{M_A,M_B,N_A,N_B=-1}^{+1}\mathrm{pr}_{M_A,M_B,N_A,N_B}M_AM_B.\end{aligned}$$

Solution to Exercise 2.45. Equation (2.26) can be written as $\langle S\rangle=\langle M_A(M_B-N_B)+N_A(M_B+N_B)\rangle$. Consider any possible set of values for $\{M_A,M_B,N_A,N_B\}$ displayed on the Fig. 2.3 in a single event. Because both M_B and N_B have values of $+1$ or -1, either (M_B-N_B) or (M_B+N_B) must be equal to zero. Because both M_A and N_A is either $+1$ or -1, we find that the value of S for that event must be either $+2$ or -2. Averaging over all events, which is equivalent to averaging over the probability distribution $\mathrm{pr}_{M_A,M_B,N_A,N_B}$, we have $|\langle S\rangle|\le 2$. This is the Bell inequality.

Solution to Exercise 2.46. We find

$$\begin{aligned}\hat\sigma_\theta &= |\theta\rangle\langle\theta|-\left|\frac{\pi}{2}+\theta\right\rangle\left\langle\frac{\pi}{2}+\theta\right|\\ &=(\cos\theta\,|H\rangle+\sin\theta\,|V\rangle)(\cos\theta\,\langle H|+\sin\theta\,\langle V|)-(-\sin\theta\,|H\rangle+\cos\theta\,|V\rangle)(-\sin\theta\,\langle H|+\cos\theta\,\langle V|)\\ &=(\cos^2\theta-\sin^2\theta)\,|H\rangle\langle H|-(\cos^2\theta-\sin^2\theta)\,|V\rangle\langle V|+2\cos\theta\sin\theta\,|H\rangle\langle V|+2\cos\theta\sin\theta\,|V\rangle\langle H|\\ &=\cos(2\theta)(|H\rangle\langle H|-|V\rangle\langle V|)+\sin(2\theta)(|H\rangle\langle V|+|V\rangle\langle H|),\end{aligned} \tag{S2.21}$$

and thus

$$\begin{aligned}\hat{M}_A &= \hat{\sigma}_0 = |H\rangle\langle H| - |V\rangle\langle V| = \hat{\sigma}_z;\\ \hat{M}_B &= \hat{\sigma}_{\pi/8} = \frac{1}{\sqrt{2}}(|H\rangle\langle H| - |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|);\\ \hat{N}_A &= \hat{\sigma}_{\pi/4} = |H\rangle\langle V| + |V\rangle\langle H| = \hat{\sigma}_x;\\ \hat{N}_B &= \hat{\sigma}_{3\pi/8} = \frac{1}{\sqrt{2}}(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|).\end{aligned} \tag{S2.22}$$

Solution to Exercise 2.47.

a) To determine $\langle\Psi^-|\hat{M}_A\otimes\hat{M}_B|\Psi^-\rangle$, we first calculate $\hat{M}_A|\Psi^-\rangle$ (because $\hat{M}_A$ and $\hat{M}_B$ live in different linear spaces, they commute, so we can apply them in any order). The operator $\hat{M}_A$ acts on Alice's photon, leaving the horizontal polarization unchanged, but multiplying the vertical polarization state by -1:

$$\hat{M}_A\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|H\rangle\langle H| - |V\rangle\langle V|)_A(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle).$$

Next, we act with the operator $\hat{M}_B$ on Bob's photon:

$$\begin{aligned}\hat{M}_A\otimes\hat{M}_B\left|\Psi^-\right\rangle &= \frac{1}{2}(|H\rangle\langle H| - |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle + |VH\rangle)\\ &= \frac{1}{2}(|VH\rangle - |HV\rangle + |HH\rangle + |VV\rangle)\end{aligned}$$

and finally

$$\left\langle\Psi^-\right|\hat{M}_A\otimes\hat{M}_B\left|\Psi^-\right\rangle = \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(|VH\rangle - |HV\rangle + |HH\rangle + |VV\rangle) = -\frac{1}{\sqrt{2}}.$$

Of course, the same calculation could also have been carried out in the matrix form, akin to Ex. 2.13.

