# Appendix SA
# Solutions to Appendix A exercises

**Solution to Exercise A.1.**

a) *Yes. No. Yes. Yes.* A field over itself is a linear space because all properties listed in Definition A.1 follow from the properties of addition and multiplication of the field elements. $\mathbb{R}$ over $\mathbb{C}$ is not a linear space because when you multiply a “vector” (real number) by a “scalar” (complex number) we may obtain a number that is not real, i.e. no longer an element of the linear space. Finally, $\mathbb{C}$ over $\mathbb{R}$ is a linear space because addition of complex numbers and multiplication of a complex number by a real number yields a complex number, which proves that these operations are defined correctly. Their properties can be readily verified to be equivalent to the axioms in Definition A.1.
b) *Yes. Yes. No.* Addition of two polynomials or their multiplication by a number (either real or complex) will yield a polynomial of power that is not higher than the original ones. The set of polynomials of power $> n$ does not form a linear space, e.g. because it does not contain a zero element.
c) *Yes. No.* In the first case, the zero element is $f(x) \equiv 0$. The set of functions such that $f(1) = 1$ does not contain this element.
d) *Yes.* A sum of two periodic functions with period $T$, or a product of such a function and a number is also a periodic function with period $T$.
e) *Yes.* It is known from geometry that addition of vectors and multiplication of a vector by a number yields a vector. The properties of these operations can be verified to satisfy the linear space axioms. Note that because an $N$-dimensional vector can be defined by a column with $N$ real numbers (the coordinates of the vector), we can say that the linear space of $N$-dimensional geometric vectors is *isomorphic* (equivalent) to the linear space of columns of $N$ real numbers.

**Solution to Exercise A.2.**

a) Suppose there are two zero elements, $|\text{zero}\rangle$ and $|\text{zero}'\rangle$. Then, according to Axiom 3, we see that, on one hand, $|\text{zero}\rangle + |\text{zero}'\rangle = |\text{zero}'\rangle$ and on the other hand, $|\text{zero}\rangle + |\text{zero}'\rangle \overset{\text{Axiom 1}}{=} |\text{zero}'\rangle + |\text{zero}\rangle = |\text{zero}\rangle$. Therefore, $|\text{zero}\rangle$ and $|\text{zero}'\rangle$ are equal to the same element of $\mathbb{V}$, thus they must be equal to each other.
b) Let us start with

$$|a\rangle + |x\rangle = |a\rangle \tag{SA.1}$$

and add $(-|a\rangle)$ to both sides of this equation. We can transform the left-hand side as follows:

$$|a\rangle + |x\rangle + (-|a\rangle) \overset{\text{Axiom 1,2}}{=} [|a\rangle + (-|a\rangle)] + |x\rangle \overset{\text{Axiom 4}}{=} |\text{zero}\rangle + |x\rangle = |x\rangle .$$

The right-hand side of Eq. (SA.1) is $|a\rangle + (-|a\rangle) = |\text{zero}\rangle$. The two sides are equal, i.e. $|x\rangle = |\text{zero}\rangle$.

c) $|a\rangle + 0\,|a\rangle \overset{\text{Axiom 8}}{=} 1\,|a\rangle + 0\,|a\rangle \overset{\text{Axiom 6}}{=} (1+0)\,|a\rangle = 1\,|a\rangle = |a\rangle$. It follows from Ex. A.2(b) that $0\,|a\rangle = |\text{zero}\rangle$.
d) $(-1)\,|a\rangle + |a\rangle \overset{\text{Axiom 6,8}}{=} (-1+1)\,|a\rangle = 0\,|a\rangle \overset{\text{Ex.A.2(c)}}{=} 0$
e) $(-|\text{zero}\rangle) + |\text{zero}\rangle \overset{\text{Axiom 4}}{=} |\text{zero}\rangle$. Using Ex. A.2(b) we see that $-|\text{zero}\rangle = |\text{zero}\rangle$
f) This is because $(-|a\rangle)$ can be written as $(-1)\,|a\rangle$, and multiplication of a vector by a number yields a unique vector.
g) Using Ex. A.2(d), we write $-(-|a\rangle) = (-1)[(-1)\,|a\rangle] \overset{\text{Axiom 7}}{=} [(-1)(-1)]\,|a\rangle = 1\,|a\rangle \overset{\text{Axiom 8}}{=} |a\rangle$.
h) If $|a\rangle = |b\rangle$ then $|a\rangle - |b\rangle = |a\rangle - |a\rangle = |a\rangle + (-|a\rangle) = 0$. Conversely, if $|a\rangle - |b\rangle = 0$ then, adding $|b\rangle$ to both sides and using associativity, we find $|a\rangle = |b\rangle$.

**Solution to Exercise A.3.** Suppose one of the vectors (without loss of generality we assume it is $|v_1\rangle$) can be expressed as a linear combination of others: $|v_1\rangle = \lambda_2\,|v_2\rangle + \ldots + \lambda_N\,|v_N\rangle$. Then the nontrivial linear combination $-|v_1\rangle + \lambda_2\,|v_2\rangle + \ldots + \lambda_N\,|v_N\rangle$ equals zero, i.e. the set is not linearly independent.

Conversely: suppose there exists a nontrivial linear combination $\lambda_1\,|v_1\rangle + \ldots + \lambda_N\,|v_N\rangle$ that is equal to zero. One of the $\lambda$'s (assume it is $\lambda_1$) is not equal to zero. Then we can express $|v_1\rangle = -(\lambda_2/\lambda_1)\,|v_2\rangle - \ldots - (\lambda_N/\lambda_1)\,|v_N\rangle$.

**Solution to Exercise A.4.**

a) Two vectors $\vec{v}_1$ and $\vec{v}_2$ being parallel means that there exists a number $\lambda$ such that $\vec{v}_1 = \lambda\vec{v}_2$. But that also means that one of the vectors can be expressed through the other, i.e. they are not linearly independent.
For the second part of the question, let us consider three vectors with coordinates $\vec{v}_1 = (x_1, y_1)$, $\vec{v}_2 = (x_2, y_2)$, $\vec{v}_3 = (x_3, y_3)$. Their linear dependence means that, for example, $\vec{v}_1 = \lambda_2\,|v_2\rangle + \lambda_3\vec{v}_3$. This translates into a set of equations

$$\begin{aligned} x_1 &= \lambda_2 x_2 + \lambda_3 x_3; \\ y_1 &= \lambda_2 y_2 + \lambda_3 y_3 \end{aligned} \tag{SA.2}$$

which we solve to obtain

$$\begin{aligned} \lambda_2 &= \frac{x_1 y_3 - y_1 x_3}{x_2 y_3 - x_3 y_2}; \\ \lambda_3 &= \frac{x_2 y_1 - y_2 x_1}{x_2 y_3 - x_3 y_2}, \end{aligned} \tag{SA.3}$$

meaning that the three vectors are linearly dependent. The above solution does not exist only if $x_2 y_3 - x_3 y_2 = 0$, i.e. $x_2/y_2 = x_3/y_3$. The latter case means that $\vec{v}_2$ and $\vec{v}_3$ are parallel to each other, i.e. these two vectors are not linearly independent.
b) The vectors being non-coplanar means that none of them is zero (because a zero vector can be ascribed to any plane). Now let us consider any two of the three vectors, $\vec{v}_1$ and $\vec{v}_2$. These two vectors form a plane, and any linear combination of theirs will lie within that plane. But the third vector, $\vec{v}_3$, is known to lie outside of that plane, and hence cannot be a linear combination of the first two.

**Solution to Exercise A.5.** As shown in the solution to Exercise A.4(a), any two non-parallel vectors are sufficient to express any other vector as their linear combination.

**Solution to Exercise A.6.** Suppose there exists a basis $V = \{|v_i\rangle\}$ in $\mathbb{V}$ containing $N$ elements and another basis $W = \{|w_i\rangle\}$ in $\mathbb{V}$ with $M > N$ elements. Vector $|w_1\rangle$ can be expressed through $|v\rangle$'s:

$$|w_1\rangle = \lambda_1 |v_1\rangle + \ldots + \lambda_N |v_N\rangle . \tag{SA.4}$$

One of the coefficients in this combination (without loss of generality we assume it is $\lambda_1$) must be nonzero. Then we can express $|v_1\rangle$ through

$$\{|w_1\rangle, |v_2\rangle, \ldots, |v_N\rangle\}, \tag{SA.5}$$

and therefore the above set spans $\mathbb{V}$.

Next, $|w_2\rangle$ can be expressed through the elements of this spanning set:

$$|w_2\rangle = \lambda_1' |w_1\rangle + \lambda_2' |v_2\rangle + \ldots + \lambda_N' |v_N\rangle . \tag{SA.6}$$

At least one of the coefficients in front of $|v_i\rangle$ must be nonzero, because otherwise set $W$ would be linearly dependent. Let this coefficient be $\lambda_2$. Then we can express $|v_2\rangle$ through

$$\{|w_1\rangle, |w_2\rangle, |v_3\rangle, \ldots, |v_N\rangle\}, \tag{SA.7}$$

and hence this set is also spanning $\mathbb{V}$.

We can repeat this procedure of replacing $|v\rangle$'s by $|w\rangle$'s and show that the set

$$\{|w_1\rangle, |w_2\rangle, \ldots, |w_N\rangle\}, \tag{SA.8}$$

is also a spanning set. But then all $|w_i\rangle$ with $N < i \leq M$ can be expressed as a linear combination of $|w_1\rangle, |w_2\rangle, \ldots, |w_N\rangle$, which means that the set $W$ is not linearly independent, i.e. is not a basis, which contradicts our initial assumption.

**Solution to Exercise A.7.**

a) Let $A = \{|v_i\rangle\}_{i=1}^N$ be some basis in $\mathbb{V}$. We need to prove that any linearly independent set of $N = \dim\mathbb{V}$ elements is a spanning set. Suppose this is not true, so there is another linearly independent set of $N$ vectors $B = \{|w_j\rangle\}_{j=1}^N$ not spanning $\mathbb{V}$.
Consider all possible sets that contain all of the $|w\rangle$'s and *some* of the $|v\rangle$'s. Among these sets let us choose one that has the largest number of elements, but is still linearly independent; denote it as $C$. Then all of the $|v\rangle$'s that are not in $C$ must be possible to express as a linear combination of the $C$ elements. Indeed, if there existed a $|v_m\rangle$ that is linearly independent of $|C\rangle$, then $|C\rangle \cup |v_m\rangle$ would also be linearly independent, and this contradicts our assumption about $C$.
Since all elements of $A$ can be expressed through elements of $C$, so must be all elements of $\mathbb{V}$ because $A$ is a basis. Hence $C$ is also a basis. But the number of elements in $C$ is greater than $N$, which contradicts the result of Ex. A.6.

b) Suppose there exists a set $B = \{|w_i\rangle\}_{i=1}^N$ of $N$ vectors that spans $\mathbb{V}$ but not linearly independent: some of the elements in this set can be expressed as a linear combination of others. Let us consider all possible subsets of $B$, and choose among them such a subset that has the smallest number of elements, but is still a spanning set in $\mathbb{V}$; denote it as $C$. Then $C$ must also be linearly independent, because if there existed an element in $C$ that can be expressed through others, then it can be removed from $C$ and $C$ would remain a spanning set, which contradicts our assumption about $C$. Hence $C$ is also a basis. But the number of elements in $C$ is smaller than $N$, which contradicts the result of Ex. A.6.

**Solution to Exercise A.8.** Let $\{|w_i\rangle\}_{i=1}^N$ be the basis we are trying to decompose our vector $|v\rangle$ into. Suppose that there is more than one decomposition, say

$$|v\rangle = \lambda_1 |w_1\rangle + \cdots + \lambda_N |w_N\rangle = \mu_1 |w_1\rangle + \cdots + \mu_N |w_N\rangle \tag{SA.9}$$

where $\lambda_l \neq \mu_l$ for at least one $l$. It follows that

$$0 = (\lambda_1 - \mu_1)|w_1\rangle + \cdots + (\lambda_N - \mu_N)|w_N\rangle \tag{SA.10}$$

where not all the coefficients on the right are equal to zero. The only way this can happen is if $\{|w_i\rangle\}$ is not linearly independent, in other words, not a basis.

**Solution to Exercise A.9.**

$$|v_k\rangle \simeq \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{th position} \tag{SA.11}$$

**Solution to Exercise A.10.** The ordered pair $(x,y)$ can also be written as the 2-vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and hence the following holds:

$$(x,y) \equiv \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{SA.12}$$

This tells us that the pair $(x,y)$ does indeed represent a decomposition into a basis consisting of unit vectors along the $x$ and $y$ axes, $\left\{\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

**Solution to Exercise A.11.**

a) According to Exercise A.4(a), any two non-parallel vectors form a linearly independent set. Because the space is two-dimensional, any linearly-independent set of two vectors must form a basis.
b) According to Exercise A.4(b), any three non-coplanar vectors form a linearly independent set. Because the space is three-dimensional, any linearly-independent set of three vectors must form a basis.

**Solution to Exercise A.12.** Vectors $\vec{a}$ and $\vec{d}$ are antiparallel, and hence linearly dependent. Pairs $\{\vec{a},\vec{c}\}$ and $\{\vec{b},\vec{d}\}$ are non-parallel, and hence bases according to Ex. A.11. The matrices of the given vectors in the $\{\hat{i},\hat{j}\}$ basis are

$$\vec{a} \simeq \begin{pmatrix} 2 \\ 0 \end{pmatrix};\ \vec{b} \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix};\ \vec{c} \simeq \begin{pmatrix} 0 \\ 3 \end{pmatrix};\ \vec{d} \simeq \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Accordingly, vector $\vec{b}$ decomposes as $\vec{b} = \vec{a}/(2\sqrt{2}) + \vec{c}/(3\sqrt{2})$ in the $\{\vec{a}, \vec{c}\}$ basis and simply as $\vec{b}$ in the $\{\vec{b}, \vec{d}\}$ basis.

**Solution to Exercise A.13.** Let subspace $\mathbb{V}'$ consist of the first $M$ elements of basis $\{|v_i\rangle\}$ with $M < \dim \mathbb{V}$. We need to prove that, when we add two elements of $\mathbb{V}'$ together or multiply an element of $\mathbb{V}'$ by a number, the result will be in $\mathbb{V}'$ as well. And indeed, for any

$$|a\rangle = \sum_{i=1}^{M} a_i |v_i\rangle \in \mathbb{V}', \; |b\rangle = \sum_{i=1}^{M} b_i |v_i\rangle \in \mathbb{V}',$$

we have, using the commutativity of addition and the distributivity of scalar sums,

$$|a\rangle + |b\rangle = \sum_{i=1}^{M} (a_i + b_i) |v_i\rangle$$

and, using the associativity of scalar multiplication,

$$\lambda |a\rangle = \sum_{i=1}^{M} (\lambda a_i) |v_i\rangle .$$

We see that both $|a\rangle + |b\rangle$ and $\lambda |a\rangle$ are linear combinations of the first $M$ elements of $\{|v_i\rangle\}$, and hence they are elements of $\mathbb{V}'$.

**Solution to Exercise A.14.** We need to apply the definition of the geometric scalar product, $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$, to verify each of the inner product properties.

1. $\vec{a} \cdot (\vec{b} + \vec{c}) = x_a(x_b + x_c) + y_a(y_b + y_c) = x_a x_b + x_a x_c + y_a y_b + y_a y_c = (x_a x_b + y_a y_b) + (x_a x_c + y_a y_c) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
2. $\vec{a} \cdot (\lambda \vec{b}) = (x_a \lambda x_b + y_a \lambda y_b) = \lambda (x_a x_b + y_a y_b) = \lambda \vec{a} \cdot \vec{b}$
3. $\vec{a} \cdot \vec{b} = (x_a x_b + y_a y_b) = \vec{b} \cdot \vec{a}$ (because this is a linear space over the field of real numbers, conjugation can be omitted)
4. $\vec{a} \cdot \vec{a} = x_a^2 + y_a^2$ is a real number greater or equal to zero. The only possibility for this number to be zero is to have $x_a = y_a = 0$, i.e. $\vec{a} = 0$.

**Solution to Exercise A.15.** For $|x\rangle = \sum_i \lambda_i |a_i\rangle$, we find, using Properties 1 and 2 of the inner product, $\langle b| x\rangle = \sum_i \langle b| (\lambda_i |a_i\rangle) = \sum_i \lambda_i \langle b| a_i\rangle$. According to Property 3, and $\langle x| b\rangle = \langle b| x\rangle^* = \sum_i \lambda_i^* \langle b| a_i\rangle^* = \sum_i \lambda_i^* \langle a_i| b\rangle$.

**Solution to Exercise A.16.** We write, for arbitrary $|b\rangle$, zero $= |b\rangle - |b\rangle$ and thus, by Property 1, $\langle a|$ zero$\rangle = \langle a| b\rangle - \langle a| b\rangle = 0$. Inner product $\langle$zero$| a\rangle$ is then also zero by Property 3.

**Solution to Exercise A.17.** Let $\{|v_i\rangle\}_{i=1}^{M}$ be a set of orthogonal vectors. Suppose these vectors are linearly dependent, i.e. one of them (say $|v_1\rangle$) can be written as a linear combination of others:

$$|v_1\rangle = \sum_{i=2}^{N} \lambda_i |v_i\rangle . \tag{SA.13}$$

We take the inner product of both sides of Eq. (SA.13) with $|v_1\rangle$. Using Property 1 of the inner product, we find

$$\langle v_1 | v_1 \rangle = \sum_{i=2}^{N} \lambda_i \langle v_1 | v_i \rangle . \tag{SA.14}$$

In the above equation, the left-hand side cannot be zero because of Property 4 of the inner product; the right-hand side is zero due to the ortogonality of the set $\{|v_i\rangle\}$. We arrive at a contradiction.

