Phys 543 Fall 2017

Homework 5

Solutions

1 a)

$$\psi(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$$

$$\rho(x\ x') = \psi(x)\,\psi^*(x') = \frac{1}{\sqrt{\pi}} e^{-x^2/2 - x'^2/2}$$

$$W(x\ p) = \frac{1}{2\pi}\int e^{ipQ}\rho\left(x-\frac{Q}{2}, x+\frac{Q}{2}\right) dQ$$

$$= \frac{1}{2\pi\sqrt{\pi}}\int e^{ipQ} e^{-x^2-\frac{Q^2}{4}} dQ$$

$$= \frac{1}{\sqrt{2}\,\pi} e^{-x^2} F\left[e^{-Q^2/4}\right](p)$$

$$= \frac{1}{\pi} e^{-x^2-p^2}$$

$$pr(x) = \int W(x\ p)\, dp = \frac{1}{\sqrt{\pi}} e^{-x^2} = \psi^2(x)$$

$$pr(p) = \int W(x\ p)\, dx = \frac{1}{\sqrt{\pi}} e^{-p^2} = \psi^2(p)$$

b)

$$\psi(x) = \frac{\sqrt{2}x}{\pi^{1/4}} e^{-x^2/2}$$

$$\rho(x\ x') = \frac{2xx'}{\sqrt{\pi}} e^{-x^2/2 - x'^2/2}$$

$$W(x\ p) = \frac{1}{2\pi}\int e^{ipQ}\rho\left(x-\frac{Q}{2}, x+\frac{Q}{2}\right) dQ$$

$$= \frac{1}{\pi\sqrt{\pi}}\int e^{ipQ}\left(x^2-\frac{Q^2}{4}\right) e^{-x^2-\frac{Q^2}{4}} dQ$$

$$= \frac{\sqrt{2}}{\pi} e^{-x^2}\left\{x^2 F\left[e^{-Q^2/4}\right](p) - \frac{1}{4} F\left[\frac{Q^2}{4} e^{-Q^2/4}\right](p)\right\}$$

$$= \frac{\sqrt{2}}{\pi} e^{-x^2}\left(x^2\sqrt{2}\, e^{-p^2} + \frac{2\sqrt{2}}{4} e^{-p^2}(-1+2p^2)\right)$$

$$= \frac{1}{\pi} e^{-x^2-p^2}\left(2x^2+2p^2-1\right)$$

$$pr(x) = \int W(x\ p)\, dp = \frac{1}{\pi} e^{-x^2}\left[(2x^2-1)\sqrt{\pi} + 2\frac{\sqrt{\pi}}{2}\right] = \frac{2x^2}{\sqrt{\pi}} e^{-x^2} = \psi^2(x)$$

$$pr(p) = \frac{2p^2}{\sqrt{\pi}} e^{-p^2} = \psi^2(p)$$

c) $\psi(x) = \frac{1}{\pi^{1/4}} e^{-(x-x_\alpha)^2/2}$ $\qquad x_\alpha = \alpha\sqrt{2}$

$\rho(x\ x') = \frac{1}{\sqrt{\pi}} e^{-(x-x_\alpha)^2/2 - (x'-x_\alpha)^2/2}$

$W(x\ p) = \frac{1}{2\pi\sqrt{\pi}} \int e^{-(x-x_\alpha)^2 - \frac{Q^2}{4}} e^{ipQ} \, dQ$

$= \frac{1}{\pi} e^{-(x-x_\alpha)^2 - p^2}$

$pr(x) = \int W(x\ p)\, dp = \frac{1}{\sqrt{\pi}} e^{-(x-x_\alpha)^2} = \psi^2_{|\alpha\rangle}(x)$

$pr(p) = \int W(x\ p)\, dx = \frac{1}{\sqrt{\pi}} e^{-p^2}$

$\psi(p) = \frac{1}{\pi^{1/4}} e^{-p^2/2} e^{-ipx_\alpha} \Rightarrow |\psi(p)|^2 = \frac{1}{\sqrt{\pi}} e^{-p^2} = pr(p)$ ✓

d) $\psi(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} e^{ip_\alpha x}$ $\qquad ip_\alpha = \alpha\sqrt{2}$