b) The second matrix element is found in a similar manner:

$$\begin{aligned}&\left\langle\Psi^-\right|\hat{M}_A\otimes\hat{N}_B\left|\Psi^-\right\rangle\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(|H\rangle\langle H| - |V\rangle\langle V|)_A(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle - |VH\rangle)\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle + |VH\rangle)\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(-|VH\rangle + |HV\rangle + |HH\rangle + |VV\rangle) = \frac{1}{\sqrt{2}}.\end{aligned}$$

c) The third and fourth matrix elements could also be found by a direct calculation. The calculation can however be avoided if we remember that the state $|\Psi^-\rangle$ is isotropic. If both Alice and Bob rotate their reference frames by an angle $\pi/8$, state $|\Psi^-\rangle$ will remain unchanged, operator $\hat{N}_A$ in Alice's space will become $\hat{M}_B$, and operator $\hat{M}_B$ in Bob's space will become $\hat{M}_A$. In the new reference frame, we thus need to calculate the expectation value of the operator $\hat{M}_B \otimes \hat{M}_A$. Because state $|\Psi^-\rangle$ is antisymmetric with respect to switching Alice and Bob, the desired expectation value equals that of $\hat{M}_A \otimes \hat{M}_B$ determined in part (a), i.e. $-1/\sqrt{2}$.

d) If we rotate Alice's and Bob's reference frames by $\pi/4$, operators $\hat{N}_A$ and $\hat{N}_B$ will become $\hat{M}_A$ and $\hat{M}_B$, respectively. The desired expectation value is once again equal to $\langle \hat{M}_A \otimes \hat{M}_B \rangle = -1/\sqrt{2}$

Solution to Exercise 2.49. Because we are playing "devil's advocate", we can make any assumptions about the operation of the particle source, the information the particles carry, and the way Alice's and Bob's apparata interpret that information, as long as these assumptions are consistent with local realism. So let us assume that each particle carries two bits of information:

- whether the apparatus receiving this particle should display a value when the observer presses the M or N button;
- whether the apparatus, in case the button pressed by the observer is consistent with the first bit, should display $+1$ or -1.

The source chooses the first bits for each particle pair randomly. The second pair of bits is chosen also randomly, but so that

- If the first bits in both Alice's and Bob's particles are M, the second pair of bits should exhibit an average correlation of $\langle M_A M_B \rangle = -1/\sqrt{2}$;
- If the first bit is M in Alice's particle and N in Bob's particle, the second pair of bits should exhibit an average correlation of $\langle M_A N_B \rangle = 1/\sqrt{2}$;
- If the first bit is N in Alice's particle and M in Bob's particle, the second pair of bits should exhibit an average correlation of $\langle N_A M_B \rangle = -1/\sqrt{2}$;
- If the first bits in both Alice's and Bob's particles are N, the second pair of bits should exhibit an average correlation of $\langle N_A N_B \rangle = -1/\sqrt{2}$.

In this way, each apparatus will display a value in one-half of all events. When both apparata do respond, the correlations of their responses will mimic those observed in the quantum case (Ex. 2.47), thereby violating the Bell inequality.

Solution to Exercise 2.50. For the events in which the detectors at both Alice's and Bob's stations function properly, which happens with the probability $\mathrm{pr}_{\text{success}} = \eta^2$, we have $\langle S \rangle_{\text{success}} = 2\sqrt{2}$. If detectors at either Alice's or Bob's station fail to register a photon, which happens with the probablility $\mathrm{pr}_{\text{fail}} = 1 - \eta^2$ the displayed values at the two stations will be completely uncorrelated, so $\langle S \rangle_{\text{fail}} = 0$. Taking both these types of events into account, we find

$$\langle S \rangle = \mathrm{pr}_{\text{success}} \langle S \rangle_{\text{success}} + \mathrm{pr}_{\text{fail}} \langle S \rangle_{\text{fail}} = 2\sqrt{2}\eta^2.$$

The critical efficiency value to violate the Bell inequality is then $\eta_{\min} = 2^{-\frac{1}{4}} = 0.84$.