**Solution to Exercise A.18.** Let $|\psi'\rangle = e^{i\phi}|\psi\rangle$. Then, using the result of Ex. A.15, we write

$$\langle \psi' | \psi' \rangle = (e^{i\phi})^* \langle \psi | \psi' \rangle = (e^{-i\phi})(e^{i\phi}) \langle \psi | \psi \rangle = \langle \psi | \psi \rangle . \tag{SA.15}$$

**Solution to Exercise A.19.** This immediately follows from Ex. A.7 and A.17.

**Solution to Exercise A.20.** Let $\{|v_i\rangle\}_{i=1}^N$ be an orthonormal basis for $\mathbb{V}$. Then $|a\rangle = \sum_i a_i |v_i\rangle$ and $|b\rangle = \sum_i b_i |v_i\rangle$. Using the result of Ex. A.15, we write

$$\begin{aligned} \langle a | b \rangle &= \sum_{i,j} a_j^* b_i \langle v_j | v_i \rangle \\ &\overset{(A.3)}{=} \sum_{i,j} a_j^* b_i \delta_{ji} \\ &= \sum_i a_i^* b_i . \end{aligned}$$

**Solution to Exercise A.21.** We begin with the decomposition

$$|a\rangle = \sum_i a_i |v_i\rangle \tag{SA.16}$$

where we have assumed that $\{|v_i\rangle\}_{i=1}^N$ is our basis. We take the inner product of both sides of Eq. (SA.16) with an arbitrary basis element $|v_j\rangle$ and find, using the basis orthonormality,

$$\langle v_j | a \rangle = \langle v_j | \left( \sum_i a_i |v_i\rangle \right) = \sum_i a_i \langle v_j | v_i \rangle = \sum_i a_i \delta_{ji} = a_j .$$

**Solution to Exercise A.22.**

a) There are two vectors in set $\{|w_1\rangle, |w_2\rangle\}$. Because our Hilbert space is two-dimensional, and according to Ex. A.19, it is enough to show that the set is orthonormal in order for it to be a basis. Using the rules for the inner product (remember to apply the complex conjugation when necessary!), we find

$$\begin{aligned} \langle w_1 | w_1 \rangle &= \frac{1}{\sqrt{2}} (\langle v_1 | w_1 \rangle - i \langle v_2 | w_1 \rangle) \\ &= \frac{\langle v_1 | v_1 \rangle - i \langle v_2 | v_1 \rangle + i \langle v_1 | v_2 \rangle - ii \langle v_2 | v_2 \rangle}{2} \\ &= \frac{1+0+0+1}{2} = 1 . \end{aligned}$$

Similarly,

$$\langle w_1|\, w_2\rangle = \frac{\langle v_1|\, v_1\rangle - i\langle v_2|\, v_1\rangle - i\langle v_1|\, v_2\rangle - i(-i)\langle v_2|\, v_2\rangle}{2} = 0,$$

and hence $\langle w_2|\, w_1\rangle = \langle w_1|\, w_2\rangle^* = 0$. It remains to test $\langle w_2|\, w_2\rangle$.

$$\langle w_2|\, w_2\rangle = \frac{\langle v_1|\, v_1\rangle + i\langle v_2|\, v_1\rangle - i\langle v_1|\, v_2\rangle + i(-i)\langle v_2|\, v_2\rangle}{2} = 1.$$

b) Using the definition A.7 of a matrix form of a vector, we find

$$|\psi\rangle \overset{v\text{-basis}}{\simeq} \begin{pmatrix} 4 \\ 5 \end{pmatrix}; \quad |\phi\rangle \overset{v\text{-basis}}{\simeq} \begin{pmatrix} -2 \\ 3i \end{pmatrix}.$$

To decompose vectors $|\psi\rangle$ and $|\phi\rangle$ in basis $\{|w_1\rangle, |w_2\rangle\}$, we find their inner products with the basis elements using the matrix multiplication rule (A.5):

$$\begin{aligned}
\langle w_1|\, \psi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2}}(4-5i);\\
\langle w_1|\, \phi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} -2 \\ 3i \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle w_2|\, \psi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2}}(4+5i);\\
\langle w_2|\, \phi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} -2 \\ 3i \end{pmatrix} = \frac{-5}{\sqrt{2}},
\end{aligned}$$

so

$$\psi \overset{w\text{-basis}}{\simeq} \frac{1}{\sqrt{2}}\begin{pmatrix} 4-5i \\ 4+5i \end{pmatrix}; \quad \phi \overset{w\text{-basis}}{\simeq} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

c) For the inner product, we have

$$\begin{aligned}
\langle \psi|\, \phi\rangle &\overset{v\text{-basis}}{=} \begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 3i \end{pmatrix} = -8+15i;\\
\langle \psi|\, \phi\rangle &\overset{w\text{-basis}}{=} \frac{1}{2}\begin{pmatrix} 4+5i & 4-5i \end{pmatrix}\begin{pmatrix} 1 \\ -5 \end{pmatrix} = -8+15i.
\end{aligned}$$

**Solution to Exercise A.23.** We begin by noting that $|a\rangle$ being a normalized vector means $\langle a|\, a\rangle = 1$. On the other hand,

$$\langle a|\, a\rangle \overset{\text{(A.4)}}{=} \sum_i a_i^* a_i = \sum_i |a_i|^2,$$

which means that $\sum_i |a_i|^2 = 1$.

**Solution to Exercise A.24.** We first notice that none of the vectors $|v_i\rangle$ defined by Eq. (A.9) can be zero because each of them is a nontrivial linear combination of linearly independent vectors $|w_1\rangle, \ldots |w_j\rangle$.

Second, we need to verify that the vectors $|v_i\rangle$ are orthogonal to each other. To this end, it is enough to show that each vector $|v_{k+1}\rangle$ is orthogonal to all $|v_j\rangle$ for $j \le k$. We proceed as follows:

$$\begin{aligned}
\langle v_j | v_{k+1}\rangle &= \langle v_j| \left(\mathcal{N}\left[|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle\right]\right) \\
&= \mathcal{N}\left[\langle v_j | w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle \langle v_j | v_i\rangle\right] \\
&= \mathcal{N}\left[\langle v_j | w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle \delta_{ji}\right] \\
&= \mathcal{N}\left[\langle v_j | w_{k+1}\rangle - \langle v_j | w_{k+1}\rangle\right] \\
&= 0,
\end{aligned}$$

hence the set $\{|v_i\rangle\}$ is orthogonal. It is also normalized and contains $N = \dim \mathbb{V}$ elements. According to Ex. A.19, such a set forms a basis in $\mathbb{V}$.

**Solution to Exercise A.25.** Let us first choose an arbitrary orthonormal basis $\{|w_i\rangle\}_{i=1}^{N}$ such that $|w_N\rangle = |\psi\rangle$. Then we define the following vectors:

$$\begin{aligned}
\left|\psi^{(1)}\right\rangle &= \frac{\sqrt{m-1}\,|\psi\rangle - |w_1\rangle}{\sqrt{m}}; \\
|v_1\rangle &= \frac{\sqrt{m-1}\,|w_1\rangle + |\psi\rangle}{\sqrt{m}}.
\end{aligned}$$

It is straightforward to verify that these vectors are normalized and orthogonal to each other as well as to $|w_2\rangle, \ldots, |w_{N-1}\rangle$, so the set $\{|v_1\rangle, |w_2\rangle, \ldots, |w_{N-1}\rangle, \left|\psi^{(1)}\right\rangle\}$ forms an orthonormal basis. We also have $\langle v_1 | \psi\rangle = 1/\sqrt{m}$ and $\left\langle \psi^{(1)} \middle| \psi\right\rangle = \sqrt{m-1/m}$.

We repeat this procedure $m-1$ more times. Specifically, for each $i$ we define

$$\begin{aligned}
\left|\psi^{(i)}\right\rangle &= \frac{\sqrt{m-i}\,|\psi\rangle^{(i-1)} - |w_i\rangle}{\sqrt{m-i+1}}; \\
|v_i\rangle &= \frac{\sqrt{m-i}\,|w_i\rangle + |\psi\rangle^{(i-1)}}{\sqrt{m-i+1}}
\end{aligned}$$

so that $\langle v_i | \psi\rangle = 1/\sqrt{m}$ and $\left\langle \psi^{(i)} \middle| \psi\right\rangle = \sqrt{m-i/m}$. After the last step, we obtain orthonormal basis $\left\{|v_1\rangle, \ldots, |v_m\rangle, |w_{i+1}\rangle, \ldots, |w_{N-1}\rangle, \left|\psi^{(m)}\right\rangle\right\}$ with $\langle v_i | \psi\rangle = 1/\sqrt{m}$ for all $1 \le i \le m$, but $\langle v_i | \psi\rangle = 0$ for all $m+1 \le i \le N-1$ as well as $\left\langle \psi^{(m)} \middle| \psi\right\rangle = 0$. According to Ex. A.21, this means that $|\psi\rangle = 1/\sqrt{m}\sum_{i=1}^{m} |v_i\rangle$.

**Solution to Exercise A.26.** To prove the Cauchy-Schwarz inequality, we first note that for any vectors $|a\rangle, |b\rangle$ and complex scalar $\lambda$, the relationship

$$0 \le \|\,|a\rangle - \lambda\,|b\rangle\,\|^2 \tag{SA.17}$$

holds. Expanding out the norm, we see that

$$0 \leq \langle a|\, a\rangle - \lambda\, \langle a|\, b\rangle - \lambda^*\, \langle b|\, a\rangle + |\lambda|^2\, \langle b|\, b\rangle$$

If $|b\rangle = 0$, the Cauchy-Schwarz inequality becomes trivial. Otherwise, we set $\lambda = \langle b|\, a\rangle \,/\, \langle b|\, b\rangle = \langle a|\, b\rangle^* \,/\, \langle b|\, b\rangle$, and the above inequality turns into the following:

$$\begin{aligned} 0 &\leq \langle a|\, a\rangle - \frac{\langle a|\, b\rangle^* \langle a|\, b\rangle}{\langle b|\, b\rangle} - \frac{\langle b|\, a\rangle^* \langle b|\, a\rangle}{\langle b|\, b\rangle} + \left|\frac{\langle b|\, a\rangle}{\langle b|\, b\rangle}\right|^2 \langle b|\, b\rangle \\ &= \langle a|\, a\rangle - 2\frac{|\langle a|\, b\rangle|^2}{\langle b|\, b\rangle} + \frac{|\langle a|\, b\rangle|^2}{\langle b|\, b\rangle} \\ &= \langle a|\, a\rangle - \frac{|\langle a|\, b\rangle|^2}{\langle b|\, b\rangle}, \end{aligned}$$

from which we find

$$|\langle a|\, b\rangle|^2 \leq \langle a|\, a\rangle\, \langle b|\, b\rangle\,. \tag{SA.18}$$

Taking the square root of each side yields the required result

$$|\langle a|\, b\rangle| \leq \|\,|a\rangle\,\| \times \|\,|b\rangle\,\|. \tag{SA.19}$$

The only case the Cauchy-Schwarz inequality can saturate is if inequality (SA.17) also saturates, which can only happen is $|a\rangle = \lambda\,|b\rangle$. Conversely, if $|a\rangle = \lambda\,|b\rangle$ for any $\lambda$, then both $|\langle a|\, b\rangle|^2 = |\lambda|^2 |\langle a|\, a\rangle|^2$ and $\langle a|\, a\rangle\, \langle b|\, b\rangle = |\lambda|^2 |\langle a|\, a\rangle|^2$, so the two sides of Eq. (SA.18) equal each other.

**Solution to Exercise A.27.** The triangle inequality is a direct consequence of the Cauchy-Schwarz inequality. To see this, we start by taking the norm of the vector $|a\rangle + |b\rangle$

$$\begin{aligned} \|\,|a\rangle + |b\rangle\,\|^2 &= \langle a|\, a\rangle + \langle a|\, b\rangle + \langle b|\, a\rangle + \langle b|\, b\rangle \\ &= \|\,|a\rangle\,\|^2 + \|\,|b\rangle\,\|^2 + \langle a|\, b\rangle^* + \langle a|\, b\rangle \\ &= \|\,|a\rangle\,\|^2 + \|\,|b\rangle\,\|^2 + 2\mathrm{Re}\{\langle a|\, b\rangle\} \\ &\leq \|\,|a\rangle\,\|^2 + \|\,|b\rangle\,\|^2 + 2|\langle a|\, b\rangle| \qquad (\text{since } \mathrm{Re}\{z\} \leq |z|) \\ &\leq \|\,|a\rangle\,\|^2 + \|\,|b\rangle\,\|^2 + 2\|\,|a\rangle\,\| \times \|\,|b\rangle\,\| \qquad (\text{by Cauchy-Schwarz}) \\ &= (\|\,|a\rangle\,\| + \|\,|b\rangle\,\|)^2. \end{aligned}$$

Taking the square root of both sides gives us the required result.

$$\|\,|a\rangle + |b\rangle\,\| \leq \|\,|a\rangle\,\| + \|\,|b\rangle\,\|. \tag{SA.20}$$

**Solution to Exercise A.28.** To show that $\mathbb{V}^\dagger$ is a linear space, we must check the linear space axioms set out in Definition A.1. Let $|a\rangle$, $|b\rangle$, $|c\rangle$ be arbitrary vectors in $\mathbb{V}^\dagger$ and $\lambda$, $\mu$ arbitrary scalars in $\mathbb{F}$. We find:

1. *Commutativity.*

$$\langle a| + \langle b| = \text{Adjoint}(|a\rangle + |b\rangle) = \text{Adjoint}(|b\rangle + |a\rangle) = \langle b| + \langle a|$$

2. *Associativity.*

$$\begin{aligned}(\langle a| + \langle b|) + \langle c| &= \text{Adjoint}((|a\rangle + |b\rangle) + |c\rangle) = \text{Adjoint}(|a\rangle + (|b\rangle + |c\rangle)) \\ &= \langle a| + (\langle b| + \langle c|).\end{aligned}$$

3. *Zero Element.* Since

$$\langle a| + \langle \text{zero}| = \text{Adjoint}(|a\rangle + |\text{zero}\rangle) = \text{Adjoint}(|a\rangle) = \langle a|,$$

$\langle \text{zero}|$ is the zero element in $\mathbb{V}^\dagger$.

4. *Opposite element.* We define $-\langle a| \equiv \text{Adjoint}(-|a\rangle)$ and verify that it is the opposite of $\langle a|$:

$$\langle a| + (-\langle a|) = \text{Adjoint}(|a\rangle + (-|a\rangle)) = \text{Adjoint}(|\text{zero}\rangle) = \langle \text{zero}|.$$

5. *Vector Distributivity.*

$$\begin{aligned}\lambda(\langle a| + \langle b|) &= \text{Adjoint}(\lambda^*(|a\rangle + |b\rangle)) = \text{Adjoint}(\lambda^*|a\rangle + \lambda^*|b\rangle) \\ &\overset{(A.12)}{=} \lambda\langle a| + \lambda\langle b|\end{aligned}$$

6. *Scalar Distributivity.*

$$\begin{aligned}(\lambda+\mu)\langle a| &= \text{Adjoint}((\lambda+\mu)^*|a\rangle) = \text{Adjoint}((\lambda^*+\mu^*)|a\rangle) \\ &= \text{Adjoint}(\lambda^*|a\rangle + \mu^*|a\rangle) = \lambda\langle a| + \mu\langle a|\end{aligned}$$

7. *Scalar Associativity.*

$$\begin{aligned}\lambda(\mu\langle a|) &= \text{Adjoint}(\lambda^*(\mu^*|a\rangle)) = \text{Adjoint}((\lambda^*\mu^*)|a\rangle) \\ &= \text{Adjoint}((\lambda\mu)^*|a\rangle) = (\lambda\mu)\langle a|\end{aligned}$$

8. *Scalar Identity.*

$$1\cdot\langle a| = \text{Adjoint}(1^*\cdot|a\rangle) = \text{Adjoint}(1\cdot|a\rangle) = \text{Adjoint}(|a\rangle) = \langle a|$$

**Solution to Exercise A.29.** Let $\{|v_i\rangle\}$ be a basis for $\mathbb{V}$. To show that $\{\langle v_i|\}$ is a basis for $\mathbb{V}^\dagger$, we must show that the set is linearly independent and spans the space.

*Spanning*. Let $\langle x| \in \mathbb{V}^\dagger$. Then, correspondingly, $|x\rangle \in \mathbb{V}$, and since $\{|v_i\rangle\}$ is a basis,

$$|x\rangle = \sum_i \lambda_i^* |v_i\rangle$$

for some set of $\lambda_i \in \mathbb{F}$. Taking the adjoint of both sides gives

$$\begin{aligned}\langle x| &= \text{Adjoint}\,(|x\rangle) \\ &= \text{Adjoint}\left(\sum_i \lambda_i\,|v_i\rangle\right) \\ &= \sum_i \lambda_i^*\,\langle v_i|\end{aligned}$$

and hence we see that $\langle x|$ is expressible via the set $\{\langle v_i|\}$. In other words, this set spans $\mathbb{V}^\dagger$.

*Linear independence*. Suppose the zero element $\langle\text{zero}|$ can be expressed as a linear combination $\langle\text{zero}| = \sum\lambda_i\,\langle v_i|$. This means that

$$\text{Adjoint}\,(|\text{zero}\rangle) = \text{Adjoint}\left(\sum_i \lambda_i^*\,|v_i\rangle\right),$$

which, in turn, implies that

$$|\text{zero}\rangle = \sum_i \lambda_i^*\,|v_i\rangle\,,$$

so the basis $\{|v_i\rangle\}$ of $\mathbb{V}$ is not linearly independent. We arrive at a contradiction.

**Solution to Exercise A.30.**

$$\text{Adjoint}\,(|v_1\rangle + i\,|v_2\rangle) \simeq \begin{pmatrix} 1 & -i \end{pmatrix}.$$

**Solution to Exercise A.31.**

a) $\hat{A}$ is linear since

$$\hat{A}(|a\rangle + |b\rangle) = 0 = 0 + 0 = \hat{A}\,|a\rangle + \hat{A}\,|b\rangle$$

and

$$\hat{A}(\lambda\,|a\rangle) = 0 = \lambda 0 = \lambda\hat{A}\,|a\rangle$$

b) $\hat{A}$ is linear since

$$\hat{A}(|a\rangle + |b\rangle) = |a\rangle + |b\rangle = \hat{A}\,|a\rangle + \hat{A}\,|b\rangle$$

and

$$\hat{A}(\lambda\,|a\rangle) = \lambda\,|a\rangle = \lambda\hat{A}\,|a\rangle$$

c) $\hat{A}$ is linear since

$$\begin{aligned}\hat{A}\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) &= \hat{A}\begin{pmatrix} x+x' \\ y+y' \end{pmatrix} = \begin{pmatrix} x+x' \\ -y-y' \end{pmatrix} \\ &= \left(\begin{pmatrix} x \\ -y \end{pmatrix} + \begin{pmatrix} x' \\ -y' \end{pmatrix}\right) \\ &= \hat{A}\begin{pmatrix} x \\ y \end{pmatrix} + \hat{A}\begin{pmatrix} x' \\ y' \end{pmatrix}\end{aligned}$$

and

$$\hat{A}\left(\lambda\begin{pmatrix}x\\y\end{pmatrix}\right)=\hat{A}\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}=\begin{pmatrix}\lambda x\\-\lambda y\end{pmatrix}$$
$$=\lambda\begin{pmatrix}x\\-y\end{pmatrix}=\lambda\hat{A}\begin{pmatrix}x\\y\end{pmatrix}$$

d) $\hat{A}$ is *not* linear. On the one hand we know that

$$\hat{A}\left(\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}x'\\y'\end{pmatrix}\right)=\hat{A}\begin{pmatrix}x+x'\\y+y'\end{pmatrix}=\begin{pmatrix}x+x'+y+y'\\(x+x')(y+y')\end{pmatrix}$$
$$=\begin{pmatrix}x+x'+y+y'\\xy+x'y+xy'+x'y'\end{pmatrix}$$

but on the other hand

$$\hat{A}\begin{pmatrix}x\\y\end{pmatrix}+\hat{A}\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}x+y\\xy\end{pmatrix}+\begin{pmatrix}x'+y'\\x'y'\end{pmatrix}$$
$$=\begin{pmatrix}x+y+x'+y'\\xy+x'y'\end{pmatrix}$$

We see that operator $\hat{A}$ is not compliant with Defn. A.15, and hence is not linear.

e) Operator $\hat{A}$ is not linear. We have

$$\hat{A}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

but

$$\hat{A}\left[\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}\right]=\hat{A}\begin{pmatrix}0\\0\end{pmatrix}+\hat{A}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}.$$

Since $\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, operator $\hat{A}$ is not compliant with Defn. A.15.

f) This operator is linear. This is easiest to picture geometrically: taking a sum of vectors $\vec{a}$ and $\vec{b}$ each rotated by an angle $\phi$ is the same as first adding the vectors and equal and rotating the sum. Similarly, rotating then scaling a vector is the same as scaling it then rotating it.