$\rho(x\ x') = \frac{1}{\pi^{1/4}} e^{-x^2/2 - x'^2/2} e^{ip_\alpha(x-x')}$

$W(x\ p) = \frac{1}{2\pi\sqrt{\pi}} \int e^{ipQ} e^{-x^2 - Q^2/4} e^{-ip_\alpha Q} \, dQ$

$= \frac{1}{2\pi\sqrt{\pi}} \int e^{i(p-p_\alpha)Q} e^{-x^2 - Q^2/4} =$

$= \frac{1}{\pi} e^{-x^2 - (p-p_\alpha)^2}$

e) $\psi(x) = \frac{1}{\pi^{1/4}} \left[e^{-(x-x_\alpha)^2/2} + e^{-(x+x_\alpha)^2/2} \right]$

$\rho(x\ x') = \frac{1}{\sqrt{\pi}} \left[e^{-(x-x_\alpha)^2/2 - (x'-x_\alpha)^2/2} + e^{-(x-x_\alpha)^2/2 - (x'+x_\alpha)^2/2} \right.$

$\left. + e^{-(x+x_\alpha)^2/2 - (x'-x_\alpha)^2/2} + e^{-(x+x_\alpha)^2/2 - (x'+x_\alpha)^2/2} \right]$

$= \frac{1}{\sqrt{\pi}} e^{-x^2/2 - x'^2/2 - x_\alpha^2} \left(e^{x_\alpha(x+x')} + e^{x_\alpha(x-x')} + e^{x_\alpha(-x+x')} + e^{x_\alpha(-x-x')} \right)$

$W(x\ p) = \frac{1}{2\pi\sqrt{\pi}} \int e^{ipQ} e^{-x^2 - x_\alpha^2 - \frac{Q^2}{4}} \left(e^{2x_\alpha x} + e^{x_\alpha Q} + e^{-x_\alpha Q} + e^{-2x_\alpha x} \right) dQ$

$= \frac{1}{\sqrt{2}\pi} \left[e^{-(x-x_\alpha)^2} + e^{-(x+x_\alpha)^2} \right] F\left[e^{-Q^2/4} \right](p)$

$+ \frac{1}{\sqrt{2}\pi} e^{-x^2} \left\{ F\left[e^{-\left(\frac{Q-2x_\alpha}{2}\right)^2} \right](p) + F\left[e^{-\left(\frac{Q+2x_\alpha}{2}\right)^2} \right](p) \right\}$

$= \frac{1}{\pi} e^{-(x-x_\alpha)^2 - p^2} + \frac{1}{\pi} e^{-(x+x_\alpha)^2 - p^2}$

$+ \frac{1}{\pi} e^{-x^2 - p^2} \left(e^{2ipx_\alpha} + e^{-2ipx_\alpha} \right)$

$= \frac{1}{\pi} e^{-(x-x_\alpha)^2 - p^2} + \frac{1}{\pi} e^{-(x+x_\alpha)^2 - p^2}$

$+ \frac{2}{\pi} e^{-x^2 - p^2} \cos 2px_\alpha$

$$pr(x) = \int W(x\ \ p)\, dp = \frac{1}{\sqrt{\pi}} e^{-(x-x_\alpha)^2} + \frac{1}{\sqrt{\pi}} e^{-(x+x_\alpha)^2} + \frac{2}{\sqrt{\pi}} e^{-x^2 - x_\alpha^2}$$

$$= \frac{1}{\sqrt{\pi}} \left(e^{-(x-x_\alpha)^2/2} + e^{-(x+x_\alpha)^2/2} \right)^2 = \left(\psi_{|\alpha\rangle}(x) + \psi_{|-\alpha\rangle}(x) \right)^2$$

$$pr(p) = \int W(x\ \ p)\, dx = \frac{2}{\sqrt{\pi}} e^{-p^2} \left(1 + \cos 2 p x_\alpha \right)$$