Solution to Exercise 2.51. The argument runs in complete analogy to Ex. 2.44. We introduce hidden parameters $\lambda_A, \lambda_B, \lambda_C$ associated with the three particles such that the values displayed on the three apparata depends on these parameters:

$$\mathrm{pr}_{\sigma_{iA},\sigma_{jB},\sigma_{kC}} = \sum_{\lambda_A,\lambda_B,\lambda_C} \mathrm{pr}_{\lambda_A,\lambda_B,\lambda_C} \mathrm{pr}_{\sigma_{iA}|\lambda_A} \mathrm{pr}_{\sigma_{jB}|\lambda_B} \mathrm{pr}_{\sigma_{kC}|\lambda_C}, \tag{S2.23}$$

where each index i, j and k can take values x or y. Then we can introduce the quantity

$$\mathrm{pr}_{\sigma_{xA},\sigma_{yA},\sigma_{xB},\sigma_{yB},\sigma_{xC},\sigma_{yC}} = \sum_{\lambda_A,\lambda_B,\lambda_C} \mathrm{pr}_{\lambda_A,\lambda_B,\lambda_C} \mathrm{pr}_{\sigma_{xA}|\lambda_A} \mathrm{pr}_{\sigma_{yA}|\lambda_A} \mathrm{pr}_{\sigma_{xB}|\lambda_B} \mathrm{pr}_{\sigma_{yB}|\lambda_B} \mathrm{pr}_{\sigma_{xC}|\lambda_C} \mathrm{pr}_{\sigma_{yC}|\lambda_C}. \tag{S2.24}$$

The sum (S2.24) must be nonnegative because so are all of its terms. Further, given that

$$\sum_{\sigma_{xA}=-1}^{1} \mathrm{pr}_{\sigma_{xA}|\lambda_A} = 1; \quad \sum_{\sigma_{xB}=-1}^{1} \mathrm{pr}_{\sigma_{xB}|\lambda_B} = 1; \quad \sum_{\sigma_{xC}=-1}^{1} \mathrm{pr}_{\sigma_{xC}|\lambda_C} = 1; \tag{S2.25}$$
$$\sum_{\sigma_{yA}=-1}^{1} \mathrm{pr}_{\sigma_{yA}|\lambda_A} = 1; \quad \sum_{\sigma_{yB}=-1}^{1} \mathrm{pr}_{\sigma_{yB}|\lambda_B} = 1; \quad \sum_{\sigma_{yC}=-1}^{1} \mathrm{pr}_{\sigma_{yC}|\lambda_C} = 1,$$

we find

$$\sum_{\sigma_{xA},\sigma_{yA},\sigma_{xB},\sigma_{yB},\sigma_{xC},\sigma_{yC}=-1}^{+1} \mathrm{pr}_{\sigma_{xA},\sigma_{yA},\sigma_{xB},\sigma_{yB},\sigma_{xC},\sigma_{yC}} = \sum_{\lambda_A,\lambda_B} \mathrm{pr}_{\lambda_A,\lambda_B} = 1.$$

This means that the quantity $\mathrm{pr}_{\sigma_{xA},\sigma_{yA},\sigma_{xB},\sigma_{yB},\sigma_{xC},\sigma_{yC}}$ can be interpreted as a probability distribution.

Solution to Exercise 2.52. Recalling that $\hat{\sigma}_x = |H\rangle\langle V| + |V\rangle\langle H|$ and $\hat{\sigma}_y = -i|H\rangle\langle V| + i|V\rangle\langle H|$, we find

a)

$$\begin{aligned}
\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B} \otimes \hat{\sigma}_{y_C} |\Psi_{GHZ}\rangle &= \frac{1}{\sqrt{2}} \hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B} \otimes \hat{\sigma}_{y_C} (|HHH\rangle + |VVV\rangle) \\
&= \frac{1}{\sqrt{2}} \hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B} (i|HHV\rangle - i|VVH\rangle) \\
&= \frac{1}{\sqrt{2}} \hat{\sigma}_{x_A} (-|HVV\rangle - |VHH\rangle) \\
&= \frac{1}{\sqrt{2}} (-|VVV\rangle - |HHH\rangle) = -|\Psi_{GHZ}\rangle.
\end{aligned}$$

For the other two operators in part (a), the proof is analogous.