**Solution to Exercise A.32.**

a) Assuming $\hat{A}$ and $\hat{B}$ are linear and recalling the definition of operator addition, we test both linearity conditions at once:

$$\begin{aligned}(\hat{A}+\hat{B})(\mu_a|a\rangle+\mu_b|b\rangle) &\overset{(A.15)}{=} \hat{A}(\mu_a|a\rangle+\mu_b|b\rangle)+\hat{B}(\mu_a|a\rangle+\mu_b|b\rangle)\\ &\overset{(A.14)}{=} \hat{A}\mu_a|a\rangle+\hat{A}\mu_b|b\rangle+\hat{B}\mu_a|a\rangle+\hat{B}\mu_b|b\rangle\\ &= \mu_a\hat{A}|a\rangle+\mu_a\hat{B}|a\rangle+\mu_b\hat{A}|b\rangle+\mu_b\hat{B}|b\rangle\\ &= \mu_a(\hat{A}+\hat{B})|a\rangle+\mu_b(\hat{A}+\hat{B})|b\rangle\end{aligned}$$

Hence $\hat{A}+\hat{B}$ is linear.
Similarly, assuming that $\hat{A}$ is linear and testing both linearity conditions on $\lambda\hat{A}$ at once, we have

$$\begin{aligned}\lambda\hat{A}(\mu_a|a\rangle+\mu_b|b\rangle) &= \lambda(\mu_a\hat{A}|a\rangle+\mu_b\hat{A}|b\rangle)\\ &\lambda\mu_a\hat{A}|a\rangle+\lambda\mu_b\hat{A}|b\rangle\\ &= \mu_a(\lambda\hat{A}|a\rangle)+\mu_b(\lambda\hat{A}|b\rangle)\end{aligned}$$

Hence $\lambda\hat{A}$ is linear.

b) We define the zero operator as the operator that maps every vector onto $|\text{zero}\rangle$. For every operator $\hat{A}$, we can define the opposite, $-\hat{A}$, according to

$$(-\hat{A})|a\rangle \equiv -(\hat{A}|a\rangle). \tag{SA.21}$$

**Solution to Exercise A.33.** Assuming $\hat{A}$ and $\hat{B}$ are linear and recalling the definition A.18 of operator multiplication, we see by testing both linearity conditions at once that

$$\begin{aligned}(\hat{A}\hat{B})(\mu_a|a\rangle+\mu_b|b\rangle) &= \hat{A}(\hat{B}(\mu_a|a\rangle+\mu_b|b\rangle))\\ &\overset{(A.14)}{=} \hat{A}(\mu_a\hat{B}|a\rangle+\mu_b\hat{B}|b\rangle)\\ &\overset{(A.14)}{=} \mu_a\hat{A}(\hat{B}|a\rangle)+\mu_b\hat{A}(\hat{B}|b\rangle)\\ &= \mu_a(\hat{A}\hat{B})|a\rangle+\mu_b(\hat{A}\hat{B})|b\rangle).\end{aligned}$$

Hence $\hat{A}\hat{B}$ is linear.

**Solution to Exercise A.34.** Consider vector $(1,0)$. If we rotate it by $\pi/2$, we obtain $(0,1)$, and a subsequent flip around the horizontal axis results in $(0,-1)$. If we perform these operations in the reverse order, the flip will have no effect so the resulting vector is $(0,1)$.

**Solution to Exercise A.35.** Let us act with operator $\hat{A}(\hat{B}\hat{C})$ on some vector $\hat{a}$. According to Definition A.18, we find

$$\hat{A}(\hat{B}\hat{C})|a\rangle = \hat{A}[(\hat{B}\hat{C})|a\rangle] = \hat{A}[\hat{B}(\hat{C}|a\rangle)]. \tag{SA.22}$$

In other words, in order to implement the action of operator $\hat{A}(\hat{B}\hat{C})$, we must first apply operator $\hat{C}$ to vector $|a\rangle$, then apply $\hat{B}$ to the result, then apply $\hat{A}$ to the result.

Now let us look at operator $(\hat{A}\hat{B})\hat{C}$. We have

$$(\hat{A}\hat{B})\hat{C}|a\rangle = (\hat{A}\hat{B})(\hat{C}|a\rangle) = \hat{A}[\hat{B}(\hat{C}|a\rangle)]. \tag{SA.23}$$

We see that operators $\hat{A}(\hat{B}\hat{C})$ and $(\hat{A}\hat{B})\hat{C}$ map any vector in the same way, i.e. they are equal to each other.

**Solution to Exercise A.36.** In any basis $\{|v_i\rangle\}$, we have the relation $\hat{\mathbf{1}}|v\rangle_j = |v\rangle_j$. Reconciling it with Eq. (A.19), we find that the matrix of the identity operator is just the identity matrix:

$$\hat{\mathbf{1}} \simeq \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}$$

**Solution to Exercise A.37.** Relation (A.19) in the matrix form is

$$\hat{A}\left|v_j\right\rangle \simeq \begin{pmatrix} A_{1j} \\ \vdots \\ A_{Nj} \end{pmatrix}$$

**Solution to Exercise A.38.** Combining Eqs. (A.18) and (A.19), we find

$$\hat{A}\left|a\right\rangle = \hat{A}\left(\sum_j a_j\left|v_j\right\rangle\right) = \sum_j A_{ij}a_j\left|v_i\right\rangle,$$

which means that the $i$th element of the decomposition of vector $\hat{A}\left|a\right\rangle$ into our operating basis is $\sum_j A_{ij}a_j$. This is in agreement with Eq. (A.20).

**Solution to Exercise A.39.**

a) Let $C_{ij}$ be the matrix of operator $\hat{C} = \hat{A} + \hat{B}$. Then, according to the definition A.19 of the operator matrix, we should have

$$\hat{C}\left|v_j\right\rangle = \sum_i C_{ij}\left|v_i\right\rangle. \tag{SA.24}$$

On the other hand,

$$(\hat{A}+\hat{B})\left|v_j\right\rangle = \hat{A}\left|v_j\right\rangle + \hat{B}\left|v_j\right\rangle \overset{\text{(A.19)}}{=} \sum_i A_{ij}\left|v_i\right\rangle + \sum_i A_{ij}\left|v_i\right\rangle = \sum_i (A_{ij}+B_{ij})\left|v_i\right\rangle$$

Comparing these results, we see that $C_{ij} = A_{ij} + B_{ij}$, so the matrix of $\hat{A}+\hat{B}$ is the sum of the matrices of the component operators.

b) Similarly,

$$(\lambda\hat{A})\left|v_j\right\rangle = \lambda\hat{A}\left|v_j\right\rangle \overset{\text{(A.19)}}{=} \lambda\sum_i A_{ij}\left|v_i\right\rangle = \sum_i(\lambda A_{ij})\left|v_i\right\rangle.$$

We see that the $(i,j)$th element of the matrix associated with operator $\lambda\hat{A}$ is $\lambda A_{ij}$.

c) Let $\hat{C} = \hat{A}\hat{B}$. According to Ex. A.19, we have $\hat{B}\left|v_j\right\rangle = \sum_k B_{kj}\left|v_k\right\rangle$. Therefore

$$\begin{aligned}
\hat{C}\left|v_j\right\rangle &\overset{\text{(A.19)}}{=} \hat{A}\left(\sum_k B_{kj}\left|v_k\right\rangle\right) \\
&= \sum_k B_{kj}\hat{A}\left|v_k\right\rangle) \\
&\overset{\text{(A.19)}}{=} \sum_k B_{kj}\sum_i A_{ik}\left|v_i\right\rangle \\
&= \sum_i\left(\sum_k A_{ik}B_{kj}\right)\left|v_i\right\rangle.
\end{aligned}$$

Comparing this with Eq. (SA.24), we find that

$$C_{ij} = \sum_k A_{ik} B_{kj},$$

which corresponds to the standard row-times-column rule for the matrix multiplication.

**Solution to Exercise A.40.** Taking the inner product of both sides of Eq. (A.19) with $\langle v_k|$, we have

$$\left\langle v_k \middle| \hat{A} \middle| v_j \right\rangle = \sum_i A_{ij} \langle v_k | v_i \rangle = \sum_i A_{ij} \delta_{ik} = A_{kj}.$$

**Solution to Exercise A.41.** We first find the matrix representation of $\hat{R_\theta}$. For our calculation we use the standard basis of $\mathbb{R}^2$, $\{\hat{i}, \hat{j}\}$, the unit vectors along the $x$- and $y$-axes, which are orthonormal under the standard dot product. The effect of rotating $\hat{i}$ by angle $\theta$ is a new unit vector forming angle $\theta$ with the $x$-axis: $\hat{R_\theta}\hat{i} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$. Similarly, $\hat{R_\theta}\hat{j} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$.

It remains to find the matrix elements of $R_\theta$. We do so using Eq. (A.21).

$$\begin{aligned}
R_{\theta xx} &= \hat{i}\cdot(\hat{R_\theta}\hat{i}) = \hat{i}\cdot(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) \\
&= \cos\theta\ \hat{i}\cdot\hat{i} + \sin\theta\ \hat{i}\cdot\hat{j} = \cos\theta; \\
R_{\theta xy} &= \hat{i}\cdot(\hat{R_\theta}\hat{j}) = \hat{i}\cdot(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}) \\
&= -\sin\theta\ \hat{i}\cdot\hat{i} + \cos\theta\ \hat{i}\cdot\hat{j} = -\sin\theta; \\
R_{\theta yx} &= \hat{j}\cdot(\hat{R_\theta}\hat{i}) = \hat{j}\cdot(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) \\
&= \cos\theta\ \hat{j}\cdot\hat{i} + \sin\theta\ \hat{j}\cdot\hat{j} = \sin\theta; \\
R_{\theta yy} &= \hat{j}\cdot(\hat{R_\theta}\hat{j}) = \hat{j}\cdot(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}) \\
&= -\sin\theta\ \hat{j}\cdot\hat{i} + \cos\theta\ \hat{j}\cdot\hat{j} = \cos\theta,
\end{aligned}$$

and hence

$$\hat{R_\theta} \simeq \begin{pmatrix} R_{\theta xx} & R_{\theta xy} \\ R_{\theta yx} & R_{\theta yy} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{SA.25}$$

Similarly,

$$\hat{R_\phi} \simeq \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{SA.26}$$

Using the rules of matrix multiplication, we find

$$\begin{aligned}
\hat{R_\phi}\hat{R_\theta} &\simeq \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}.
\end{aligned}$$

As expected, the matrix of $\hat{R_\phi}\hat{R_\theta}$ is identical to that of the rotation by angle $\theta+\phi$.

**Solution to Exercise A.42.** A basis in the space of linear operators is formed by $N^2$ operators $|v_i\rangle\langle v_j|$, where $\{|v_i\rangle\}$ is any orthonormal basis in $\mathbb{V}$. The fact that this set is spanning follows from Ex. A.46. To show linear independence, suppose that there exist a pair $(k,l)$ such that

$$|v_k\rangle\langle v_l| = \sum_{i,j}\lambda_{i,j}\,|v_i\rangle\langle v_j|\,,$$

with either $i\neq k$ or $j\neq l$ for each element of the sum in the right-hand side. Let us sandwich the operators in the left- and right-hand sides of the above equation by $\langle v_k|$ and $|v_l\rangle$. We then have

$$\langle v_k|\,v_k\rangle\,\langle v_l|\,v_l\rangle = \sum_{i,j}\lambda_{i,j}\,\langle v_k|\,v_i\rangle\,\langle v_j|\,v_l\rangle\,.$$

The left-hand side of this expression is 1, but the right-hand side is zero because $\{|v_i\rangle\}$ is orthonormal and hence $\langle v_k|\,v_i\rangle=0$ when $i\neq k$ and $\langle v_j|\,v_l\rangle=0$ when $j\neq l$. We have arrived at a contradiction.

To visualize this argument, we can remember that each operator is represented by an $N\times N$ square matrix. Treating these matrices as ordered sets of $N^2$ numbers, we can see that they form a linear space. A basis in this space is a set of $N^2$ matrices where all the elements are zero except the element in the $i$th row and $j$th column, which is one. Each such matrix corresponds to the operator $|v_i\rangle\langle v_j|$.

**Solution to Exercise A.43.** We test both linearity properties at once. Let $|x\rangle, |y\rangle\in\mathbb{V}$ and $\lambda,\mu\in\mathbb{F}$. Then

$$\begin{aligned}|a\rangle\langle b|\,(\lambda\,|x\rangle+\mu\,|y\rangle) &= \langle b|\,(\lambda\,|x\rangle+\mu\,|y\rangle)\,|a\rangle\\ &= (\lambda\,\langle b|\,x\rangle+\mu\,\langle b|\,y\rangle)\,|a\rangle\\ &= \lambda\,(\langle b|\,x\rangle\,|a\rangle)+\mu\,(\langle b|\,y\rangle\,|a\rangle)\\ &= \lambda\,(|a\rangle\langle b|)\,|x\rangle+\mu\,(|a\rangle\langle b|)\,|y\rangle\end{aligned}$$

hence $|a\rangle\langle b|$ is linear.

**Solution to Exercise A.44.** This follows from Definition A.20 of the outer product operator:

$$\langle a|\,(|b\rangle\langle c|)\,|d\rangle = \langle a|\,(\langle c|\,d\rangle\,|b\rangle) = (\langle a|\,b\rangle)(\langle c|\,d\rangle).$$

**Solution to Exercise A.45.** Let $\{|v_i\rangle\}$ be the orthonormal basis in which we seek to find the matrix. Then the $(i,j)$th matrix element equals, according to Eq. (A.21)

$$\langle v_i|\,(|a\rangle\langle b|)\,|v_j\rangle \overset{\text{Ex. A.44}}{=} \langle v_i|\,a\rangle\,\langle b|\,v_j\rangle \overset{(A.6)}{=} a_i b_j^*.$$

**Solution to Exercise A.46.** The matrix of the operator in the right-hand side of Eq. (A.24),

$$\langle v_k| \left( \sum_{i,j} A_{ij} |v_i\rangle\langle v_j| \right) |v_l\rangle = \sum_{i,j} A_{ij} \langle v_k| v_i\rangle \langle v_j| v_l\rangle = \sum_{i,j} A_{ij} \delta_{ki}\delta_{jl} = A_{kl},$$

equals the matrix of the operator $\hat{A}$.

**Solution to Exercise A.47.** Let us define $\hat{B} \equiv \sum_i |w_i\rangle\langle v_i|$ and show that $\hat{B} = \hat{A}$. We notice that, for any $m$,

$$\hat{B}|v_m\rangle \stackrel{(A.22)}{=} \sum_i |w_i\rangle \langle v_i| v_m\rangle = \sum_i |w_i\rangle \delta_{im} = |w_m\rangle,$$

so $\hat{B}|v_m\rangle = \hat{A}|v_m\rangle$ for all elements of basis $|v_m\rangle$. The fact that the two operators map all elements of a basis identically means that they in fact map all vectors identically, i.e. they are identical.

**Solution to Exercise A.48.** Referring to Ex. A.46, we have

$$\begin{pmatrix} 1 & -3i \\ 3i & 4 \end{pmatrix} \simeq 1\,|H\rangle\langle H| + (-3i)\,|H\rangle\langle V| + 3i\,|V\rangle\langle H| + 4\,|V\rangle\langle V|$$

**Solution to Exercise A.49.** Using Eq. (A.25), we write

$$\begin{aligned} \hat{A} &= |w_1\rangle\langle u_1| + |w_2\rangle\langle u_2| \\ &= \sqrt{2}\,|v_1\rangle \frac{\langle v_1| + \langle v_2|}{\sqrt{2}} + \sqrt{2}(|v_1\rangle + 3i\,|v_2\rangle)\frac{\langle v_1| - \langle v_2|}{\sqrt{2}} \\ &= 2\,|v_1\rangle\langle v_1| + 3i\,|v_2\rangle\langle v_1| - 3i\,|v_2\rangle\langle v_2| \\ &\stackrel{(A.24)}{\simeq} \begin{pmatrix} 2 & 0 \\ 3i & -3i \end{pmatrix}. \end{aligned}$$

**Solution to Exercise A.50.**

$$\begin{aligned} \hat{\mathbf{1}} &\stackrel{(A.24)}{=} \sum_{i,j} \langle v_i| \hat{\mathbf{1}} |v_j\rangle\, |v_i\rangle\langle v_j| \\ &= \sum_{i,j} \langle v_i| v_j\rangle\, |v_i\rangle\langle v_j| \\ &= \sum_{i,j} \delta_{ij}\, |v_i\rangle\langle v_j| \\ &= \sum_i |v_i\rangle\langle v_i|. \end{aligned}$$

**Solution to Exercise A.51.**

a) It follows from Eqs. A.28 that

$$|v_1\rangle = \frac{1}{\sqrt{2}}(|w_1\rangle + |w_2\rangle);$$
$$|v_2\rangle = \frac{1}{i\sqrt{2}}(|w_1\rangle - |w_2\rangle).$$

Using the expansion of $\hat{A}$ we found in Ex. A.48, we substitute

$$\begin{aligned}
\hat{A} &= 1\,|v_1\rangle\langle v_1| - 3i\,|v_1\rangle\langle v_2| + 3i\,|v_2\rangle\langle v_1| + 4\,|v_2\rangle\langle v_2| \\
&= 1\left(\frac{|w_1\rangle + |w_2\rangle}{\sqrt{2}}\right)\left(\frac{\langle w_1| + \langle w_2|}{\sqrt{2}}\right) - 3i\left(\frac{|w_1\rangle + |w_2\rangle}{\sqrt{2}}\right)\left(\frac{\langle w_1| - \langle w_2|}{-\sqrt{2}i}\right) \\
&\quad +3i\left(\frac{|w_1\rangle - |w_2\rangle}{\sqrt{2}i}\right)\left(\frac{\langle w_1| + \langle w_2|}{\sqrt{2}}\right) + 4\left(\frac{|w_1\rangle - |w_2\rangle}{\sqrt{2}i}\right)\left(\frac{\langle w_1| - \langle w_2|}{-\sqrt{2}i}\right) \\
&= \frac{1}{2}(|w_1\rangle\langle w_1| + |w_1\rangle\langle w_2| + |w_2\rangle\langle w_1| + |w_2\rangle\langle w_2|) \\
&\quad + \frac{3}{2}(|w_1\rangle\langle w_1| - |w_1\rangle\langle w_2| + |w_2\rangle\langle w_1| - |w_2\rangle\langle w_2|) \\
&\quad + \frac{3}{2}(|w_1\rangle\langle w_1| + |w_1\rangle\langle w_2| - |w_2\rangle\langle w_1| - |w_2\rangle\langle w_2|) \\
&\quad + \frac{4}{2}(|w_1\rangle\langle w_1| - |w_1\rangle\langle w_2| - |w_2\rangle\langle w_1| + |w_2\rangle\langle w_2|) \\
&= \frac{11}{2}\,|w_1\rangle\langle w_1| - \frac{3}{2}\,|w_1\rangle\langle w_2| - \frac{3}{2}\,|w_2\rangle\langle w_1| + \frac{1}{2}\,|w_2\rangle\langle w_2|
\end{aligned}$$

and hence

$$\hat{A} \overset{w\text{-basis}}{\simeq} \frac{1}{2}\begin{pmatrix} 11 & -3 \\ -3 & -1 \end{pmatrix}.$$

b) Using the second method, we must first find the inner product matrices.