$$|\psi_{|\alpha\rangle}(p) + \psi_{|-\alpha\rangle}(p)|^2 = \left| \frac{1}{\pi^{1/4}} e^{-p^2/2} \left(e^{ipx_\alpha} + e^{-ipx_\alpha} \right) \right|^2$$

$$= \frac{1}{\sqrt{\pi}} e^{-p^2} \left(2 \cos p x_\alpha \right)^2$$

$$= \frac{1}{\sqrt{\pi}} e^{-p^2} \left(1 + \cos 2 p x_\alpha \right)$$

f) $W(x\ \ p) = W_{|\alpha\rangle}(x\ \ p) + W_{|-\alpha\rangle}(x\ \ p)$

$$= \frac{1}{\pi} \left(e^{-(x-x_\alpha)^2 - p^2} + e^{-(x+x_\alpha)^2 - p^2} \right)$$

$$pr(x) = \frac{1}{\sqrt{\pi}} \left(e^{-(x-x_\alpha)^2} + e^{-(x+x_\alpha)^2} \right)$$

$$pr(p) = \frac{2}{\sqrt{\pi}} e^{-p^2}$$

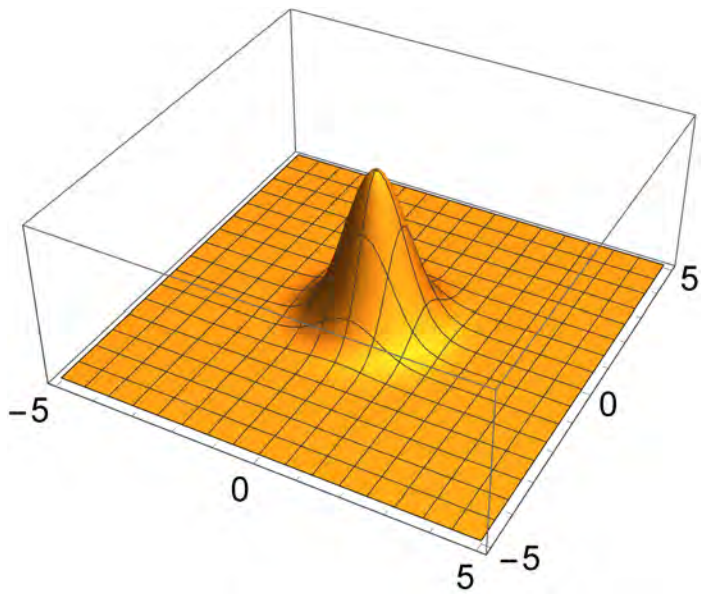
Problem 6.1

a)

In[149]:=
```
Wig[X_, P_] = E^(-P^2 - X^2)/π;
```
$$\text{Wig}[X\_, P\_] = \frac{e^{-P^2-X^2}}{\pi};$$

In[150]:=
```
P3 = Plot3D[Wig[x, p], {x, -5, 5}, {p, -5, 5},
  PlotRange → All, PlotPoints → 64, Ticks → {Automatic, Automatic, None},
  BaseStyle → {FontFamily → "Helvetica", FontSize → 18}]
```