b)

$$\begin{aligned}
\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{x_B} \otimes \hat{\sigma}_{x_C} |\Psi_{GHZ}\rangle &= \frac{1}{\sqrt{2}} \hat{\sigma}_{x_A} \otimes \hat{\sigma}_{x_B} (|HHV\rangle + |VVH\rangle) \\
&= \frac{1}{\sqrt{2}} \hat{\sigma}_{x_A} (|HVV\rangle + |VHH\rangle) \\
&= \frac{1}{\sqrt{2}} (|VVV\rangle + |HHH\rangle) = |\Psi_{GHZ}\rangle.
\end{aligned}$$

Solution to Exercise 2.53. The decoherence consists in losing the information about the atom's entanglement partner, the environment. Following the argument of Sec. 2.2.4, we find that, after that information has been lost, the atom can be in any of the states $|x_i\rangle$ with the probability $\mathrm{pr}_i = |\psi_i|^2$.

Solution to Exercise 2.54. The initial state of the photon pair is

$$\left|\Psi^-\right\rangle \overset{(2.6)}{=} \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}\left(|\theta\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle - \left|\frac{\pi}{2}+\theta\right\rangle \otimes |\theta\rangle\right). \tag{S2.26}$$

Let us assume that Alice's measurement takes place first. Because Alice's measurement is in the $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$ basis, the entanglement between the system and Alice's apparatus will be as follows:

$$\begin{aligned}|\Psi_{SA}\rangle &= \frac{1}{\sqrt{2}}\left(|w_{A1}\rangle \otimes |\theta\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle - |w_{A2}\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle \otimes |\theta\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left[|w_{A1}\rangle \otimes |\theta\rangle \otimes (-\sin\theta\,|H\rangle + \cos\theta\,|V\rangle) - |w_{A2}\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle \otimes (\cos\theta\,|H\rangle + \sin\theta\,|V\rangle)\right]\end{aligned}$$

where $|w_{1,2}\rangle$ can correspond to avalanches in detectors 1 and 2, respectively. Now Bob entangles his apparatus with this state, producing

$$\begin{aligned}|\Psi_{SAB}\rangle = \frac{1}{\sqrt{2}}\Big[&|w_{A1}\rangle \otimes |\theta\rangle \otimes (-\sin\theta\,|H\rangle \otimes |w_{B1}\rangle + \cos\theta\,|V\rangle \otimes |w_{B2}\rangle) \\ &- |w_{A2}\rangle \otimes \left|\frac{\pi}{2}+\theta\right\rangle \otimes (\cos\theta\,|H\rangle \otimes |w_{B1}\rangle + \sin\theta\,|V\rangle \otimes |w_{B2}\rangle)\Big].\end{aligned}$$

Solution to Exercise 2.55. The number of branches that contain k out of n results with horizontal polarization is given by the combinatoric expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Since the total number of the superposition terms equals 2^n, the fraction of the terms we are interested in equals

$$\binom{n}{k}\Big/2^n = \frac{n!}{2^n k!(n-k)!} \tag{S2.27}$$

Solution to Exercise 2.57.

Without loss of generality, let us assume that n is even and find the logarithm of the ratio between the number of terms that contain k horizontal polarization components and those that contain $n/2$ such components. Using the Stirling approximation we obtain

$$\begin{aligned}r &= \log\left[\binom{n}{k}\Big/\binom{n}{n/2}\right] \\ &= \log\frac{[(n/2)!]^2}{k!(n-k)!} \\ &= 2\log[(n/2)!] - \log[(n/2+\delta)!] - \log[(n/2-\delta)!] \\ &\approx n[\log(n/2)-1] - (n/2+\delta)[\log(n/2+\delta)-1] - (n/2-\delta)[\log(n/2-\delta)-1]\end{aligned} \tag{S2.28}$$

where $\delta \equiv k - n/2$.

Now we use the Taylor decomposition to approximate

$$\log(x \pm \delta) = \log x + \log\left(1 \pm \frac{\delta}{x}\right) = \log x \pm \frac{\delta}{x} - \frac{\delta^2}{2x^2} + O(\delta^3). \tag{S2.29}$$

Substituting this result into Eq. (S2.28) we find

$$r \approx n[\log(n/2) - 1] - (n/2+\delta)\left[\log(n/2) + \frac{2\delta}{n} - \frac{2\delta^2}{n^2} - 1\right] - (n/2-\delta)\left[\log(n/2) - \frac{2\delta}{n} - \frac{2\delta^2}{n^2} - 1\right] \approx -\frac{2\delta^2}{n},$$

from which Eq. (2.41) follows.