$$\begin{aligned}
\langle v_1|\, w_1\rangle &\simeq \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \\
\langle v_1|\, w_2\rangle &\simeq \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \\
\langle v_2|\, w_1\rangle &\simeq \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{i}{\sqrt{2}} \\
\langle v_2|\, w_2\rangle &\simeq \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{-i}{\sqrt{2}}
\end{aligned}$$

Now we can apply Eq. (A.27) to write

$$\begin{aligned}
\left\langle w_1\right|\hat{A}\left|w_1\right\rangle &= \langle w_1|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_1\rangle+\langle w_1|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_1\rangle\\
&\quad+\langle w_1|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_1\rangle+\langle w_1|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_1\rangle\\
&=\frac{1}{\sqrt{2}}1\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-3i)\frac{i}{\sqrt{2}}+\frac{-i}{\sqrt{2}}(3i)\frac{1}{\sqrt{2}}+\frac{-i}{\sqrt{2}}4\frac{i}{\sqrt{2}}=\frac{11}{2};\\
\left\langle w_1\right|\hat{A}\left|w_2\right\rangle &= \langle w_1|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_2\rangle+\langle w_1|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_2\rangle\\
&\quad+\langle w_1|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_2\rangle+\langle w_1|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_2\rangle\\
&=\frac{1}{\sqrt{2}}1\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-3i)\frac{-i}{\sqrt{2}}+\frac{-i}{\sqrt{2}}(3i)\frac{1}{\sqrt{2}}+\frac{-i}{\sqrt{2}}4\frac{-i}{\sqrt{2}}=\frac{-3}{2};\\
\left\langle w_2\right|\hat{A}\left|w_1\right\rangle &= \langle w_2|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_1\rangle+\langle w_2|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_1\rangle\\
&\quad+\langle w_2|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_1\rangle+\langle w_2|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_1\rangle\\
&=\frac{1}{\sqrt{2}}1\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-3i)\frac{i}{\sqrt{2}}+\frac{i}{\sqrt{2}}(3i)\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}4\frac{i}{\sqrt{2}}=\frac{-3}{2};\\
\left\langle w_2\right|\hat{A}\left|w_2\right\rangle &= \langle w_2|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_2\rangle+\langle w_2|\,v_1\rangle\left\langle v_1\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_2\rangle\\
&\quad+\langle w_2|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_1\right\rangle\langle v_1|\,w_2\rangle+\langle w_2|\,v_2\rangle\left\langle v_2\right|\hat{A}\left|v_2\right\rangle\langle v_2|\,w_2\rangle\\
&=\frac{1}{\sqrt{2}}1\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-3i)\frac{-i}{\sqrt{2}}+\frac{i}{\sqrt{2}}(3i)\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}4\frac{-i}{\sqrt{2}}=\frac{-1}{2}.
\end{aligned}$$

This calculation can be abbreviated if we write the last line of Eq. (A.27) as a product of matrices with

$$\langle w_i|\,v_k\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix};\quad \left\langle v_m\right|w_j\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$$

and hence

$$\begin{aligned}
\hat{A}_{w\text{-basis}} &\simeq \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}1 & -3i\\ 3i & 4\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}4 & -2\\ 7i & -i\end{pmatrix}=\frac{1}{2}\begin{pmatrix}11 & -3\\ -3 & -1\end{pmatrix}.
\end{aligned}$$

**Solution to Exercise A.52.**

a) Utilizing Eq. (A.29), we write

$$\begin{aligned}
(\lambda\langle b|+\mu\langle c|)\hat{A} &= \sum_{ij}(\lambda b_i+\mu c_i)^*A_{ij}\left\langle v_j\right|\\
&=\lambda\sum_{ij}b_i^*A_{ij}\left\langle v_j\right|+\mu\sum_{ij}c_i^*A_{ij}\left\langle v_j\right|\\
&=\lambda\langle b|\hat{A}+\mu\langle c|\hat{A},
\end{aligned}$$

which implies linear mapping according to Defn. A.15.

b) We know from Ex. A.45 that the matrix of the outer product operator $|b\rangle\langle c|$ is $\langle v_i|\,b\rangle\left\langle c\right|v_j\rangle=b_ic_j^*$. Now using Eq. (A.29), we find

$$\langle a|\left(|b\rangle\langle c|\right)=\sum_{ij}a_i^*b_ic_j^*\left\langle v_j\right|.$$

This is the same as

$$\langle a|\,b\rangle\,\langle c|=\left(\sum_i a_i^*b_i\right)\left(\sum_j c_j^*\left\langle v_j\right|\right).$$

c) We implement the proof in the matrix form. For the left-hand side of Eq. (A.30), we find

$$\left(\langle a|\hat{A}\right)|c\rangle\simeq\left(\sum_i a_i^*A_{i1}\ \dots\ \sum_i a_i^*A_{iN}\right)\begin{pmatrix}c_1\\ \vdots\\ c_N\end{pmatrix}=\sum_{ik}A_{ik}a_i^*c_k,$$

and for the right hand side,

$$\langle a|\left(\hat{A}\,|c\rangle\right)\simeq\left(a_1^*\ \dots\ a_N^*\right)\begin{pmatrix}\sum_k A_{1k}c_k\\ \vdots\\ \sum_k A_{Nk}c_k\end{pmatrix}=\sum_{ik}A_{ik}a_i^*c_k,$$

i.e. the same expression.

d) According to the result of part (c), the inner product of $\langle a|\hat{A}$ with any arbitrary vector $|c\rangle$ equals $\langle a|\left(\hat{A}\,|c\rangle\right)$, i.e. is independent of the basis $\{|v_i\rangle\}$ used in Eq. (A.29). This means that $\langle a|\hat{A}$ itself is basis-independent, too.

**Solution to Exercise A.53.** The matrix of operator $\hat{A}$ is found from Eq. (A.21) as $A_{ij}=\langle v_i|\hat{A}\left|v_j\right\rangle$. If we denote $|b\rangle=\hat{A}\left|v_j\right\rangle$, it follows from the definition of the adjoint operator that $\langle b|=\left\langle v_j\right|\hat{A}^\dagger$. Therefore,

$$A_{ij}=\langle v_i|\hat{A}\left|v_j\right\rangle=\langle v_i|\,b\rangle=\langle b|\,v_i\rangle^*=\left\langle v_j\right|\hat{A}^\dagger\,|v_i\rangle^*=(A^\dagger)_{ji}^*,$$

where $(A^\dagger)_{ji}$ is the martix element of the operator $\hat{A}^\dagger$ in the $j$th row, $i$th column. We see that the matrix of $\hat{A}^\dagger$ obtains from the matrix of $\hat{A}$ by exchanging the row and column numbers (i.e. transposition) and complex conjugation.

**Solution to Exercise A.54.** Double transposition of a matrix, combined with a double complex conjugation of each of its elements, will produce the same matrix.

**Solution to Exercise A.55.** Transposing and conjugating each of the matrices (1.7) will produce the same matrix. According to Ex. A.53, this indicates that the corresponding Pauli operators are Hermitian.

**Solution to Exercise A.56.** For a simple counterexample, we will use the $\hat{\sigma}_z$ and $\hat{\sigma}_y$ operators, which are both Hermitian:

$$\hat{\sigma}_z\hat{\sigma}_y\simeq\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$$

The resulting matrix is not Hermitian:

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

**Solution to Exercise A.57.** According to Ex. A.45, the matrices of the outer product operators $|b\rangle\langle c|$ and $|c\rangle\langle b|$ are, respectively is $\langle v_i|\ b\rangle\left\langle c\right|\ v_j\rangle = b_i c_j^*$ and $\langle v_i|\ c\rangle\left\langle b\right|\ v_j\rangle = b_j^* c_i$. These matrices are transpose and conjugate to each other.

**Solution to Exercise A.58.**

a) Let $\hat{C} = \hat{A} + \hat{B}$. Then for the matrix of $\hat{C}^\dagger$ we have

$$(C^\dagger)_{ij} = C_{ji}^* = A_{ji}^* + B_{ji}^* = (A^\dagger)_{ij} + (B^\dagger)_{ij},$$

where $(A^\dagger)_{ij}$ and $(B^\dagger)_{ij}$ are the matrices of operators $\hat{A}^\dagger$ and $\hat{B}^\dagger$, respectively.

b) Similarly, for the matrix of $\hat{C}^\dagger = (\lambda\hat{A})^\dagger$,

$$(C^\dagger)_{ij} = C_{ji}^* = \lambda^* A_{ji}^* = \lambda^*(A^\dagger)_{ij}.$$

c) The matrix of of the operator $\hat{A}\hat{B}$ is the product of the individual matrices [see Ex. A.39(c)]:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}.$$

For the adjoint matrix, we have

$$\left[(AB)^\dagger\right]_{ij} = (AB)_{ji}^* = \sum_k A_{jk}^* B_{ki}^*. \tag{SA.27}$$

On the other hand, the matrix product of $\hat{B}^\dagger$ and $\hat{A}^\dagger$ equals

$$(B^\dagger A^\dagger)_{ij} = \sum_k (B^\dagger)_{ik}(A^\dagger)_{kj} = \sum_k B_{ki}^* A_{jk}^*,$$

which is the same as (SA.27).

**Solution to Exercise A.59.** Let $\hat{A}\,|\psi\rangle = |\chi\rangle$. Then $\langle\psi|\hat{A}^\dagger = \langle\chi|$ and thus

$$\langle\psi|\hat{A}^\dagger\,|\phi\rangle^* = \langle\chi|\ \phi\rangle^* = \langle\phi|\ \chi\rangle = \langle\phi|\hat{A}\,|\psi\rangle\,.$$

One can also obtain this result by arguing that the objects $\langle\psi|\hat{A}^\dagger\,|\phi\rangle$ and $\langle\phi|\hat{A}\,|\psi\rangle$ are adjoint to each other because they are related by reversing the order and replacing the operator by its adjoint. Because these two objects are adjoint to each other and are numbers, they must be complex conjugate with respect to each other.

**Solution to Exercise A.60.** Let us look for the eigenvalues and eigenvectors of this matrix such that $\hat{A}\,|v\rangle = v\,|v\rangle$, or

$$\left(\hat{A} - v\hat{\mathbf{1}}\right)|v\rangle = 0. \tag{SA.28}$$

This equation can be satisfied for a nonzero $|v\rangle$ only if the determinant of the matrix in the left-hand side vanishes:

$$\left|\hat{A} - v\hat{\mathbf{1}}\right| = 0 \tag{SA.29}$$

Equation (SA.29) is called the *characteristic equation* for matrix $\hat{A}$.

According to the fundamental theorem of algebra, this equation has at least one root, so $\hat{A}$ has at least one eigenvalue $v_1$ and a corresponding eigenvector $|v_1\rangle$:

$$\hat{A}\,|v_1\rangle = v_1\,|v_1\rangle\,.$$

We first notice that because $\hat{A}$ is Hermitian,

$$\langle v_1|\hat{A}\,|v_1\rangle = \langle v_1|\hat{A}\,|v_1\rangle^*$$

according to Eq. (A.37), so the quantity

$$\langle v_1|\hat{A}\,|v_1\rangle = v_1\,\langle v_1|\,v_1\rangle = v_1$$

is real.

We proceed by selecting vectors $|v_2\rangle\,,\ldots,|v_N\rangle$ such that, with the addition of the previously found eigenvector $|v_1\rangle$, they form an orthonormal basis in our Hilbert space $\mathbb{V}$. Because the basis is orthonormal, we find for the first column of the matrix of $\hat{A}$ in this basis

$$A_{i1} = \langle v_i|\hat{A}\,|v_1\rangle = v_1\,\langle v_i|\,v_1\rangle = 0, \qquad i \neq 1. \tag{SA.30a}$$

The first row of this matrix has the same property because $\hat{A}$ is Hermitian:

$$A_{1i} = A_{i1}^* = 0, \qquad i \neq 1, \tag{SA.30b}$$

We conclude that the matrix of $\hat{A}$ in the basis $\{|v_i\rangle\}$ has the form

$$\hat{A} \simeq \left(\begin{array}{c|ccc} v_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \hat{A}_1 & \\ 0 & & & \end{array}\right),$$

where $\hat{A}_1$ is an $(N-1)\times(N-1)$ matrix. Because of the relations (SA.30), the operator associated with this matrix maps the subspace $\mathbb{V}_1 \subset \mathbb{V}$ spanned by the set $\{|v_2\rangle\,,\ldots,|v_N\rangle\}$ onto itself. The above argument can thus be repeated for the operator $\hat{A}_1$ in $\mathbb{V}_1$ to obtain a basis $\{|v_2'\rangle\,,\ldots,|v_N'\rangle\}$ in which $|v_2'\rangle$ is an eigenvector of $\hat{A}_1$, and at the same time an eigenvector of $\hat{A}$. In the basis $\{|v_1\rangle\,,|v_2'\rangle\,,\ldots,|v_N'\rangle\}$ this operator takes the form

$$\hat{A} \simeq \left(\begin{array}{cc|ccc} v_1 & 0 & 0 & \cdots & 0 \\ 0 & v_2' & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \hat{A}_2 & \\ 0 & 0 & & & \end{array}\right).$$

Repeating this procedure additional $N-2$ times, we fully diagonalize $\hat{A}$ and find a set of eigenvectors $\{|v_i\rangle\}$ that forms an orthonormal basis.

**Solution to Exercise A.61.** By comparing Eq. (A.38) with Eq. (A.24), we find

$$\hat{V} \simeq \begin{pmatrix} v_1 & 0 & 0 & . & 0 \\ 0 & v_2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & v_N \end{pmatrix}. \tag{SA.31}$$

**Solution to Exercise A.62.** Using the definition provided by Eq. (A.38), we write out the expression for the operator $\hat{V}$ acting on one of the elements of its eigenbasis

$$\hat{V}\left|v_j\right\rangle = \left(\sum_i v_i |v_i\rangle \langle v_i|\right)\left|v_j\right\rangle \stackrel{(A.22)}{=} \sum_i v_i |v_i\rangle \left\langle v_i \middle| v_j\right\rangle = \sum_i v_i |v_i\rangle \delta_{ij} = v_j\left|v_j\right\rangle. \tag{SA.32}$$

**Solution to Exercise A.64.**

The matrix representation for the rotation operator in $\mathbb{R}^2$ is (Ex. A.41)

$$\hat{R_\phi} \simeq \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{SA.33}$$

By transposing this matrix we find that it is not Hermitian. To find its eigenvalues, we write the characteristic equation:

$$\begin{aligned} \left|\hat{R}_\phi - v\hat{\mathbf{1}}\right| &= \begin{vmatrix} \cos\phi - v & -\sin\phi \\ \sin\phi & \cos\phi - v \end{vmatrix} \\ &= \cos^2\phi - 2v\cos\phi + v^2 + \sin^2\phi \\ &= v^2 - 2v\cos\phi + 1 = 0. \end{aligned} \tag{SA.34}$$

Thus, our eigenvalues are

$$\begin{aligned} v_{1,2} &= \cos\phi \pm \sqrt{\cos^2\phi - 1} \\ &= \cos\phi \pm i\sin\phi = e^{\pm i\phi} \end{aligned} \tag{SA.35}$$

The eigenvalues are complex; therefore, unless $\phi = 0$ or $\phi = \pi$, the matrix $\hat{R_\phi}$ has no eigenvectors in the two-dimensional geometric space $\mathbb{R}^2$. This is not surprising: when we rotate a vector by an angle other than 0 or $\pi$, we will not produce a collinear vector. However, if we consider this matrix in the linear space $\mathbb{C}^2$ over the field of complex numbers, it has two eigenvalues $v_{1,2}$ and two corresponding eigenvectors, which we find in the next step.

We begin with the eigenvalue $v_1 = e^{i\phi} = \cos\phi + i\sin\phi$. Equation $(\hat{R_\phi} - v\hat{\mathbf{1}})\,|v\rangle = 0$ then becomes

$$\begin{pmatrix} -i\sin\phi & -\sin\phi \\ \sin\phi & -i\sin\phi \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$i\alpha\sin\phi + \beta\sin\phi = 0.$$

Solving this equation with the normalization condition $\alpha^2 + \beta^2 = 1$ we determine the eigenvector

$$|v_1\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{SA.36}$$

Similarly, for the $v_2 = e^{-i\phi}$ eigenvalue:

$$|v/_2\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{SA.37}$$

This result can be illustrated in terms of the polarization vector (Appendix C): the circularly polarized state (i.e. such that the trajectory of the tip of the electric field vector is a circle) remains circularly polarized when the reference frame is rotated.

**Solution to Exercise A.66.** Let $\hat{V} = \sum_i v_i\,|v_i\rangle\langle v_i|$ be the spectral decomposition of operator $\hat{V}$. Let us decompose vector $|\psi\rangle$ into the eigenbasis of $\hat{V}$: $|\psi\rangle = \sum_i \psi_i\,|v_i\rangle$. Then

$$\hat{V}\,|\psi\rangle = \sum_i \psi_i\hat{V}\,|v_i\rangle = \sum_i \psi_i v_i\,|v_i\rangle\,.$$

Since $|\psi\rangle$ is an eigenvector of $\hat{V}$, we also have

$$\hat{V}\,|\psi\rangle = v\,|\psi\rangle = \sum_i v\psi_i\,|v_i\rangle\,.$$

But a vector can be decomposed into the same basis in only one way, so $v\psi_i = v_i\psi_i$ for all $i$. Hence $v_i = v$ whenever $\psi_i \neq 0$, so the only nonzero coefficients in the decomposition of $|\psi\rangle$ are for those basis elements for which $\hat{V}\,|v_i\rangle = v\,|v_i\rangle$.

**Solution to Exercise A.67.**

a) Suppose there are two eigenbases, $\{|v_i\rangle\}$ and $\{|w_i\rangle\}$. According to Ex. A.66, each of the $|w_i\rangle$'s must be proportional to one of the $|v_i\rangle$'s. Because both bases are normalized orthogonal sets, they must be identical to each other up to phase factors.
b) Similarly, each of the elements of the set must be proportional to one of the eigenbasis elements. Because the set is normalized and linearly independent, it must then be identical to the eigenbasis.

**Solution to Exercise A.68.** By definition, every vector is an eigenvector of the identity operator with eigenvalue 1. This also means that *any* basis is an eigenbasis of that operator: for example, the canonical and diagonal bases.