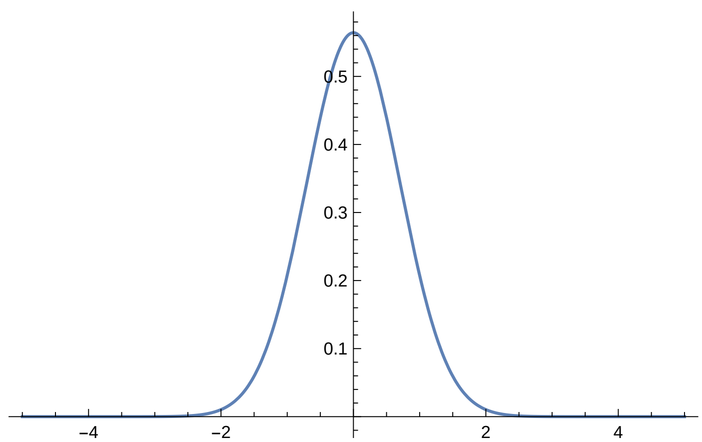
Out[150]=



In[152]:=
$$\text{prX} = \frac{e^{-X^2}}{\sqrt{\pi}};$$

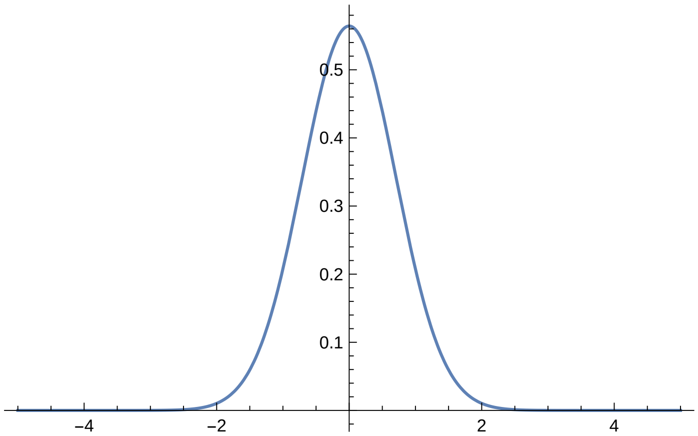
In[153]:=
```
Plot[prX, {X, -5, 5}]
```

Out[153]=



In[155]:=
```
prP = E^(-P^2)/Sqrt[π];
```
$$prP = \frac{e^{-P^2}}{\sqrt{\pi}};$$

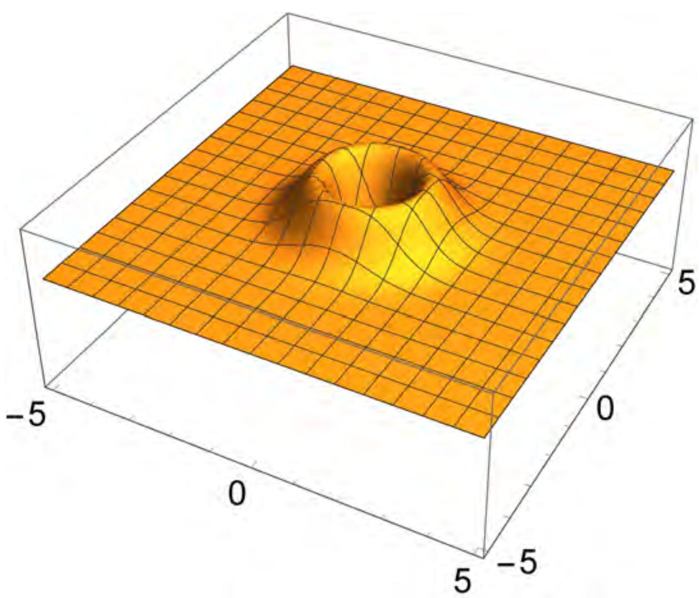
In[156]:=
```
Plot[prP, {P, -5, 5}, PlotRange → All]
```

Out[156]=


b)

In[157]:=
$$Wig[X\_, P\_] = \frac{e^{-P^2-X^2}\,(-1 + 2\,P^2 + 2\,X^2)}{\pi};$$

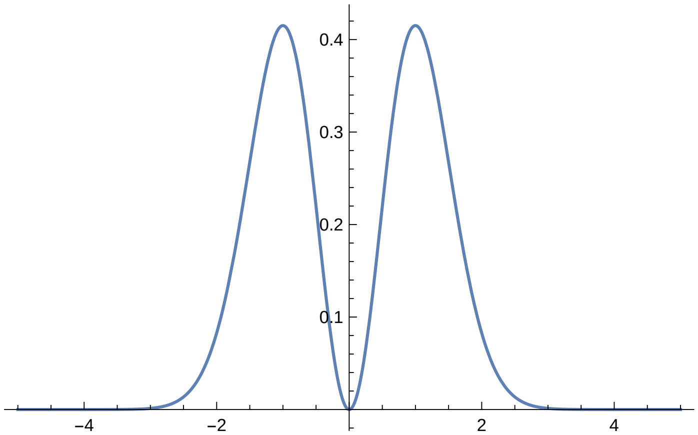
In[158]:=
```
P3 = Plot3D[Wig[x, p], {x, -5, 5}, {p, -5, 5},
  PlotRange → All, PlotPoints → 64, Ticks → {Automatic, Automatic, None},
  BaseStyle → {FontFamily → "Helvetica", FontSize → 18}]
```