Solution to Exercise 2.58.

a) In Ex. 2.55, we found that, in the tree shown in Fig. 2.5(a), the number of paths containing k solid branches (corresponding to an observation of the horizontal polarization) and $n-k$ dashed branches (vertical polarization) is $\binom{n}{k}$. Each solid branch in Fig. 2.5(a) is replaced by m_H branches in Fig. 2.5(b), while each dashed branch is replaced by m_V branches. Therefore the number of paths with k solid branches and $n-k$ dashed branches in Fig. 2.5(b) is $\binom{n}{k} m_H^k m_V^{n-k}$.
b) See Fig. 2.6(b);
c) Following in the footsteps of the previous exercise, we are looking for the logarithm of the ratio between the number of terms that contain k horizontal polarization components and those that contain $\alpha^2 n$ such components. We set $\delta = k - \alpha^2 n$. Using the result from part (a), we have

$$\begin{aligned} r &= \log\left[m_H^{k-\alpha^2 n} m_V^{\alpha^2 n-k} \binom{n}{k} \Big/ \binom{n}{\alpha^2 n}\right] \\ &= \log\left[\frac{\alpha^{2\delta}}{\beta^{-2\delta}} \frac{(\alpha^2 n)!(\beta^2 n)!}{(\alpha^2 n+\delta)!(\beta^2 n-\delta)}\right] \\ &\overset{\text{Stirling}}{\approx} (\log\alpha^2 - \log\beta^2)\delta \\ &\quad + \alpha^2 n(\log\alpha^2 n - 1) + \beta^2 n(\log\beta^2 n - 1) \\ &\quad - \left(\alpha^2 n+\delta\right)\left(\log(\alpha^2 n+\delta) - 1\right) \\ &\quad - \left(\beta^2 n-\delta\right)\left(\log(\beta^2 n-\delta) - 1\right) \\ &\overset{\text{(S2.29)}}{\approx} (\log\alpha^2 - \log\beta^2)\delta \\ &\quad + \alpha^2 n(\log\alpha^2 + \log n - 1) + \beta^2 n(\log\beta^2 + \log n - 1) \\ &\quad - \left(\alpha^2 n+\delta\right)\left(\log\alpha^2 + \log n + \frac{\delta}{\alpha^2 n} - \frac{\delta^2}{2\alpha^4 n^2} - 1\right) \\ &\quad - \left(\beta^2 n-\delta\right)\left(\log\beta^2 + \log n - \frac{\delta}{\beta^2 n} - \frac{\delta^2}{2\beta^4 n^2} - 1\right) \\ &\approx -\frac{\delta^2}{2\alpha^2\beta^2 n}. \end{aligned} \tag{S2.30}$$

In the above transformation, we used $m_H/m_V = \alpha^2/\beta^2$ and $\alpha^2 + \beta^2 = 1$.

Solution to Exercise 2.59.

a) From the description of the operator, we immediately write

$$\widehat{\mathrm{C}-\mathrm{NOT}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| \simeq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \tag{S2.31}$$

b) Similarly,

$$\widehat{\mathrm{C}-\mathrm{PHASE}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11| \simeq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{S2.32}$$

c) The Hadamard gate in the local space maps $|0\rangle \to |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle \to |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. In the tensor product space, the Bob's local Hadamard operator maps

$$|00\rangle \to |0+\rangle\,;$$

$$|01\rangle \to |0-\rangle\,;$$

$$|10\rangle \to |1+\rangle\,;$$

$$|11\rangle \to |1-\rangle$$

and can thus be written as

$$\hat{\mathbf{1}} \otimes \hat{H} = |0+\rangle\langle 00| + |0-\rangle\langle 01| + |1+\rangle\langle 10| + |1-\rangle\langle 11|\,.$$

Now using Eq. (A.21) we find, in the canonical basis,

$$\hat{\mathbf{1}} \otimes \hat{H} \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix}. \tag{S2.33}$$

All these operators are unitary (we can verify this against the definition of unitarity or simply notice that each of them maps an orthonormal basis onto an orthonormal basis). This means one can implement them in a physical process.