**Solution to Exercise A.69.** Let vectors $|v\rangle$ and $|w\rangle$ be eigenvectors of operator $\hat{V}$ with eigenvalues $v$ and $w$, respectively. Suppose the spectral decomposition of $\hat{V}$ has basis elements $|v_1\rangle, |v_2\rangle, \dots$ associated with eigenvalue $v$ and basis elements $|w_1\rangle, |w_2\rangle, \dots$ with eigenvalue $w$. Then we can decompose, according to Ex. A.66,

$$|v\rangle = \sum_i v_i |v_i\rangle;$$
$$|w\rangle = \sum_j w_j |w_j\rangle.$$

Because the spectral decomposition produces an orthonormal basis, all $|v_i\rangle$'s and $|w_i\rangle$'s are mutually orthogonal. Therefore

$$\langle v|\, w\rangle = \sum_{i,j} v_i^* w_j \langle v_i|\, w_j\rangle = 0.$$

**Solution to Exercise A.70.** We need to show that any linear combination of eigenstates of $\hat{V}$ with a given eigenvalue $v$ is also an eigenstate of $\hat{V}$ with the same eigenvalue. This follows from the definition A.15 of a linear operator. Indeed, for any two eigenvectors $|v_1\rangle$ and $|v_2\rangle$ of $\hat{V}$ with eigenvalue $v$, we have

$$\begin{aligned}\hat{V}(a_1|v_1\rangle + a_2|v_2\rangle) &= a_1\hat{V}|v_1\rangle + a_2\hat{V}|v_2\rangle \\ &= a_1 v|v_1\rangle + a_2 v|v_2\rangle \\ &= v(a_1|v_1\rangle + a_2|v_2\rangle).\end{aligned}$$

**Solution to Exercise A.71.**

a) Let $\hat{C} = \hat{A} - \hat{B}$. The condition $\langle\psi|\,\hat{A}\,|\psi\rangle = \langle\psi|\,\hat{B}\,|\psi\rangle$ is equivalent to

$$\langle\psi|\,\hat{C}\,|\psi\rangle = 0 \qquad \text{(SA.38)}$$

for all $|\psi\rangle$. Suppose $\hat{C} \neq 0$ — that is, there exists vector $|a\rangle$ such that $\hat{C}|a\rangle \neq 0$. Let us denote $|b\rangle = \hat{C}|a\rangle$ and $|c\rangle = \hat{C}|b\rangle$. It follows from Eq. (SA.38) that $\langle a|\, b\rangle = 0$ and $\langle b|\, c\rangle = 0$.
The linearity of operator $\hat{C}$ implies that

$$\hat{C}(|a\rangle + |b\rangle) = |b\rangle + |c\rangle.$$

Taking the inner product of both sides of this equation with $|a\rangle + |b\rangle$ and using Eq. (SA.38) as well as $\langle a|\, b\rangle = \langle b|\, c\rangle = 0$, we have $\langle a|\, c\rangle + \langle b|\, b\rangle = 0$.
On the other hand, we also have

$$\hat{C}(|a\rangle + i|b\rangle) = |b\rangle + i|c\rangle.$$

Taking the inner product of both sides of this equation with $|a\rangle + i|b\rangle$, we find $\langle a|\, c\rangle - \langle b|\, b\rangle = 0$, so $\langle b|\, b\rangle$ is equal to $\langle c|\, a\rangle$ and $-\langle c|\, a\rangle$ at the same time. This can only happen when $|b\rangle = 0$, which contradicts our assumption.

b) Using Eq. (A.37), we have $\left\langle\psi\right|\hat{A}\left|\psi\right\rangle=\left\langle\psi\right|\hat{A}^{\dagger}\left|\psi\right\rangle^{*}$ for all $|\psi\rangle$. Since $\left\langle\psi\right|\hat{A}\left|\psi\right\rangle$ is known to be real, this means $\left\langle\psi\right|\hat{A}\left|\psi\right\rangle=\left\langle\psi\right|\hat{A}^{\dagger}\left|\psi\right\rangle$, leading to $\hat{A}=\hat{A}^{\dagger}$ according to part (a).

**Solution to Exercise A.72.**

- Suppose all eigenvalues in the spectral decomposition $\hat{A}=\sum_i v_i\left|v_i\right\rangle\left\langle v_i\right|$ are positive (non-negative). We can decompose any arbitrary nonzero vector $|\psi\rangle$ into $\hat{A}$'s eigenbasis: $|\psi\rangle=\sum_i\psi_i\left|v_i\right\rangle$. Then we have

$$\left\langle\psi\right|\hat{A}\left|\psi\right\rangle=\left(\sum_i\psi_i^*\left\langle v_i\right|\right)\left(\sum_j v_j\psi_j\left|v_j\right\rangle\right)=\sum_i|\psi_i|^2v_i.$$

  Because $|\psi\rangle$ is nonzero, so is at least one of the $\psi_i$'s. Then, if all $v_i$'s are positive (non-negative), so is $\left\langle\psi\right|\hat{A}\left|\psi\right\rangle$, therefore $\hat{A}$ is a positive (non-negative) operator.
- Suppose $\hat{A}$ is a positive (non-negative) operator. For any nonzero eigenvector $|v\rangle$ of $\hat{A}$ with eigenvalue $v$ we have $\left\langle v\right|\hat{A}\left|v\right\rangle=v\left\langle v|v\right\rangle=v$. If $\left\langle v\right|\hat{A}\left|v\right\rangle$ is positive (non-negative), so is $v$.

**Solution to Exercise A.73.** For any arbitrary vector $|\psi\rangle$, according to the definitions of the linear operator and inner product,

$$\left\langle\psi\right|\left(\hat{A}+\hat{B}\right)\left|\psi\right\rangle=\left\langle\psi\right|\hat{A}\left|\psi\right\rangle+\left\langle\psi\right|\hat{B}\left|\psi\right\rangle.$$

If both terms in the right-hand side are positive (non-negative), so is the left-hand side.

**Solution to Exercise A.74.**

a)

$$\frac{1}{2}\left([\hat{A},\hat{B}]+\{\hat{A},\hat{B}\}\right)=\frac{1}{2}\left(\hat{A}\hat{B}-\hat{B}\hat{A}+\hat{A}\hat{B}+\hat{B}\hat{A}\right)=\hat{A}\hat{B}$$

.

b)

$$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}=-(\hat{B}\hat{A}-\hat{A}\hat{B})=-[\hat{B},\hat{A}]$$

.

c)

$$[\hat{A},\hat{B}]^{\dagger}=(\hat{A}\hat{B}-\hat{B}\hat{A})^{\dagger}\overset{(A.35)}{=}\hat{B}^{\dagger}\hat{A}^{\dagger}-\hat{A}^{\dagger}B^{\dagger}=\left[\hat{B}^{\dagger},\hat{A}^{\dagger}\right].$$

d)

$$[\hat{A},\hat{B}+\hat{C}]=\hat{A}(\hat{B}+\hat{C})-(\hat{B}+\hat{C})\hat{A}=\hat{A}\hat{B}+\hat{A}\hat{C}-\hat{B}\hat{A}-\hat{C}\hat{A}=[\hat{A},\hat{B}]+[\hat{A},\hat{C}];$$
$$[\hat{A}+\hat{B},\hat{C}]=(\hat{A}+\hat{B})\hat{C}-\hat{C}(\hat{A}+\hat{B})=\hat{A}\hat{C}+\hat{B}\hat{C}-\hat{C}\hat{A}-\hat{C}\hat{B}=[\hat{A},\hat{C}]+[\hat{B},\hat{C}].$$

e)

$$[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}=[\hat{A},\hat{B}\hat{C}];$$
$$[\hat{A},\hat{C}]\hat{B}+\hat{A}[\hat{B},\hat{C}]=\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}+\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}=\hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}=[\hat{A}\hat{B},\hat{C}].$$

f)

$$[\hat{A}\hat{B},\hat{C}\hat{D}] \stackrel{(A.44a)}{=} \hat{C}[\hat{A}\hat{B},\hat{D}]+[\hat{A}\hat{B},\hat{C}]\hat{D}$$
$$\stackrel{(A.44b)}{=} \hat{C}\hat{A}[\hat{B},\hat{D}]+\hat{C}[\hat{A},\hat{D}]\hat{B}+\hat{A}[\hat{B},\hat{C}]\hat{D}+[\hat{A},\hat{C}]\hat{B}\hat{D}$$

or alternatively

$$[\hat{A}\hat{B},\hat{C}\hat{D}] \stackrel{(A.44b)}{=} \hat{A}[\hat{B},\hat{C}\hat{D}]+[\hat{A},\hat{C}\hat{D}]\hat{B}$$
$$\stackrel{(A.44a)}{=} \hat{A}\hat{C}[\hat{B},\hat{D}]+\hat{A}[\hat{B},\hat{C}]\hat{D}+\hat{C}[\hat{A},\hat{D}]\hat{B}+[\hat{A},\hat{C}]\hat{D}\hat{B}.$$

**Solution to Exercise A.75.**

a)

$$[\hat{A}\hat{B}\hat{C},\hat{D}] \stackrel{(A.44b)}{=} \hat{A}[\hat{B}\hat{C},\hat{D}]+[\hat{A}\hat{B},\hat{D}]\hat{C}$$
$$=\hat{A}\hat{B}[\hat{C},\hat{D}]+2\hat{A}[\hat{B},\hat{D}]\hat{C}+[\hat{A},\hat{D}]\hat{B}\hat{C}.$$

b)

$$[\hat{A}^2+\hat{B}^2,\hat{A}+i\hat{B}] \stackrel{(A.43)}{=} [\hat{A}^2,\hat{A}]+[\hat{B}^2,\hat{A}]+i[\hat{A}^2,\hat{B}]+i[\hat{B}^2,\hat{B}]$$
$$=0+\hat{B}[\hat{B},\hat{A}]+[\hat{B},\hat{A}]\hat{B}+i\hat{A}[\hat{A},\hat{B}]+i[\hat{A},\hat{B}]\hat{A}+0$$
$$=(-\hat{B}+i\hat{A})[\hat{A},\hat{B}]+[\hat{A},\hat{B}](-\hat{B}+i\hat{A}).$$

**Solution to Exercise A.76.** Since $\hat{A}\hat{B}=\hat{B}\hat{A}+c$, we have

$$\begin{aligned}\hat{A}\hat{B}^n &= \hat{A}\underbrace{\hat{B}\hat{B}\hat{B}\ldots\hat{B}}_{n\text{ times}} \\ &= (\hat{B}\hat{A}+c)\underbrace{\hat{B}\hat{B}\ldots\hat{B}}_{n-1\text{ times}} \\ &= \hat{B}\hat{A}\underbrace{\hat{B}\hat{B}\ldots\hat{B}}_{n-1\text{ times}}+c\hat{B}^{n-1} \\ &= \hat{B}B\hat{A}\ \underbrace{\hat{B}\ldots\hat{B}}_{n-2\text{ times}}\ +2c\hat{B}^{n-1} \\ &= \ldots = \underbrace{\hat{B}\hat{B}\hat{B}\ldots\hat{B}}_{n\text{ times}}\hat{A}+nc\hat{B}^{n-1} \\ &= \hat{B}^n\hat{A}+nc\hat{B}^{n-1}. \end{aligned} \tag{SA.39}$$

Therefore

$$[\hat{A},\hat{B}^n]=\hat{A}\hat{B}^n-\hat{B}^n\hat{A}=nc\hat{B}^{n-1}. \tag{SA.40}$$

**Solution to Exercise A.77.**

a) Using Eq. (A.42) we find:

$$\begin{aligned}\left(i[\hat{A},\hat{B}]\right)^{\dagger} &= -i\left[\hat{B}^{\dagger},\hat{A}^{\dagger}\right]\\ &= -i[\hat{B},\hat{A}] \qquad \text{[because } \hat{A} \text{ and } \hat{B} \text{ are Hermitian]}\\ &= i[\hat{A},\hat{B}],\end{aligned}$$

which shows that $(i[\hat{A},\hat{B}])^{\dagger}$ is Hermitian.

b) Similarly,

$$\{\hat{A},\hat{B}\}^{\dagger} = (\hat{A}\hat{B})^{\dagger} + (\hat{B}\hat{A})^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger} + \hat{A}^{\dagger}\hat{B}^{\dagger} = \hat{B}\hat{A} + \hat{A}\hat{B} = \{\hat{A},\hat{B}\}.$$

**Solution to Exercise A.78.** We work out the commutator relations according to Eq. (1.7).

$$\begin{aligned}[\hat{\sigma}_z,\hat{\sigma}_x] &= \hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z\\ &\simeq \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &= 2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \simeq 2i\sigma_y\end{aligned} \tag{SA.41}$$

Thus, we also know that $[\hat{\sigma}_x,\hat{\sigma}_z] = -2i\sigma_y$.

$$\begin{aligned}[\hat{\sigma}_y,\hat{\sigma}_z] = -[\hat{\sigma}_z,\hat{\sigma}_y] &= \hat{\sigma}_y\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_y\\ &\simeq \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\\ &= 2\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \simeq 2i\sigma_x\end{aligned} \tag{SA.42}$$

Finally,

$$\begin{aligned}[\hat{\sigma}_x,\hat{\sigma}_y] = -[\hat{\sigma}_y,\hat{\sigma}_x] &= \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x\\ &\simeq \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\ &= 2\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \simeq 2i\sigma_z\end{aligned} \tag{SA.43}$$

Also, $[\hat{\sigma}_x, \hat{\sigma}_x] = [\hat{\sigma}_y, \hat{\sigma}_y] = [\hat{\sigma}_z, \hat{\sigma}_z] = 0$ because any operator commutes with itself.

**Solution to Exercise A.79.** For any nonzero vector $|a\rangle$ there exists vector $|a_1\rangle = |a\rangle / \| |a\rangle \|$ of length 1. Operator $\hat{U}$ maps this vector onto $|a_1'\rangle = \hat{U}|a_1\rangle$, which also has length 1. Because $\hat{U}$ is unitary, we have $|a'\rangle = \hat{U} \| |a\rangle \| |a_1\rangle = \| |a\rangle \| |a_1'\rangle$ and accordingly

$$\| |a'\rangle \|^2 = \langle a' | a' \rangle = \| |a\rangle \|^2 \langle a_1' | a_1' \rangle = \| |a\rangle \|^2.$$

**Solution to Exercise A.80.** If an operator preserves the inner product, it will also preserve the norm of a vector because the norm is the square root of the inner product of the vector with itself.

To prove the converse statement, let us consider arbitrary two vectors, $|a\rangle$ and $|b\rangle$. Then for $|c\rangle = |a\rangle + |b\rangle$ we have

$$\langle c | c \rangle = \langle a | a \rangle + \langle b | b \rangle + \langle a | b \rangle + \langle a | b \rangle^* . \tag{SA.44}$$

At the same time, for $|a'\rangle = \hat{U}|a\rangle$, $|b'\rangle = \hat{U}|b\rangle$ and $|c'\rangle = \hat{U}|c\rangle$, we have

$$\langle c' | c' \rangle = \langle a' | a' \rangle + \langle b' | b' \rangle + \langle a' | b' \rangle + \langle a' | b' \rangle^* . \tag{SA.45}$$

Since $\langle a' | a' \rangle = \langle a | a \rangle$, $\langle b' | b' \rangle = \langle b | b \rangle$, $\langle c' | c' \rangle = \langle c | c \rangle$, we see from Eqs. (SA.44) and (SA.45) that $\langle a' | b' \rangle + \langle a' | b' \rangle^* = \langle a | b \rangle + \langle a | b \rangle^*$, i.e. $\mathrm{Re}\langle a' | b' \rangle = \mathrm{Re}\langle a | b \rangle$.

By conducting a similar argument with $|c\rangle = |a\rangle + i|b\rangle$ , we obtain $\mathrm{Im}\langle a' | b' \rangle = \mathrm{Im}\langle a | b \rangle$.

**Solution to Exercise A.81.**

a) Since a unitary operator preserves inner products, it maps an orthonormal basis onto an orthonormal set. According to Ex. A.19, such a set comprises a basis.
b) For any ket $|a\rangle = \sum_i a_i |w_i\rangle$, we have $|a'\rangle = \hat{U}|a\rangle = \sum_i a_i |v_i\rangle$. Accordingly,

$$\langle a' | a' \rangle = \sum_i |a_i|^2 = \langle a | a \rangle .$$

We see that operator $\hat{U}$ preserves the norm of $|a\rangle$ and hence it is unitary.

**Solution to Exercise A.82.** If operator $\hat{U}$ is unitary, it maps some orthonormal basis $\{|w_i\rangle\}$ onto another orthonormal basis $\{|v_i\rangle\}$ (Ex. A.81). Hence it can be written in the form $\hat{U} = \sum_i |v_i\rangle\langle w_i|$ (Ex. A.25). Then $\hat{U}^\dagger = \sum_i |w_i\rangle\langle v_i|$ (Ex. A.35). Accordingly,

$$\hat{U}\hat{U}^\dagger = \sum_{ij} |v_i\rangle \langle w_i | w_j \rangle \langle v_j | = \sum_{ij} |v_i\rangle \delta_{ij} \langle v_j | = \sum_i |v_i\rangle\langle v_j| \overset{(A.26)}{=} \hat{\mathbf{1}}.$$

The argument that $\hat{U}^\dagger\hat{U} = \hat{\mathbf{1}}$ is similar.

Now let us prove that any operator $\hat{U}$ that satisfies $\hat{U}^\dagger\hat{U} = \hat{\mathbf{1}}$ preserves the inner product between two arbitrary vectors $|a\rangle$ and $|b\rangle$. Defining $|a'\rangle = \hat{U}|a\rangle$ and $|b'\rangle = \hat{U}|b\rangle$, we have

$$\langle a' | b' \rangle = \langle a | \hat{U}^\dagger\hat{U} | b \rangle = \langle a | b \rangle .$$

**Solution to Exercise A.83.**

a) Because every unitary operator $\hat{U}$ satisfies $\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbf{1}}$, the statement of Ex. A.63 applies, so $\hat{U}$ is diagonalizable. For any eigenvalue $u$ with an associated eigenvector $|u\rangle$, we have $|u'\rangle = \hat{U}|u\rangle = u|u\rangle$ and hence

$$\langle u' | u' \rangle = u^* u \langle u | u \rangle .$$

Because a unitary operator preserves the norm, we must have $u^* u = |u|^2 = 1$. This can be satisfied by any $u = e^{i\theta}$ with $\theta \in \mathbb{R}$.

b) If an operator $\hat{U}$ is diagonalizable, its matrix in its eigenbasis has the form

$$\hat{U} \simeq \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N \end{pmatrix}, \tag{SA.46}$$

where $u_i$ are the eigenvalues of absolute value 1 (i.e. such that $u_i^* u_i = 1$). The adjoint matrix is then

$$\hat{U}^\dagger \simeq \begin{pmatrix} u_1^* & 0 & \cdots & 0 \\ 0 & u_2^* & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N^* \end{pmatrix} \tag{SA.47}$$

and their product is

$$\hat{U}^\dagger U = UU^\dagger \simeq \begin{pmatrix} u_1^* u_1 & 0 & \cdots & 0 \\ 0 & u_2^* u_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N^* u_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \simeq \hat{\mathbf{1}}. \tag{SA.48}$$

This shows that operator $\hat{U}$ is unitary.

**Solution to Exercise A.84.**

a) For the Pauli operators:

$$\sigma_x^\dagger \sigma_x \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \simeq \hat{\mathbf{1}};$$
$$\sigma_y^\dagger \sigma_y \simeq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \simeq \hat{\mathbf{1}};$$
$$\sigma_z^\dagger \sigma_z \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \simeq \hat{\mathbf{1}}.$$

So all three Pauli operators are unitary.

b) For the rotation operator:

$$\hat{R_\phi}\hat{R_\phi}^\dagger \simeq \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^\dagger$$
$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\simeq \hat{\mathbf{1}},$$

so this operator is unitary, too. This can be understood intuitively: when vectors are rotated, their lengths (norms) do not change.