Out[158]=


In[160]:=
$$prX = \frac{2\,e^{-X^2}\,X^2}{\sqrt{\pi}};$$

In[161]:=
```
Plot[prX, {X, -5, 5}]
```

Out[161]=

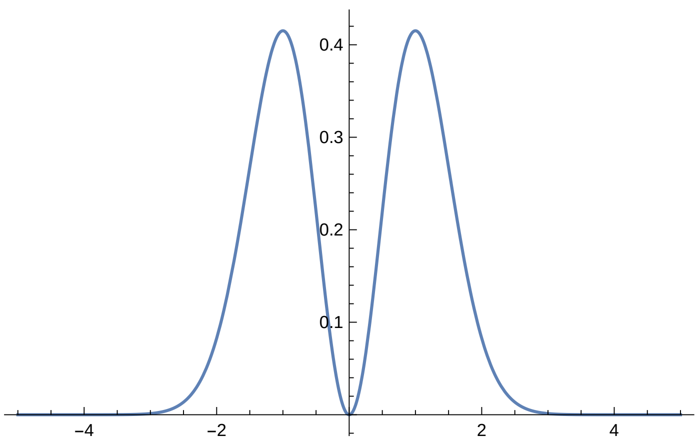


In[162]:=

$$\text{prP} = \frac{2\, e^{-P^2}\, P^2}{\sqrt{\pi}};$$

In[163]:=
```
Plot[prP, {P, -5, 5}, PlotRange → All]
```

Out[163]=

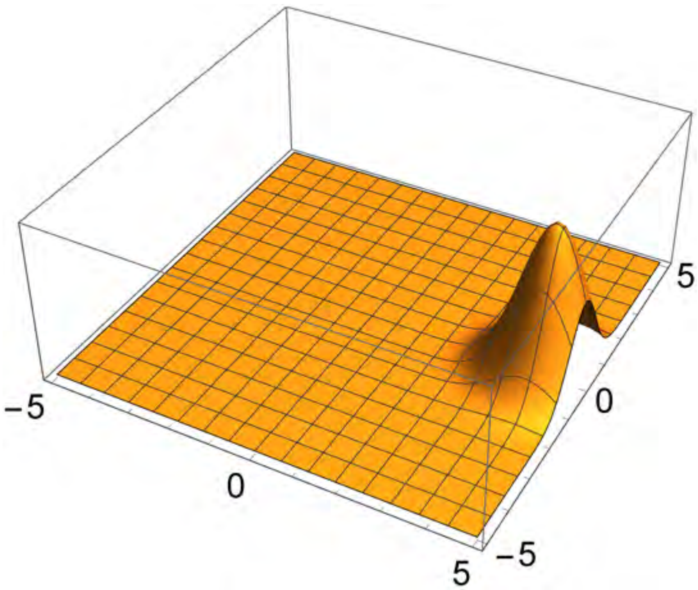


c)

$$\text{Wig}[X\_, P\_] = \frac{e^{-18-P^2+6\sqrt{2}\, X-X^2}}{\pi};$$

In[171]:=
```
P3 = Plot3D[Wig[x, p], {x, -5, 5}, {p, -5, 5},
  PlotRange → All, PlotPoints → 64, Ticks → {Automatic, Automatic, None},
  BaseStyle → {FontFamily → "Helvetica", FontSize → 18}]
```