Solution to Exercise 2.60. Multiplying matrix (S2.33) by (S2.32) and again by (S2.33), we obtain matrix (S2.31).

Solution to Exercise 2.61. Because the Hamiltonian can be written as

$$\hat{H} = 0\,|HH\rangle\langle HH| + 0\,|HV\rangle\langle HV| + 0\,|VH\rangle\langle VH| + \hbar\omega\,|VV\rangle\langle VV|\,,$$

the evolution operator is

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{0}\,|HH\rangle\langle HH| + e^{0}\,|HV\rangle\langle HV| + e^{0}\,|VH\rangle\langle VH| + e^{-i\omega t}\,|VV\rangle\langle VV|\,.$$

For $\omega t = \pi$,

$$e^{-\frac{i}{\hbar}\hat{H}t} = |HH\rangle\langle HH| + |HV\rangle\langle HV| + |VH\rangle\langle VH| - |VV\rangle\langle VV|\,,$$

which constitutes the c-phase gate.

Solution to Exercise 2.62. Applied to the system and apparatus together, the c-not gate (S2.31) takes the form

$$\widehat{\mathrm{C-NOT}} = |v_1 w_1\rangle\langle v_1 w_1| + |v_1 w_2\rangle\langle v_1 w_2| + |v_2 w_2\rangle\langle v_2 w_1| + |v_2 w_1\rangle\langle v_2 w_2|\,. \qquad \text{(S2.34)}$$

It will transform the system in a state such as (2.32) and the apparatus in state $|w_1\rangle$ into

$$\widehat{\mathrm{C-NOT}}(\alpha\,|v_1\rangle + \beta\,|v_2\rangle) \otimes |w_1\rangle = (\alpha\,|v_1 w_1\rangle + \beta\,|v_2 w_2\rangle$$

in agreement with the von Neumann expression Eq. (2.33).

Solution to Exercise 2.63.

Acting with the c-not gate (S2.31) upon a separable state $(|0\rangle + |1\rangle) \otimes |0\rangle/\sqrt{2}$, we obtain $(|00\rangle + |11\rangle)/\sqrt{2} = |\Phi^+\rangle$, which, as we know (Ex. 2.6), is entangled.

The fact that c-phase can also create entanglement follows from the fact that it can be expressed as a product of local unitaries (Hadamard gates) and the c-not gate (Ex. 2.59). As we know from Ex. 2.21, a local unitary operator cannot change the state's property of being entangled. Therefore, if the c-not gate creates entanglement, so does the c-phase gate.

Here is a specific example: acting with the c-phase gate (S2.32) upon separable state $|{+}{+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, we obtain $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$. This state is entangled, because it is obtained from Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ by means of a Hadamard operation on the second photon.

Solution to Exercise 2.64. Subjecting Bell states to the c-not gate, we obtain

$$\begin{aligned}
\left|\Psi^+\right\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \to \frac{1}{\sqrt{2}}(|HV\rangle + |VV\rangle) = |{+}V\rangle\,;\\
\left|\Psi^-\right\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) \to \frac{1}{\sqrt{2}}(|HV\rangle - |VV\rangle) = |{-}V\rangle\,;\\
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle) \to \frac{1}{\sqrt{2}}(|HH\rangle + |VH\rangle) = |{+}H\rangle\,;\\
\left|\Phi^-\right\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle) \to \frac{1}{\sqrt{2}}(|HH\rangle - |VH\rangle) = |{-}H\rangle\,.
\end{aligned}$$

Now measuring the first photon in the diagonal basis, and the second photon in the canonical basis, we can distinguish all four states.