**Solution to Exercise A.85.** Operator $f(\hat{A})$ acting on vector $|a\rangle$ yields

$$f(\hat{A})\,|a\rangle \overset{(A.49)}{=} \sum_i f(a_i)\,|a_i\rangle\,\langle a_i|\,a\rangle\,.$$

Because $\hat{A}$ is Hermitian, its eigenvectors are orthonormal. Hence all $\langle a_i|\,a\rangle = 0$ except when $|a_i\rangle = |a\rangle$, in which case this inner product is 1. Hence

$$f(\hat{A})\,|a\rangle = f(a)\,|a\rangle\,\langle a|\,a\rangle = f(a)\,|a\rangle\,.$$

**Solution to Exercise A.86.** The matrix of the operator function (A.49) in its eignebasis is diagonal with real values, i.e. self-adjoint.

**Solution to Exercise A.87.** For a non-negative function $f(x)$, all eigenvalues $f(a_i)$ of the operator function (A.49) are non-negative, which means that the operator is also non-negative according to Ex. A.72.

**Solution to Exercise A.88.** The first step is to diagonalize $\hat{A}$. The characteristic equation for this matrix is:

$$|\hat{A} - v\hat{1}| = \begin{vmatrix} 1-v & 3 \\ 3 & 1-v \end{vmatrix} = 0,$$

from which we find our eigenvalues $v_{1,2} = \{4, -2\}$. The normalized eigenvector associated with the first eigenvalue is

$$|v_1\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and with the second

$$|v_2\rangle \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This means that our operator can be written as

$$\hat{A} = 4\,|v_1\rangle\langle v_1| - 2\,|v_2\rangle\langle v_2|.$$

Now we apply Eq. (A.49) to express $\sqrt{\hat{A}}$ as

$$\begin{aligned}\sqrt{\hat{A}} &= \sqrt{4}\,|v_1\rangle\langle v_1| + \sqrt{-2}\,|v_2\rangle\langle v_2| \\ &= 2\,|v_1\rangle\langle v_1| \pm i\sqrt{2}\,|v_2\rangle\langle v_2| \\ &\simeq 2\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} \pm i\sqrt{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix} \\ &= \begin{pmatrix}(1 \pm \frac{i}{\sqrt{2}}) & (1 \mp \frac{i}{\sqrt{2}}) \\ (1 \mp \frac{i}{\sqrt{2}}) & (1 \pm \frac{i}{\sqrt{2}})\end{pmatrix},\end{aligned}$$

where all matrices are in the same basis as the matrix of $\hat{A}$.

To determine $\ln\hat{A}$, we need to find the logarithm of its eigenvalues, one of which ($v_2$) is negative. The logarithm of negative numbers is not defined in the real number space. In the complex number space, we can use the fact that $e^{(2m+1)i\pi} = -1$ (where $m$ is an arbitrary integer) and thus $e^{(2m+1)i\pi+\ln 2} = (-1)\times 2 = -2$. Hence, $\ln(-2) = (2m+1)i\pi + \ln 2$.[1] Thus,

$$\begin{aligned}\ln\hat{A} &= \ln 4\,|v_1\rangle\langle v_1| + \ln(-2)\,|v_2\rangle\langle v_2| \\ &= \frac{1}{2}\begin{pmatrix}\ln 4 + \ln 2 + (2m+1)i\pi & \ln 4 - \ln 2 - (2m+1)i\pi \\ \ln 4 - \ln 2 - (2m+1)i\pi & \ln 4 + \ln 2 + (2m+1)i\pi\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}\ln 8 + (2m+1)i\pi & \ln 2 - (2m+1)i\pi \\ \ln 2 - (2m+1)i\pi & \ln 8 + (2m+1)i\pi\end{pmatrix}.\end{aligned}$$

**Solution to Exercise A.89.** The eigenvalues of $\hat{A}$ are $a_1 = 0$ and $a_2 = 1$ with corresponding eigenvectors $|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and $|a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. Therefore

$$e^{i\theta\hat{A}} = e^0\,|a_1\rangle\langle a_1| + e^{i\theta}\,|a_2\rangle\langle a_2| = \frac{1}{2}\begin{pmatrix}e^{i\theta}+1 & e^{i\theta}-1 \\ e^{i\theta}-1 & e^{i\theta}+1\end{pmatrix}.$$

**Solution to Exercise A.90.** The matrices of $\hat{A}$ and $f(\hat{A})$ in the eigenbasis of $\hat{A}$ are

$$\hat{A} \simeq \begin{pmatrix}a_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_N\end{pmatrix}; \qquad f(\hat{A}) \simeq \begin{pmatrix}f(a_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & f(a_N)\end{pmatrix}$$

(with $a_i$ being the eigenvalues) and therefore

[1] The logarithm and the square root are examples of multivalued functions that are common in complex analysis.

$$\hat{A}f(\hat{A}) = f(\hat{A})\hat{A} \simeq \begin{pmatrix} a_1 f(a_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_N f(a_N) \end{pmatrix}.$$

Hence $[\hat{A}, f(\hat{A})] = \hat{A}f(\hat{A}) - f(\hat{A})\hat{A} = 0$.

**Solution to Exercise A.91.**

$$\begin{aligned} f(\hat{A}) &= \sum_i f(a_i)\, |a_i\rangle\langle a_i| \\ &= \sum_i \left(f_0 + f_1 a_i + f_2 a_i{}^2 + \cdots\right) |a_i\rangle\langle a_i| \\ &= f_0 \sum_i |a_i\rangle\langle a_i| + f_1 \sum_i a_i\, |a_i\rangle\langle a_i| + f_2 \sum_i a_i{}^2\, |a_i\rangle\langle a_i| + \cdots \\ &= f_0 \hat{1} + f_1 \hat{A} + f_2 \hat{A}^2 + \cdots . \end{aligned}$$

**Solution to Exercise A.92.** Any Hermitian operator may be diagonalized with real eigenvalues $a_i$ (see Ex.A.60):

$$\hat{A} = \sum_i a_i\, |a_i\rangle\langle a_i|\, .$$

The exponent of this operator,

$$e^{i\hat{A}} = \sum_i e^{ia_i}\, |a_i\rangle\langle a_i|\, ,$$

has the same eigenvectors, but eigenvalues $e^{ia_i}$. Because all $a_i$ are real, all $e^{ia_i}$ have absolute values equal to 1, so $e^{i\hat{A}}$ is unitary according to Ex. A.83.

At the same time, $e^{-i\hat{A}} = \sum\limits_i e^{-ia_i}\, |a_i\rangle\langle a_i|$, so

$$e^{i\hat{A}} e^{-i\hat{A}} = \sum_i e^{ia_i} e^{-ia_i}\, |a_i\rangle\langle a_i| = \sum_i |a_i\rangle\langle a_i| \overset{\text{(A.26)}}{=} \hat{\mathbf{1}}.$$

**Solution to Exercise A.93.** In the canonical basis, the operator $\vec{s}\hat{\vec{\sigma}}$ has the following matrix:

$$\begin{aligned} \vec{s}\hat{\vec{\sigma}} &\simeq s_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + s_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + s_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} s_z & s_x - is_y \\ s_x + is_y & -s_z \end{pmatrix}. \end{aligned}$$

This matrix is Hermitian; hence (Ex. A.60) $\vec{s}\hat{\vec{\sigma}}$ has two eigenvalues $v_{1,2}$ and two associated orthogonal eigenvectors $|v_{1,2}\rangle$. The eigenvalues of $\vec{s}\hat{\vec{\sigma}}$ are found by solving the charateristic equation:

$$\begin{aligned}|\vec{s}\hat{\vec{\sigma}} - v\hat{\mathbf{1}}| &= \begin{vmatrix} s_z - v & s_x - is_y \\ s_x + is_y & -s_z - v \end{vmatrix} \\ &= -(s_z - v)(s_z + v) - (s_x - is_y)(s_x + is_y) \\ &= v^2 - s_x^2 - s_y^2 - s_z^2 \\ &= v^2 - |\vec{s}|^2 = 0.\end{aligned}$$

Because $\vec{s}$ is a unit length vector, the eigenvalues are $v_{1,2} = \pm 1$ and thus

$$\vec{s}\hat{\vec{\sigma}} = |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|\,. \tag{SA.49}$$

We can now write the exponent of the operator as:

$$\begin{aligned}e^{i\theta\vec{s}\hat{\vec{\sigma}}} &= e^{i\theta}\,|v_1\rangle\langle v_1| + e^{-i\theta}\,|v_2\rangle\langle v_2| \\ &= (\cos\theta + i\sin\theta)\,|v_1\rangle\langle v_1| + (\cos\theta - i\sin\theta)\,|v_2\rangle\langle v_2| \\ &= \cos\theta(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|) + i\sin\theta(|v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|).\end{aligned} \tag{SA.50}$$

Although we have not found explicit expressions for $|v_1\rangle$ and $|v_2\rangle$, we know from Eq. (A.50) that $|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| = \hat{\mathbf{1}}$. Using this and Eq. (SA.49), we can rewrite Eq. (SA.50) as

$$e^{i\theta\vec{s}\hat{\vec{\sigma}}}\cos\theta\,\hat{1} + i\sin\theta\,\vec{s}\hat{\vec{\sigma}}.$$

**Solution to Exercise A.95.** Let us decompose $|\psi(t)\rangle = \sum_i \psi_i(t)\,|v_i\rangle$, where $\{|v_i\rangle\}$ is an orthonormal basis that is constant with respect to $t$. Using the Hilbert space linearity, we find

$$\frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t} = \lim_{\Delta t\to 0}\sum_i \frac{\psi_i(t+\Delta t) - \psi_i(t)}{\Delta t}\,|v_i\rangle = \sum_i \frac{\mathrm{d}\psi_i}{\mathrm{d}t}\,|v_i\rangle\,.$$

By a similar token, the derivative of an operator with matrix $(Y_{ij}(t))$ is the matrix $(\mathrm{d}Y_{ij}(t)/\mathrm{d}t)$.

**Solution to Exercise A.96.** In the orthonormal basis $\{|a_i\rangle\}$ that diagonalizes $\hat{A}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{i\hat{A}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \left(e^{ia_i t}\right)|a_i\rangle\langle a_i| \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{ia_i t}\right)|a_i\rangle\langle a_i| \qquad \text{(SA.51)} \\ &= \sum_i ia_i e^{ia_i t}\,|a_i\rangle\langle a_i|\,. \qquad \text{(SA.52)}\end{aligned}$$

Operators $i\hat{A}e^{i\hat{A}t}$ and $ie^{i\hat{A}t}\hat{A}$ have the same spectral decomposition.

**Solution to Exercise A.97.**

a) We use the Taylor decomposition of the exponential function of the operator to write

$$
\begin{aligned}
[\hat{A}, e^{\hat{B}}] &= \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, \hat{B}^n] \\
&\overset{(A.46)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} nc\hat{B}^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} c\hat{B}^{n-1} \\
&\overset{n'=n-1}{=} \sum_{n'=0}^{\infty} \frac{1}{(n')!} c\hat{B}^{n'} \\
&= c\hat{e}^{\hat{B}}.
\end{aligned}
\tag{SA.53}
$$

b) We begin by using the result of Ex. A.96 and writing

$$\frac{\mathrm{d}\hat{G}(\lambda)}{\mathrm{d}\lambda} = \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\hat{B}e^{\lambda\hat{B}}.$$

In order to bring the above result to the form of the right-hand side of Eq. (A.56), we need to move both $\hat{A}$ and $\hat{B}$ to the right of the exponentials. Each operator commutes with the exponential of itself (Ex. A.90), but in order to commute operators $\hat{A}$ and $e^{\lambda\hat{B}}$, the result of part (a) must be used, which we write as $\hat{A}e^{\lambda\hat{B}} = e^{\lambda\hat{B}}(\hat{A}+\lambda c)$. We have

$$\frac{\mathrm{d}\hat{G}(\lambda)}{\mathrm{d}\lambda} = e^{\lambda\hat{A}}e^{\lambda\hat{B}}(\hat{A}+\lambda c) + e^{\lambda\hat{A}}e^{\lambda\hat{B}}\hat{B} = \hat{G}(\lambda)(\hat{A}+\hat{B}+\lambda c).$$

c) Let $\hat{G}'(\lambda) = e^{\lambda\hat{A}+\lambda\hat{B}+\lambda^2 c/2}$ By taking the derivative of both sides of this equation, we obtain Eq. (A.56):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{G}'(\lambda) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} e^{\lambda\hat{A}+\lambda\hat{B}+\lambda^2 c/2} \\
&= e^{\lambda\hat{A}+\lambda\hat{B}+\lambda^2 c/2}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda\hat{A}+\lambda\hat{B}+\lambda^2 c/2)\right] \\
&= e^{\lambda\hat{A}+\lambda\hat{B}+\lambda^2 c/2}(\hat{A}+\hat{B}+\lambda c) \\
&= \hat{G}'(\lambda)(\hat{A}+\hat{B}+\lambda c).
\end{aligned}
$$

We see that both operators $\hat{G}(\lambda)$ and $\hat{G}'(\lambda)$ satisfy Eq. (A.56). In order to verify that these two operators are equal, we also need to check the Cauchy boundary condition, e.g.. that $\hat{G}(\lambda) = \hat{G}'(\lambda)$ for $\lambda = 0$. And indeed, in this case both $\hat{G}(\lambda)$ and $\hat{G}'(\lambda)$ become the identity operator, so the equation holds.

d) For $\lambda = 1$, Eq. (A.57) becomes

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+c/2}. \tag{SA.54}$$

Because $c$ is a number, this equation is equivalent to the Baker-Hausdorff-Campbell formula.

# Appendix SB
# Solutions to Appendix B exercises

**Solution to Exercise B.1.** If we have a fair, six-sided die the chance of it landing on any given side will be $\frac{1}{6}$. Thus, $\mathrm{pr}_i = \frac{1}{6}$ for all $i$. The quantity $Q_i$ is the value displayed on the die. Inserting these values into the equation for the expectation value we obtain:

$$\langle Q\rangle = \sum_{i=1}^{6} \mathrm{pr}_i Q_i = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3\frac{1}{2} \tag{SB.1}$$

**Solution to Exercise B.2.** We expand the right-hand side of Eq. (B.2) to write

$$\left\langle \Delta Q^2\right\rangle = \sum_i \mathrm{pr}_i Q_i^2 - 2\sum_i \mathrm{pr}_i Q_i \langle Q\rangle + \sum_i \mathrm{pr}_i \langle Q\rangle^2 . \tag{SB.2}$$

In the last two terms of the above expression, the quantity $\langle Q\rangle$ is the same for all values of $i$, so it can be factored out of the sum:

$$\left\langle \Delta Q^2\right\rangle = \sum_i \mathrm{pr}_i Q_i^2 - 2\langle Q\rangle \sum_i \mathrm{pr}_i Q_i + \langle Q\rangle^2 \sum_i \mathrm{pr}_i . \tag{SB.3}$$

Using

$$\sum_i \mathrm{pr}_i Q_i^2 = \left\langle Q^2\right\rangle; \quad \sum_i \mathrm{pr}_i Q_i = \langle Q\rangle; \quad \sum_i \mathrm{pr}_i = 1, \tag{SB.4}$$

we obtain

$$\left\langle \Delta Q^2\right\rangle = \left\langle Q^2\right\rangle - 2\langle Q\rangle\langle Q\rangle + \langle Q\rangle^2 = \left\langle Q^2\right\rangle - \langle Q\rangle^2 . \tag{SB.5}$$

**Solution to Exercise B.3.** The expectation value of the value on top of the die $\langle Q\rangle = 7/2$ (see Ex. B.1) and the probability of each event is $\frac{1}{6}$. Applying the definition of the uncertainty, we calculate

$$
\begin{aligned}
\left\langle \Delta Q^2 \right\rangle &= \sum_i \mathrm{pr}_i \left(Q_i - \langle Q \rangle\right)^2 \\
&= \frac{1}{6}\left(1 - 3\frac{1}{2}\right)^2 + \frac{1}{6}\left(2 - 3\frac{1}{2}\right)^2 + \frac{1}{6}\left(3 - 3\frac{1}{2}\right)^2 \\
&+ \frac{1}{6}\left(4 - 3\frac{1}{2}\right)^2 + \frac{1}{6}\left(5 - 3\frac{1}{2}\right)^2 + \frac{1}{6}\left(6 - 3\frac{1}{2}\right)^2 \\
&= \frac{35}{12}
\end{aligned}
$$

We can also solve this problem using the result of the previous exercise:

$$
\begin{aligned}
\left\langle \Delta Q^2 \right\rangle &= \left\langle Q^2 \right\rangle - \langle Q \rangle^2 = \sum_i \mathrm{pr}_i Q_i^2 - \langle Q \rangle^2 \\
&= \frac{1}{6}1^2 + \frac{1}{6}2^2 + \frac{1}{6}3^2 + \frac{1}{6}4^2 + \frac{1}{6}5^2 + \frac{1}{6}6^2 - \left(\frac{7}{2}\right)^2 \\
&= \frac{35}{12}.
\end{aligned}
$$

**Solution to Exercise B.4.** The quantity $QR$ can be viewed as a random variable that takes on value $Q_iR_j$ when $Q_i$ and $R_j$ occur at the same time, which happens with probability $\mathrm{pr}_i^Q \mathrm{pr}_j^R$ for each pair $(i, j)$. Now applying the definition of the expectation value we find

$$
\begin{aligned}
\langle QR \rangle &= \sum_i \sum_j \mathrm{pr}(Q_iR_j) Q_iR_j \\
&= \sum_i \sum_j \mathrm{pr}_i^Q \mathrm{pr}_j^R Q_iR_j \\
&= \left(\sum_i \mathrm{pr}_i^Q Q_i\right)\left(\sum_j \mathrm{pr}_j^R R_j\right) \\
&= \langle Q \rangle \langle R \rangle .
\end{aligned}
$$

If $Q$ and $R$ are not independent, the statement that $Q_i$ and $R_j$ occur at the same time with probability $\mathrm{pr}_i^Q \mathrm{pr}_j^R$ no longer holds, and neither does the identity $\langle QR \rangle = \langle Q \rangle \langle R \rangle$. For example, if $Q = R$, then $\langle QR \rangle = \left\langle Q^2 \right\rangle \neq \langle Q \rangle^2 = \langle Q \rangle \langle R \rangle$.