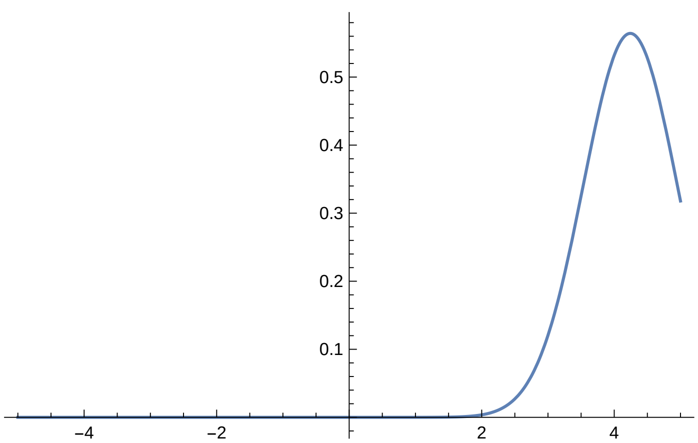
Out[171]=



In[173]:= prX = $\frac{e^{-18+6\sqrt{2}\,X-X^2}}{\sqrt{\pi}}$;

In[174]:=
```
Plot[prX, {X, -5, 5}]
```

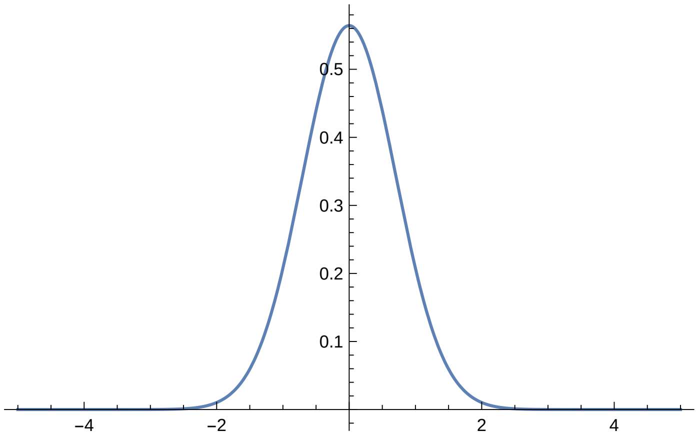
Out[174]=



In[176]:= prP = $\frac{e^{-P^2}}{\sqrt{\pi}}$;

In[177]:= `Plot[prP, {P, -5, 5}, PlotRange → All]`

Out[177]=



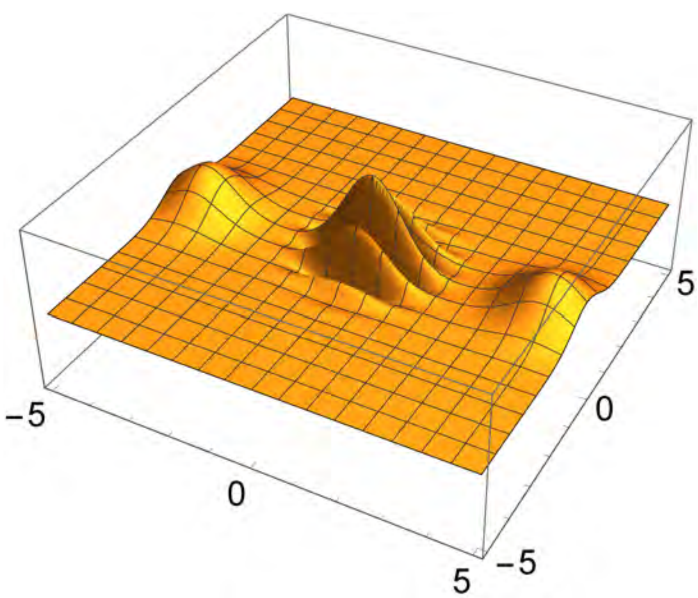
e)

In[178]:= $$\text{Wig}[X\_, P\_] = \frac{e^{-P^2-(X+X\alpha)^2}}{\pi} + \frac{e^{-P^2-(X-X\alpha)^2}}{\pi} + \frac{2\, e^{-P^2-X^2}}{\pi} \text{Cos}[2\, P\, X\alpha] \;/.\; X\alpha \to 3\sqrt{2}$$

Out[178]= $$\frac{e^{-P^2-\left(-3\sqrt{2}+X\right)^2}}{\pi} + \frac{e^{-P^2-\left(3\sqrt{2}+X\right)^2}}{\pi} + \frac{2\, e^{-P^2-X^2}\,\text{Cos}\left[6\sqrt{2}\, P\right]}{\pi}$$