Solution to Exercise 2.65.

a)

$$|\chi\rangle \otimes |\Psi^-\rangle = (\alpha|H\rangle + \beta|V\rangle) \otimes \frac{1}{\sqrt{2}}(|H\rangle \otimes |V\rangle - |V\rangle \otimes |H\rangle)$$
$$= \frac{1}{\sqrt{2}}(\alpha|HHV\rangle - \alpha|HVH\rangle + \beta|VHV\rangle - \beta|VVH\rangle). \tag{S2.35}$$

b) From the definition of Bell states, we find

$$|HH\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}; \tag{S2.36}$$
$$|HV\rangle = \frac{|\Psi^+\rangle + |\Psi^-\rangle}{\sqrt{2}}; \tag{S2.37}$$
$$|VH\rangle = \frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}; \tag{S2.38}$$
$$|VV\rangle = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}. \tag{S2.39}$$

c) Using the results of the two previous parts, we find

$$|\chi\rangle \otimes |\Psi^-\rangle = \frac{1}{2}\left(\alpha|\Phi^+V\rangle + \alpha|\Phi^-V\rangle - \alpha|\Psi^+H\rangle - \alpha|\Psi^-H\rangle\right.$$
$$\left. + \beta|\Psi^+V\rangle - \beta|\Psi^-V\rangle - \beta|\Phi^+H\rangle + \beta|\Phi^-H\rangle\right). \tag{S2.40}$$

d) Factoring out the Bell states in Eq. (S2.40), we obtain

$$|\text{input}\rangle = \frac{1}{2}|\Psi^-\rangle(-\alpha|H\rangle - \beta|V\rangle) + \frac{1}{2}|\Psi^+\rangle(-\alpha|H\rangle + \beta|V\rangle)$$
$$+ \frac{1}{2}|\Phi^-\rangle(\alpha|V\rangle + \beta|H\rangle) + \frac{1}{2}|\Phi^+\rangle(\alpha|V\rangle - \beta|H\rangle). \tag{S2.41}$$

This equation has the form of Eq. (2.15). A measurement by Alice will randomly select one of the four terms in the above equation and prepare the corresponding state at Bob's station. Because the norm of each term is 1/2, the probability of each result is $(1/2)^2 = 1/4$.

e)
- If Alice detects $|\Psi^-\rangle$, Bob's photon will project onto $-(\alpha|H\rangle + \beta|V\rangle)$, which, up to an overall phase factor, is identical to the source state $|\chi\rangle$. In this case Bob does not need to do anything.
- If Alice detects $|\Psi^+\rangle$, Bob's photon will project onto $-(\alpha|H\rangle - \beta|V\rangle)$. To obtain $|\chi\rangle$, Bob will need to perform an operation which does not change the horizontally polarized photon, but applies a phase factor of (-1) to the vertically polarized. This operation is achieved by Pauli operator $\hat{\sigma}_z = |H\rangle\langle H| - |V\rangle\langle V|$, and physically implemented by means of a half-waveplate with its optic axis oriented horizontally or vertically (Ex. 1.26).
- If Alice detects $|\Phi^-\rangle$, Bob's photon will project onto $(\beta|H\rangle + \alpha|V\rangle)$. In this case, Bob needs to flip between horizontal and vertical polarizations, which is done by Pauli operator $\hat{\sigma}_x = |H\rangle\langle V| + |V\rangle\langle H|$. This corresponds to a half-wave plate at 45°.
- If Alice detects $|\Phi^+\rangle$, Bob's photon will project onto $(-\beta|H\rangle + \alpha|V\rangle)$. Bob must *both* flip the polarizations and shift the phase of one of the polarizations, i.e. apply $\hat{\sigma}_z\hat{\sigma}_x = |H\rangle\langle V| - |V\rangle\langle H|$ by means of two half-waveplate, one oriented at 45° and the other at 0°. Note that we can write this operator as $\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_y$.

Solution to Exercise 2.66. Proceeding in a similar fashion to the quantum teleportation argument, we find:

$$\begin{aligned}
\left|\Psi^-\Psi^-\right\rangle_{1234} &= \frac{1}{2}(|HVHV\rangle - |HVVH\rangle - |VHHV\rangle + |VHVH\rangle)_{1234} \\
&= \frac{1}{2\sqrt{2}}\left[|H\rangle_1\left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right)_{23}|V\rangle_4 - |H\rangle_1\left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)_{23}|H\rangle_4\right. \\
&- \left.|V\rangle_1\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)_{23}|V\rangle_4 + |V\rangle_1\left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)_{23}|H\rangle_4\right] \\
&= \frac{1}{2\sqrt{2}}\left[\left|\Psi^-\right\rangle_{23}(-|HV\rangle + |VH\rangle)_{14} + \left|\Psi^+\right\rangle_{23}(|HV\rangle + |VH\rangle)_{14}\right. \\
&+ \left.\left|\Phi^-\right\rangle_{23}(|HH\rangle - |VV\rangle)_{14} + \left|\Phi^+\right\rangle_{23}(-|HH\rangle - |VV\rangle)_{14}\right] \\
&= \frac{1}{2}\left[-\left|\Psi^-\right\rangle_{23}\left|\Psi^-\right\rangle_{14} + \left|\Psi^+\right\rangle_{23}\left|\Psi^+\right\rangle_{14} + \left|\Phi^-\right\rangle_{23}\left|\Phi^-\right\rangle_{14} - \left|\Phi^+\right\rangle_{23}\left|\Phi^+\right\rangle_{14}\right].
\end{aligned}$$

Detecting photons 2 and 3 in a particular Bell state will entangle the remaining two photons, projecting them onto the same Bell state. As in the case of quantum teleportation, the probability of each measurement outcome is $1/4$.

Solution to Exercise 2.67. Using the result of Ex. 2.66, we find that, when the initial states $|\Psi^-\Psi^-\rangle_{1234}$ are projected onto the detected states $|\Phi^+\rangle_{23}$ and $|\Phi^-\rangle_{23}$ in the first and second links, the states of the stored photons become $|\Phi^+\rangle_{14}$ and $|\Phi^-\rangle_{14}$, respectively. Relabeling these photons with letters from A to D, we find their joint state to be

$$\left|\Phi^+\right\rangle_{AB} \otimes \left|\Phi^-\right\rangle_{CD} = \frac{1}{2}(|HHHH\rangle - |HHVV\rangle + |VVHH\rangle - |VVVV\rangle)_{ABCD}.$$

Projecting this state onto $|\Psi^+\rangle_{BC}$, we obtain $|\Psi^-\rangle_{AD}$.

Solution to Exercise 2.68.

a) According to Beer's law (Sec. 1.6.2), the probability for each photon to reach the Bell-basis analyzer is $e^{-\beta L/2k}$. The probability that both photons reach the Bell-basis analyzer is therefore $\mathrm{pr}_1 = (e^{-\beta L/2k})^2 = e^{-\beta L/k} = 0.082$.
To find the probability of success after n attempts, we notice that the probability of failure after one attempt is $1 - \mathrm{pr}_1$, and hence the probability that all n attempts fail is $(1 - \mathrm{pr}_1)^n$. Hence the probability that at least one of the n attempts does not fail is $\mathrm{pr}_n = 1 - (1 - \mathrm{pr}_1)^n = 1 - (1 - e^{-\beta L/k})^n$.

b) Here the event whose probability is pr_n must occur simultaneously in k links. The probability of this is $\mathrm{pr}_n^k = [1 - (1 - e^{-\beta L/k})^n]^k$.

c) Solving $\mathrm{pr}_n^k = 1/2$, we find for the required number of attempts

$$n = \log_{1-\mathrm{pr}_1}\left[1 - \left(\frac{1}{2}\right)^{\frac{1}{k}}\right] = \frac{\ln(1 - 2^{-\frac{1}{k}})}{\ln(1 - e^{-\frac{\beta L}{k}})} = 31.6.$$

The required time is therefore $n/f = 31.6\ \mu$s.

d) The probability for a single photon sent directly from Alice to reach Bob is $\mathrm{pr}_1' = e^{-\beta L} = 1.39 \times 10^{-11}$. The probability of success for n' attempts is then $\mathrm{pr}_n' = 1 - (1 - e^{-\beta L})^n$. Setting $\mathrm{pr}_n' = 1/2$, we have

$$n' = \ln_{1-e^{-\beta L}} \frac{1}{2} = \frac{-\ln 2}{\ln(1 - e^{-\beta L})} \approx \frac{\ln 2}{e^{-\beta L}} = 5.0 \times 10^{10},$$

so the expected time is $t' = n'/f = 50,000$ s.