**Solution to Exercise B.5.** We can treat each $k$th time we toss the die as an independent random variable $Q^{(k)}$. Then $\tilde{Q} = \sum\limits_{k=1}^{N} Q^{(k)}$ and

$$
\left\langle \tilde{Q} \right\rangle = \sum_{k=1}^{N} \left\langle Q^{(k)} \right\rangle = N \langle Q \rangle
$$

and

$$\left\langle \tilde{Q}^2 \right\rangle = \left\langle \sum_{k=1}^{N} \sum_{\ell=1}^{N} Q^{(k)} Q^{(\ell)} \right\rangle$$
$$= \sum_{k=1}^{N} \sum_{\ell=1}^{N} \left\langle Q^{(k)} Q^{(\ell)} \right\rangle.$$

The latter expression has $N^2$ terms, of which $N$ terms have $k = \ell$ and $N(N-1)$ terms have $k \neq \ell$. For $k = \ell$, we have $\left\langle Q^{(k)} Q^{(\ell)} \right\rangle = \left\langle Q^2 \right\rangle$; otherwise $\left\langle Q^{(k)} Q^{(\ell)} \right\rangle = \langle Q \rangle^2$ according to Ex. B.4. Hence

$$\left\langle \tilde{Q}^2 \right\rangle = N \left\langle Q^2 \right\rangle + N(N-1) \langle Q \rangle^2.$$

For the variance of $\tilde{Q}$, we use Eq. (B.3) to write

$$\begin{aligned}
\left\langle \Delta \tilde{Q}^2 \right\rangle &= \left\langle \tilde{Q}^2 \right\rangle - \left\langle \tilde{Q} \right\rangle^2 \\
&= N \left\langle Q^2 \right\rangle + N(N-1) \langle Q \rangle^2 - N^2 \langle Q \rangle^2 \\
&= N \left\langle Q^2 \right\rangle - N \langle Q \rangle^2 \\
&= N \left\langle \Delta Q^2 \right\rangle,
\end{aligned}$$

and hence for the standard deviation

$$\sqrt{\left\langle \Delta \tilde{Q}^2 \right\rangle} = \sqrt{N} \sqrt{\langle \Delta Q^2 \rangle}.$$

**Solution to Exercise B.6.** Using Eq. (B.5), we find $\mathrm{pr}_{A|B_i} \mathrm{pr}_{B_i} = \mathrm{pr}_{A \text{ and } B_i}$. Because events $B_i$ are mutually exclusive we have $\sum_i \mathrm{pr}_{A \text{ and } B_i} = \sum_i \mathrm{pr}_{A \text{ and } (B_1 \text{ or } \dots \text{ or } B_n)}$. The latter quantity equals $\mathrm{pr}_A$ because events $B_i$ are collectively exhaustive, so event ($B_1$ or $\dots$ or $B_n$) occurs with certainty.

**Solution to Exercise B.7.**

a) According to Eq. (B.5), we have

$$\mathrm{pr}_{\text{positive}|\text{not infected}} = \mathrm{pr}_{\text{positive and not infected}} / \mathrm{pr}_{\text{not infected}},$$

so

$$\begin{aligned}
\mathrm{pr}_{\text{positive and not infected}} &= \mathrm{pr}_{\text{positive}|\text{not infected}} \times \mathrm{pr}_{\text{not infected}} \\
&= \mathrm{pr}_{\text{positive}|\text{not infected}} \times [1 - \mathrm{pr}_{\text{infected}}] = 0.04995.
\end{aligned}$$

b) We divide all people with positive results into two subsets: those who are infected and those who aren't:

$$\begin{aligned}
\mathrm{pr}_{\text{positive}} &= \mathrm{pr}_{\text{positive and not infected}} + \mathrm{pr}_{\text{positive and infected}} \\
&= \mathrm{pr}_{\text{positive and not infected}} + \mathrm{pr}_{\text{infected}} \\
&= 0.051.
\end{aligned}$$

The second equality above is because the test has no false negatives, i.e. the set of people who are infected *and* show positive results is the same as the set of people who are *just* infected.

c) Using the previous two results, we find:

$$\mathrm{pr}_{\text{not infected}|\text{positive}} = \frac{\mathrm{pr}_{\text{positive and not infected}}}{\mathrm{pr}_{\text{positive}}} \approx 0.98.$$

This result may appear surprising. Even though Alice's result is positive, the probability that she is actually infected is very low — because such is the fraction of people who are actually infected. The positive result for a random person is most likely false, in spite of a formally specified low rate of false positives.

**Solution to Exercise B.8.**

a) Each of the $n$ tosses is an independent random event. Therefore there are $2^n$ possible sequences of outcomes of length $n$, and the probability of any specific sequence is $1/2^n$. Among these sequences, there are $\binom{n}{k}$ those that have $k$ heads and $n-k$ tails. Hence the answer is $\mathrm{pr}_k = \binom{n}{k} \Big/ 2^n$.

b) In this case, the probability of any specific sequence containing $k$ heads and $n-k$ tails outcomes is $p^k(1-p)^{n-k}$. Hence the answer to part (a) becomes $\mathrm{pr}_k = \binom{n}{k} p^k(1-p)^{n-k}$.

**Solution to Exercise B.10.** For the mean, we have

$$\begin{aligned}\langle k\rangle &= \sum_{k=0}^{n} \frac{n!k}{k!(n-k)!} p^k(1-p)^{n-k} \\ &= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k(1-p)^{n-k}\end{aligned}$$

(note that we changed the lower summation limit because the term corresponding to $k=0$ equals zero). Now replacing the summation variable to $m=k-1$ we find

$$\begin{aligned}\langle k\rangle &= \sum_{m=0}^{n-1} np \frac{(n-1)!}{m!(n-1-m)!} p^m(1-p)^{n-1-m} \\ &= np \sum_{m=0}^{n-1} \binom{n-1}{m} p^m(1-p)^{n-1-m}.\end{aligned}$$

In the above, the expression subjected to summation is the binomial probability corresponding to $m$ successes out of $n-1$ events. The sum of these probabilities for all values of $m$ equals 1. Therefore $\langle k\rangle = np$.

For the mean square, we find, acting in a similar fashion:

$$\begin{aligned}
\langle k^2 \rangle &= \sum_{k=0}^{n} \frac{n!k^2}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \left[ \frac{n!k(k-1)}{k!(n-k)!} + \frac{n!k}{k!(n-k)!} \right] p^k (1-p)^{n-k} \\
&= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} + \langle k \rangle \\
&\overset{m:=k-2}{=} \sum_{m=0}^{n-2} p^2 n(n-1) \frac{(n-2)!}{m!(n-2-m)!} p^m (1-p)^{n-2-m} + \langle k \rangle \\
&= p^2 n(n-1) \sum_{m=0}^{n-2} \binom{n-2}{m} p^m (1-p)^{n-2-m} + \langle k \rangle \\
&= n(n-1)p^2 + np.
\end{aligned}$$

Hence

$$\langle \Delta k^2 \rangle = \langle k^2 \rangle - \langle k \rangle^2 = np - np^2.$$

**Solution to Exercise B.11.**

a) The probability for a baby to be born per unit of population per day is $p = 10/100000 = 10^{-4}$. Using the binomial distribution with $n = 100000$, we find

$$\begin{aligned}
\mathrm{pr}_{12} &= \frac{n!}{12!n-12!} p^{12} (1-p)^{n-12} \\
&= \frac{n(n-1)\dots(n-11)}{12!} e^{12\ln p + (n-12)\ln(1-p)} && = 0.0947841.
\end{aligned}$$

b) In a similar way, $\mathrm{pr}_{12} = 0.0947807$.

**Solution to Exercise B.12.**

a)

$$\frac{1}{n^k} \binom{n}{k} = \lim_{n\to\infty} \frac{n(n-1)\dots(n-k)}{n^k} \frac{1}{k!} = \frac{1}{k!}.$$

b) We know from calculus that $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$. Using this, we find

$$\lim_{n\to\infty} (1-p)^{n-k} = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n-k} = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = e^{-\lambda}.$$

c) From the above results, we obtain

$$\lim_{n\to\infty} \mathrm{pr}_k = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty} \frac{n^k}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}.$$

**Solution to Exercise B.13.**

$$e^{-12}\frac{10^{12}}{12!} = 0.0947803.$$

**Solution to Exercise B.15.** In the limit $p \to 0$, $n \to \infty$, $\lambda = pn = \text{const}$, Eq. (B.8) becomes

$$\langle k \rangle = np = \lambda; \quad \left\langle \Delta k^2 \right\rangle = np - np^2 \to \lambda.$$

**Solution to Exercise B.16.**

a) For a given discretized distribution, the probability that $Q$ falls within the range between $Q'$ and $Q''$ is the sum of the probabilities for all the bins located between these values:

$$\text{pr}_{[Q',Q'']} \approx \sum_{i(Q')}^{i(Q'')} \text{pr}_{\tilde{Q}_i}.$$

In the limit $\delta Q \to 0$, this approximation becomes equality because $\tilde{Q}_{i(Q')} \to Q'$ and $\tilde{Q}_{i(Q'')} \to Q''$. Hence, according to the definition (B.10) of the continuous probability density as well as the definition of the integral, we have

$$\lim_{\delta Q \to 0} \sum_{i(Q')}^{i(Q'')} \tilde{\text{pr}}_{\tilde{Q}_i} = \lim_{\delta Q \to 0} \sum_{i(Q')}^{i(Q'')} \text{pr}(Q)\delta Q = \int\limits_{Q'}^{Q''} \text{pr}(Q)\text{d}Q,$$

where $i(Q)$ is the number of the bin to which the value $Q$ belongs.

b) According to part (a), integral (B.12) corresponds to the probability to detect any value of $Q$ between $-\infty$ and $+\infty$, which is one.

c) In the discretized case,

$$\langle Q \rangle = \sum_i \tilde{Q}_i \tilde{\text{pr}}_{\tilde{Q}_i},$$

where the summation is performed over all bins. The transition from summation to integration in the limit $\delta Q \to 0$ is performed similarly to part (a).

**Solution to Exercise B.17.** The probability for the nucleus to remain undecayed after time $t$ from the start of the experiment is $2^{-t/\tau}$. The probability that the decay event occurs between the moments $t$ and $t + \delta t$ must then also be proportional to $2^{-t/\tau}$. Accordingly, $\text{pr}(t) = C \times 2^{-t/\tau}$, where $C$ is the normalization constant, which we can find using Eq. (B.12):

$$1 = \int_0^\infty \text{pr}(t)\text{d}t = C \int_0^\infty \times e^{-t\frac{\ln 2}{\tau}}\text{d}t = C\frac{\tau}{\ln 2},$$

so $C = \frac{\ln 2}{\tau}$.

For the expectation, we have

$$\langle t \rangle \overset{\text{(B.13)}}{=} \int_0^\infty t\,\text{pr}(t)\text{d}t = \frac{\ln 2}{\tau} \int_0^\infty t e^{-t\frac{\ln 2}{\tau}}\text{d}t = \frac{\tau}{\ln 2} = 1.44 \text{ ms}$$

and for the mean square

$$\left\langle t^2 \right\rangle = \int_0^\infty t^2 \mathrm{pr}(t)\mathrm{d}t = \frac{\ln 2}{\tau} \int_0^\infty t^2 e^{-t\frac{\ln 2}{\tau}} \mathrm{d}t = 2\left(\frac{\tau}{\ln 2}\right)^2 = 4.16 \text{ ms}^2.$$

Hence the uncertainty is

$$\sqrt{\langle \Delta t^2 \rangle} = \sqrt{\langle t^2 \rangle - \langle t \rangle^2} = \frac{\tau}{\ln 2} = 1.44 \text{ ms}.$$

**Solution to Exercise B.18.**

a) This follows directly from Eqs. (B.15) and (B.17).
b)

$$\begin{aligned}\langle x \rangle &= \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} x e^{-(x-a)^2/b^2} \mathrm{d}x \\ &= \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} (x-a) e^{-(x-a)^2/b^2} \mathrm{d}x + \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} a e^{-(x-a)^2/b^2} \mathrm{d}x \\ &= \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} t e^{-t^2/b^2} \mathrm{d}t + a \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-a)^2/b^2} \mathrm{d}x,\end{aligned}$$

where we replaced the integration variable according to $t = x - a$. The first term in the expression above vanishes because it is an integral of an odd function. The second term equals to $a$ as per part (a).

c) We have

$$\left\langle \Delta x^2 \right\rangle = \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} (x-a)^2 e^{-(x-a)^2/b^2} \mathrm{d}x \overset{(\mathrm{B.18})}{=} \frac{b^3\sqrt{\pi}}{2b\sqrt{\pi}} = \frac{b^2}{2}.$$

# Appendix SC
# Solutions to Appendix C exercises

**Solution to Exercise C.2.** See Fig. SC.1.

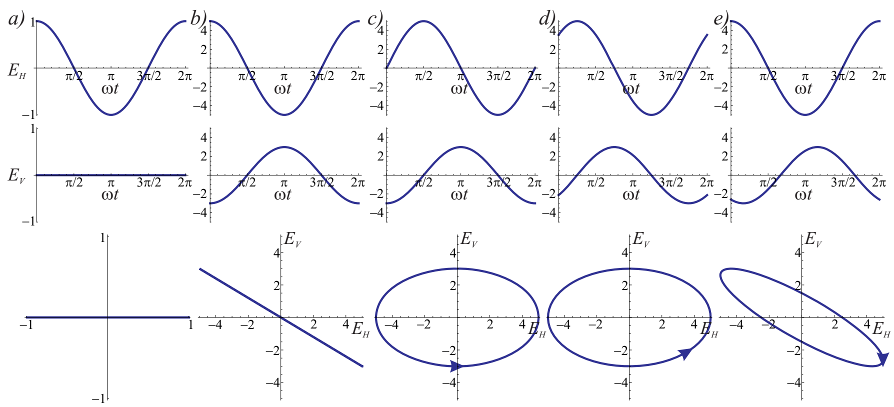

**Fig. SC.1** Solution to Ex. C.2.

**Solution to Exercise C.3.** The polarization pattern (C.2) at point $z+\Delta z$ at moment $t$ is the same as that at point $z$ at moment $t-(k/\omega)\Delta z = t-\Delta z/c$. Because the field vector is a periodic function of time, a shift in time will not change the shape of its trajectory.

**Solution to Exercise C.4.**

a) According to Eq. (C.1),

$$\begin{aligned} E_H(z,t) &= A_H \cos(kz-\omega t+\varphi_H), \\ E_V(z,t) &= A_V \cos(kz-\omega t+\varphi_V). \end{aligned} \tag{SC.1}$$

The polarization is linear if and only if $E_H(z,t) = 0$ or $E_V(z,t) = 0$ or $E_H(z,t) = \lambda E_V(z,t)$ with some coefficient $\lambda$. The first two conditions fulfill if and only if $A_H = 0$ or $A_V = 0$, respectively. The third condition implies that the two cosine function are proportional to each other, which can happen if and only if they are shifted in phase by $m\pi$.

b) Notice first that a circular pattern implies that the maximum absolute values of the horizontal and vertical components of the field must be the same, so $A_H = \pm A_V$. Further, circular pattern means $E_H^2 + E_V^2 = \text{const}$, which implies

$$\cos^2(kz - \omega t + \varphi_H) + \cos^2(kz - \omega t + \varphi_V) = \text{const}.$$

Because $\cos^2\varphi = (1 + \cos 2\varphi)/2$ for any $\varphi$, this condition is equivalent to

$$\cos[2(kz - \omega t + \varphi_H)] + \cos[2(kz - \omega t + \varphi_V)] = \text{const}.$$

Using another trigonometric identity $\cos\varphi + \cos\theta = 2\cos[(\varphi + \theta)/2]\cos[(\varphi - \theta)/2]$, we have

$$\cos[2(kz - \omega t) + \varphi_H + \varphi_V]\cos(\varphi_H - \varphi_V) = \text{const}.$$

Because the first factor in the left-hand side of the above condition cannot be a constant, this condition is satisfied if and only if $\cos(\varphi_H - \varphi_V) = 0$, i.e. $\varphi_H = \varphi_V \frac{\pi}{2} + m\pi$.

**Solution to Exercise C.5.** We will try and prove that there exist a set of numbers $\{A, B, D, F\}$ that are independent of $u$, such that

$$AE_H^2 + BE_V^2 + DE_H E_V = F, \tag{SC.2}$$

where $E_H(z,t)$ and $E_V(z,t)$ are given by Eq. (C.1). It is known from analytic geometry that Eq. (SC.2) represents one of the conic sections: hyperbola, parabola, or ellipse. Because both $E_H$ and $E_V$ are bounded functions, Eq. (SC.2) can only describe an ellipse, with the circle and linear trajectory being extreme cases thereof.

We use trigonometric identities to write Eq. (C.1) as follows:

$$\begin{aligned} E_H &= A_H(c_H c - s_H s); \\ E_V &= A_H(c_H c - s_V s), \end{aligned} \tag{SC.3}$$

where we have defined $c = \cos(kz - \omega t)$, $s = \sin(kz - \omega t)$, $c_{H,V} = \cos\varphi_{V,H}$ and $s_{H,V} = \sin\varphi_{V,H}$ Next, we transform the left-hand side of Eq. (SC.2):

$$
\begin{aligned}
&AE_H^2+BE_V^2+DE_HE_V \qquad \text{(SC.4)}\\
&=AA_H^2[c_H^2c^2+s_H^2s^2-2c_Hs_Hcs]\\
&+BA_V^2[c_V^2c^2+s_V^2s^2-2c_Vs_Vcs]\\
&+DA_HA_V[c_Hc_Vc^2+s_Hs_Vs^2-(c_Hs_V+s_Hc_V)cs]\\
&=AA_H^2\left[\frac{1}{2}(c_H^2+s_H^2)(c^2+s^2)+\frac{1}{2}(c_H^2-s_H^2)(c^2-s^2)-2c_Hs_Hcs\right]\\
&+BA_V^2\left[\frac{1}{2}(c_V^2+s_V^2)(c^2+s^2)+\frac{1}{2}(c_V^2-s_V^2)(c^2-s^2)-2c_Vs_Vcs\right]\\
&+DA_HA_V\left[\frac{1}{2}(c_Hc_V+s_Hs_V)(c^2+s^2)+\frac{1}{2}(c_Hc_V-s_Hs_V)(c^2-s^2)-(c_Hs_V+s_Hc_V)cs\right]\\
&=AA_H^2\left[\frac{1}{2}+\frac{1}{2}(c_H^2-s_H^2)(c^2-s^2)-2c_Hs_Hcs\right]\\
&+BA_V^2\left[\frac{1}{2}+\frac{1}{2}(c_V^2-s_V^2)(c^2-s^2)-2c_Vs_Vcs\right]\\
&+DA_HA_V\left[\frac{1}{2}+\frac{1}{2}(c_Hc_V-s_Hs_V)(c^2-s^2)-(c_Hs_V+s_Hc_V)cs\right],
\end{aligned}
$$

where we used $c_H^2+s_H^2=c_V^2+s_V^2=c^2+s^2=1$. The above result will simplify to

$$AE_H^2+BE_V^2+DE_HE_V=\frac{1}{2}(AA_H^2+BA_V^2+DA_HA_V) \qquad \text{(SC.5)}$$

if $A$, $B$ and $D$ are such that the coefficients in front of the $(z,t)$-dependent variables $c^2-s^2$ and $cs$ in Eq. (SC.4) vanish:

$$\begin{cases} AA_H^2(c_H^2-s_H^2)+BA_V^2(c_V^2-s_V^2)+DA_HA_V(c_Hc_V-s_Hs_V)=0\\ 2AA_H^2c_Hs_H+2BA_V^2c_Vs_V+DA_HA_V(c_Hs_V+s_Hc_V)=0. \end{cases}$$

The above is a system of two equations with three unknown variables, so it always has a nontrivial solution. For this solution, Eq. (SC.5) holds, which is identical to Eq. (SC.2) with $F=\frac{1}{2}(AA_H^2+BA_V^2+DA_HA_V)$.

**Solution to Exercise C.6.** Refractive indices $n_e$ and $n_o$ modify the wavelengths of the ordinary and extraordinary waves according to $\lambda_e=\lambda/n_e$ and $\lambda_o=\lambda/n_o$, respectively, which corresponds to wavenumbers $k_e=2\pi n_e/\lambda$ and $k_o=2\pi n_o/\lambda$. Propagating through the crystal, these waves acquire phases $\varphi_e=k_eL$ and $\varphi_o=k_oL$, so $\Delta\varphi=2\pi(n_e-n_o)L/\lambda$.