In[179]:=
```
P3 = Plot3D[Wig[x, p] /. α → 3, {x, -5, 5}, {p, -5, 5},
  PlotRange → All, PlotPoints → 64, Ticks → {Automatic, Automatic, None},
  BaseStyle → {FontFamily → "Helvetica", FontSize → 18}]
```

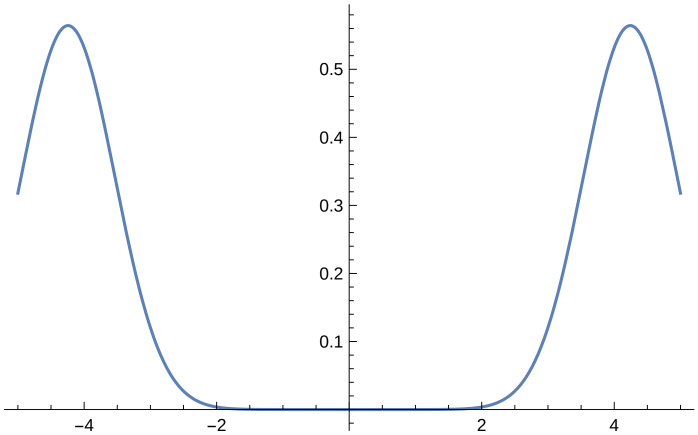
Out[179]=



In[182]:= $$\text{prX} = \frac{4\, e^{-18-X^2}\,\text{Cosh}\left[3\sqrt{2}\, X\right]^2}{\sqrt{\pi}};$$

In[183]:= `Plot[prX, {X, -5, 5}]`

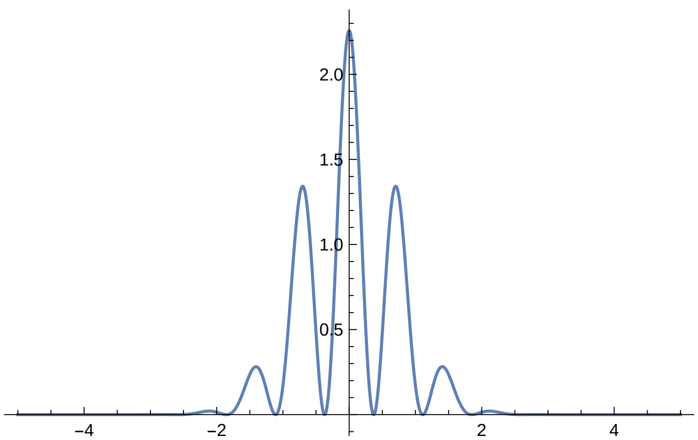
Out[183]=



In[186]:= $prP = \frac{4\, e^{-P^2} \text{Cos}\left[3\sqrt{2}\, P\right]^2}{\sqrt{\pi}};$

In[188]:= `Plot[prP, {P, -5, 5}, PlotRange → All]`

Out[188]=

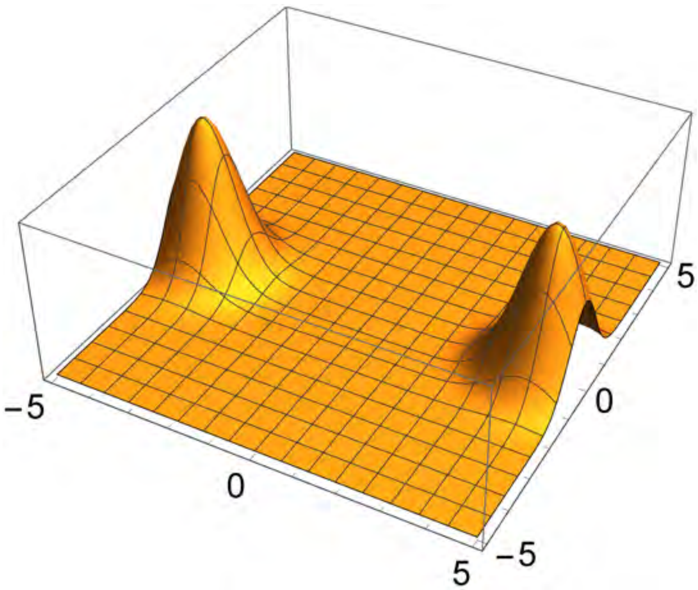


g)