**Solution to Exercise C.7.** The half- and quarter-waveplates with vertical optic axes will shift the phase of the vertical field component by $\pi$ and $\pi/2$, respectively. See Fig. SC.2.

**Solution to Exercise C.9.** The $\pm 45^\circ$ polarization patterns correspond to $A_H=\pm A_V$ and $\varphi_H=\varphi_V+m\pi$ with an integer $m$. By comparing this condition with that of Ex. C.4(b), we find that the $\pm 45^\circ$ and circular polarization waves obtain from each other by adding or subtracting $\pi/2$ to/from $\varphi_V$, which is exactly what the quarter-wave plate does.

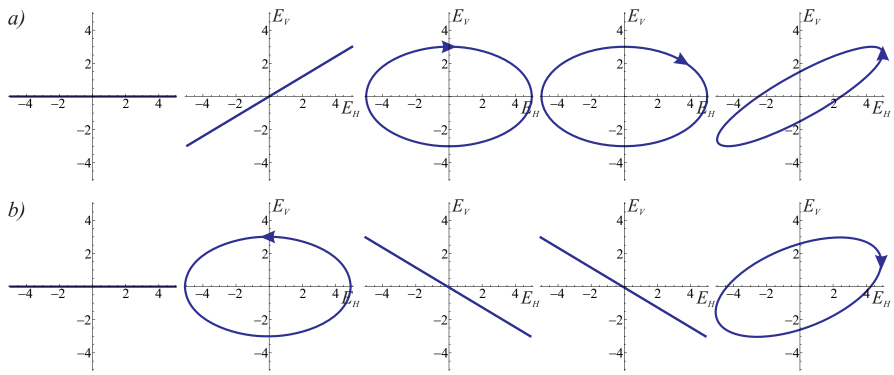

**Fig. SC.2** Solution to Ex. C.2.

**Solution to Exercise C.10.** Linear polarization at angle $\theta$ implies that $A_H = A\cos\theta$, $A_V = A\sin\theta$, where $A$ is real and positive, and $\varphi_H - \varphi_V = 0$. Without loss of generality, we assume $\varphi_H = \varphi_V = 0$. Before the waveplate, we have

$$\begin{aligned} E_H(z,t) &= A\cos\theta\cos(kz-\omega t), \\ E_V(z,t) &= A\sin\theta\cos(kz-\omega t); \end{aligned} \tag{SC.6}$$

and after the waveplate

$$\begin{aligned} E_H(z,t) &= A\cos\theta\cos(kz-\omega t), \\ E_V(z,t) &= A\sin\theta\cos(kz-\omega t+\pi/2) = -A\sin\theta\sin(kz-\omega t). \end{aligned} \tag{SC.7}$$

It follows from the latter result that

$$\frac{E_H^2(z,t)}{\cos^2\theta} + \frac{E_V^2(z,t)}{\sin^2\theta} = A^2,$$

which is the equation of an ellipse whose axes are oriented vertically and horizontally, and the ratio between these axes equal to $\cos\theta/\sin\theta$ (Fig. SC.3).

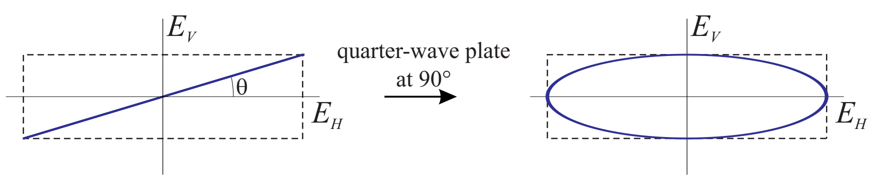


**Fig. SC.3** A $\lambda/4$-plate with the optic axis oriented vertically transforms a linearly polarized light into elliptically polarized while preserving both the vertical and horizontal amplitudes (Ex. C.10).

**Solution to Exercise C.11.** As we know from Ex. C.5, the general polarization pattern is elliptical. Suppose the amplitudes along the major and minor semiaxes are $A_1$ and $A_2$ and the major axis is oriented at angle $\beta$ to horizontal. Let us denote $\theta = \tan^{-1}(A_2/A_1)$ and $A = \sqrt{A_1^2 + A_2^2}$. Let us start with horizontally polarized light of amplitude $A$ and apply a quarter-wave plate at angle $\theta$ to horizontal. In the reference frame of the waveplate, this action is equivalent to applying the quarter-wave plate with a horizontal optic axis to a linear pattern at angle $-\theta$. Using the logic of the previous exercise, we will obtain an elliptical pattern with the axes oriented along and perpendicular to the optic axis, the ratio between these axes being equal to $\cos\theta/\sin\theta = A_1/A_2$. In the lab frame, this corresponds to an elliptical pattern at angle $\theta$ to horizontal. What remains is to turn this pattern; this is achieved by means of a half-wave plate at angle $(\beta+\theta)/2$ (Fig. SC.4).

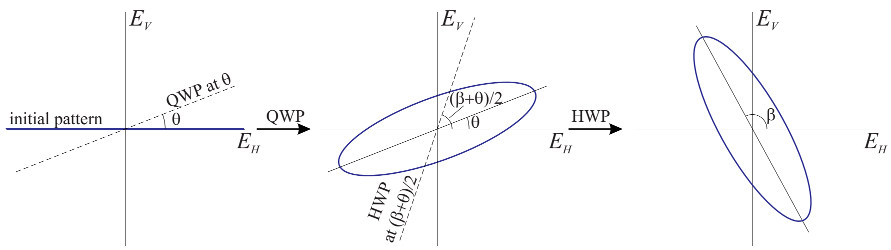


**Fig. SC.4** Obtaining an arbitrary polarization pattern from horizontal using two waveplates (Ex. C.11). HWP/QWP: half- and quarter-wave plates

**Solution to Exercise C.12.** In the reference frame oriented at angle 45° with respect to the lab frame, the optic axis of the quarter wave plate is vertical. Linearly polarized light, propagating through this waveplate, generates a pattern described by

$$\vec{E}(z,t) = A\mathrm{Re}[(\cos\theta\hat{i} + i\sin\theta\hat{j})e^{ikz-i\omega t}],$$

where $\theta$ is the angle between the polarization and the waveplate axis and $A = \sqrt{A_H^2 + A_V^2}$ (see Ex. C.10). In order to convert to the lab frame, we perform rotation of the field vector in the $x-y$ plane by 45° using the matrix found in Ex. A.41

$$\hat{R}_{45^\circ} \simeq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

to find

$$\vec{E}'(z,t) = \frac{A}{\sqrt{2}}\mathrm{Re}[(\cos\theta - i\sin\theta)\hat{i} + (\cos\theta + i\sin\theta)\hat{j}]e^{ikz-i\omega t}.$$

This corresponds to the same intensity $A^2(\cos^2\theta + \sin^2\theta)/2 = A^2/2$ for the horizontal and vertical polarizations.

This result is easily visualized by observing that the transformation of a linear pattern in the reference frame of the quarter-wave plate (Fig. SC.3) results in an elliptical pattern that is symmetric with respect to ±45° axes (which correspond to the horizontal and vertical in the lab frame), and hence will yield equal amounts of energy when projected onto these axes.

# Appendix SD
# Solutions to Appendix D exercises

**Solution to Exercise D.1.** Using integration by parts, we find

$$\int_{-\infty}^{+\infty} G_b(x) f(x) = \Gamma_b(x) f(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \Gamma_b(x) f'(x)\mathrm{d}x, \tag{SD.1}$$

where

$$\Gamma_b(x) = \int_{-\infty}^{x} G_b(x')\mathrm{d}x'.$$

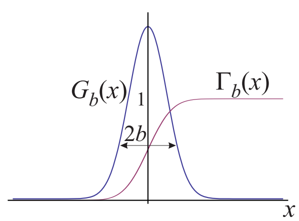


**Fig. SD.1** Normalized Gaussian function $G_b(x)$ and its integral $\Gamma_b(x)$.

The first term of Eq. (SD.1) equals $f(+\infty)$ for a bounded $f(x)$. To evaluate the second term, we analyze the behavior of function $\Gamma_b(x)$ (Fig. SD.1). It approaches 0 at $-\infty$, 1 at $+\infty$, and is significantly different from these values in the region where $G_b(x)$ is substantially different from zero. The width of this region vanishes with $b \to 0$. In this limit, $\Gamma_b(x)$ behaves like the Heaviside step function (D.7). Therefore, for a smooth $f(x)$,

$$\int_{-\infty}^{+\infty} \Gamma_b(x) f'(x)\mathrm{d}x = \int_{0}^{+\infty} f'(x)\mathrm{d}x = f(x)\Big|_{0}^{+\infty} = f(+\infty) - f(0)$$

according to the fundamental theorem of calculus. Substituting both terms into Eq. (SD.1), we obtain $f(0)$.

**Solution to Exercise D.2.**

a) We obtain Eq. (D.4) by substituting $f(x) = 1$ into Eq. (D.3).
b) Make a replacement of the integration variable $x - a = t$. Then $\mathrm{d}t = \mathrm{d}x$ and

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \delta(t) f(t+a)\mathrm{d}t \overset{\text{(D.3)}}{=} f(a).$$

c) Consider a smooth function $f(x)$ and the integral

$$I = \int_{-\infty}^{+\infty} \delta(ax) f(x)\mathrm{d}x.$$

To calculate this integral, we replace the integration variable $ax = t$, so $\mathrm{d}x = \mathrm{d}t/a$. Then for a positive $a$,

$$I = \int_{-\infty}^{+\infty} \delta(t) f(t/a)\mathrm{d}t/a = f(0)/a.$$

If $a$ is negative, $x = \pm\infty$ corresponds to $t = \mp\infty$, so we have to change the integration limits:

$$I = \int_{+\infty}^{-\infty} \delta(t) f(t/a)\mathrm{d}t/a = -\int_{-\infty}^{+\infty} \delta(t) f(t/a)\mathrm{d}t/a = -f(0)/a.$$

The two equations above can be combined by writing

$$\int_{-\infty}^{+\infty} \delta(ax) f(x)\mathrm{d}x = f(0)/|a|. \tag{SD.2}$$

Comparing Eqs. (D.3) and (SD.2) we obtain

$$\delta(ax) = \frac{\delta(x)}{|a|}.$$

**Solution to Exercise D.3.** Let $\mathrm{d}\theta(x)/\mathrm{d}x = \alpha(x)$ and consider the integral

$$I = \int_{-\infty}^{+\infty} \alpha(x) f(x)\mathrm{d}x$$

for a smooth bounded function $f(x)$. Integrating by parts, we find

$$I = \theta(x) f(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \theta(x) f'(x) \mathrm{d}x.$$

The first term in the above expression is $f(+\infty)$. The second term, according to the definition of the Heaviside function, is

$$\int_{0}^{+\infty} f'(x) \mathrm{d}x = f(+\infty) - f(0),$$

so $I = f(0)$. Thus the generalized function $\alpha(x)$ behaves according to the definition (D.3) of the delta function, so it *is* the delta function.

**Solution to Exercise D.4.**

$$\int_{c}^{d} \delta(x) \mathrm{d}x = \int_{c}^{d} \theta'(x) \mathrm{d}x = \theta(x)|_{-\infty}^{+\infty} = 1,$$

where $\theta(x)$ is the Heaviside function and we have used the fundamental theorem of calculus.

**Solution to Exercise D.5.**

a) This follows from the definition (D.10) of the Fourier transform for $k = 0$.
b)

$$\begin{aligned}
\tilde{f}(-k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(-k)x} f(x) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{-ikx})^* f(x) \mathrm{d}x \\
&= \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) \mathrm{d}x \right]^* \quad \text{(because } f(x) \text{ is real)} \\
&= \tilde{f}^*(k).
\end{aligned}$$

c) We introduce a new integration variable $t = ax$ and act by analogy to Ex. D.2(c)

$$\begin{aligned}
\mathscr{F}[f(ax)] &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(ax)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(ax)e^{-i\frac{k}{a}ax}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\,|a|}\int\limits_{-\infty}^{+\infty} f(t)e^{-i\frac{k}{a}t}\mathrm{d}t \\
&= \frac{1}{|a|}\tilde{f}(k/a).
\end{aligned}$$

d) Similarly to the above, replacing the integration variable according to $t = x - a$, we have

$$\begin{aligned}
\mathscr{F}[f(x-a)] &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x-a)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x-a)e^{-ik(x-a)-ika}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-ika}\int\limits_{-\infty}^{+\infty} f(t)e^{-ikt}\mathrm{d}t \\
&= e^{-ika}\tilde{f}(k).
\end{aligned}$$

e)

$$\begin{aligned}
\mathscr{F}[e^{i\xi x}f(x)] &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}e^{i\xi x}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-i(k-\xi)x}\mathrm{d}x \\
&= \tilde{f}(k-\xi).
\end{aligned}$$

f) We use integration by parts and the fact that $f(x)$ vanishes at $\pm\infty$:

$$\begin{aligned}\mathscr{F}[\mathrm{d}f(x)/\mathrm{d}x] &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}f(x)}{\mathrm{d}x}\,e^{-ikx}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}f(x)e^{-ikx}\Big|_{-\infty}^{+\infty} - \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)\frac{\mathrm{d}e^{-ikx}}{\mathrm{d}x}\mathrm{d}x \\ &= 0-(-ik)\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x \\ &= ik\tilde{f}(k).\end{aligned}$$

**Solution to Exercise D.6.** In order to calculate the integral

$$\mathscr{F}[e^{-x^2/b^2}] = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{-ikx}e^{-x^2/b^2}\mathrm{d}x, \tag{SD.3}$$

we express the exponent in Eq. (SD.3) as a quadratic function of $x$ and then apply Eq. (B.17):

$$\begin{aligned}\mathscr{F}[e^{-x^2/b^2}] &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{b^2}+ikx+\frac{k^2b^2}{4}-\frac{k^2b^2}{4}\right)}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}\int\limits_{-\infty}^{+\infty} e^{-\left(\frac{x}{b}+\frac{ikb}{2}\right)^2}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}\int\limits_{-\infty}^{+\infty} e^{-\frac{\left(x+ikb^2/2\right)^2}{b^2}}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}b\sqrt{\pi} \\ &= \frac{b}{\sqrt{2}}e^{-\frac{k^2b^2}{4}}.\end{aligned}$$

**Solution to Exercise D.7.**

a)

$$\mathscr{F}[\delta(x)](k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\delta(x)e^{-ikx} \overset{\text{(D.3)}}{=} \frac{1}{\sqrt{2\pi}}.$$

b) By setting $d = 2/b$, Eq. (D.1) can be rewritten in the form

$$\frac{d}{2\sqrt{\pi}}e^{\frac{-k^2d^2}{4}} \to \delta(k) \qquad \text{for} \qquad d\to\infty. \tag{SD.4}$$

We also notice that, in the limit $d \to \infty$, the Gaussian function $e^{-x^2/d^2}$ becomes constant 1. Hence

$$\mathscr{F}[1] = \lim_{d\to\infty} \mathscr{F}[e^{-x^2/d^2}] \overset{(D.16)}{=} \lim_{d\to\infty} \frac{d}{\sqrt{2}} e^{-\frac{k^2d^2}{4}} \overset{(SD.4)}{=} \sqrt{2\pi}\,\delta(k).$$

**Solution to Exercise D.8.** Let us first set $a = 1$. We notice that the required integral is, up to a factor of $\sqrt{2\pi}$, the Fourier transform of the function $f(x) = 1$ at point $-k$. Applying Eq. (D.18), we find

$$\int_{-\infty}^{+\infty} e^{ikx} \mathrm{d}x = \sqrt{2\pi}\mathscr{F}[1](-k) = 2\pi\,\delta(-k) = 2\pi\,\delta(k).$$

Here we used the fact that the Delta function is even, which is evident from Eq. (D.1). To generalize this result to an arbitrary $a$, we use Eq. (D.12).

**Solution to Exercise D.9.**

a) Using results (D.13) and (D.17) we have

$$\mathscr{F}[\delta(x+a)+\delta(x-a)] = \frac{e^{ika}+e^{-ika}}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\cos ka.$$

b)

$$\begin{aligned}\mathscr{F}[\cos(ax+b)](k) &= \mathscr{F}\left[\frac{e^{iax}e^{ib}+e^{-iax}e^{-ib}}{2}\right](k)\\ &\overset{(D.14)}{=} \frac{e^{ib}}{2}\mathscr{F}[1](k-a) + \frac{e^{-ib}}{2}\mathscr{F}[1](k+a)\\ &\overset{(D.18)}{=} \sqrt{\frac{\pi}{2}}[e^{ib}\delta(k-a)+e^{-ib}\delta(k+a)].\end{aligned}$$

c)

$$\begin{aligned}\mathscr{F}[e^{-ax^2}\cos bx](k) &= \mathscr{F}\left[\frac{e^{-ax^2}e^{ibx}+e^{-ax^2}e^{-ibx}}{2}\right](k)\\ &\overset{(D.14)}{=} \frac{1}{2}\left\{\mathscr{F}[e^{-ax^2}](k-b)+\mathscr{F}[e^{-ax^2}](k+b)\right\}\\ &\overset{(D.16)}{=} \frac{1}{2\sqrt{2a}}\left\{e^{-\frac{(k-b)^2}{4a}}+e^{-\frac{(k+b)^2}{4a}}\right\}.\end{aligned}$$

d)

$$\begin{aligned}\mathscr{F}[e^{-a(x+b)^2}+e^{-a(x-b)^2}](k) &\overset{(D.13)}{=} \mathscr{F}[e^{-ax^2}](e^{ikb}+e^{-ikb})\\ &\overset{(D.16)}{=} \sqrt{\frac{2}{a}}e^{-\frac{k^2}{4a}}\cos kb.\end{aligned}$$

e)

$$\mathscr{F}[\theta(x)e^{-ax}](k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-ax-ikx} \mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{a+ik}.$$

f)

$$\begin{aligned}\mathscr{F}[\text{top-hat}(x)](k) &= \frac{A}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \mathrm{d}x \\ &= -\frac{A}{\sqrt{2\pi}ik}(e^{-ika} - e^{ika}) \\ &= \frac{\sqrt{2}A}{\sqrt{\pi}k} \sin(ka) \\ &= \frac{\sqrt{2}Aa}{\sqrt{\pi}} \operatorname{sinc}(ka),\end{aligned}$$

where $\operatorname{sinc} t = \sin t / t$.

**Solution to Exercise D.10.**

$$\begin{aligned}\mathscr{F}^{-1}[\mathscr{F}[f]](x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(x') e^{-ikx'} \mathrm{d}x' \right) e^{ikx} \mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x') e^{ik(x-x')} \mathrm{d}k \, \mathrm{d}x' \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x') \left[ \int_{-\infty}^{+\infty} e^{ik(x-x')} \mathrm{d}k \right] \mathrm{d}x' \\ &\overset{(D.19)}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x') \left[ 2\pi\delta(x-x') \right] \mathrm{d}x' \\ &\overset{(D.5)}{=} f(x).\end{aligned}$$

**Solution to Exercise D.11.** We start with the definition (D.21) of the inverse Fourier transformation:

$$\mathscr{F}^{-1}[f(x)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(-k)x} f(x) \mathrm{d}x = \mathscr{F}[f(x)](-k)$$

[here we exchanged the variables $x$ and $k$ with respect to Eq. (D.21)].