In[189]:= $\text{Wig}[X\_, P\_] = \frac{e^{-P^2-(X+X\alpha)^2}}{\pi} + \frac{e^{-P^2-(X-X\alpha)^2}}{\pi} \text{ /. } X\alpha \to 3\sqrt{2};$

In[190]:=
```
P3 = Plot3D[Wig[x, p] /. α → 3, {x, -5, 5}, {p, -5, 5},
  PlotRange → All, PlotPoints → 64, Ticks → {Automatic, Automatic, None},
  BaseStyle → {FontFamily → "Helvetica", FontSize → 18}]
```

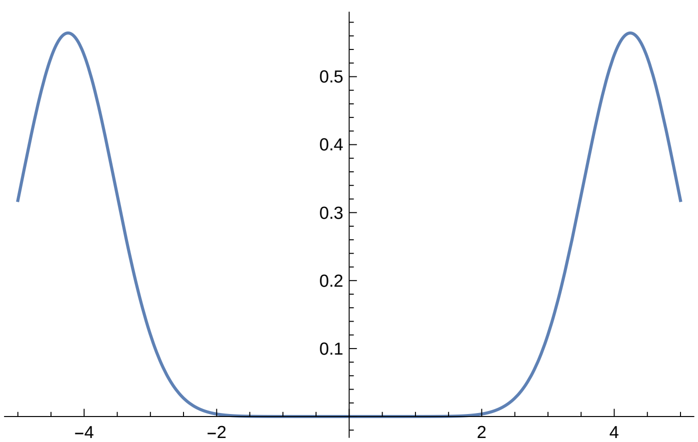
Out[190]=



In[193]:= $prX = \frac{2\, e^{-18-X^2}\, \text{Cosh}\left[6\sqrt{2}\, X\right]}{\sqrt{\pi}};$

In[194]:=
```
Plot[prX, {X, -5, 5}]
```

Out[194]=

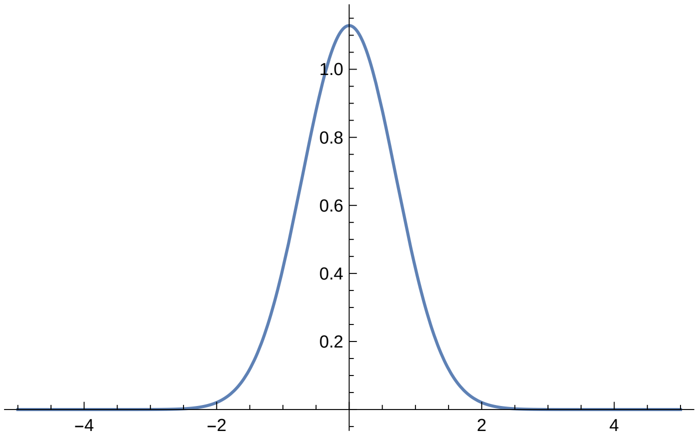


In[196]:= $prP = \frac{2\, e^{-P^2}}{\sqrt{\pi}}$

Out[196]= $\frac{2\, e^{-P^2}}{\sqrt{\pi}}$

In[197]:=
```
Plot[prP, {P, -5, 5}, PlotRange → All]
```

Out[197]=



2 a) $$N_1 = \frac{1}{\langle\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle + \langle\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle} = \frac{1}{2+2e^{-2\alpha^2}}$$

$$N_2 = 2$$

b) $$|\alpha\rangle = e^{-\alpha^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|-\alpha\rangle = e^{-\alpha^2/2} \sum_n \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$\sqrt{N_1}\,(|\alpha\rangle + |-\alpha\rangle) = \sqrt{N_1}\, e^{-\alpha^2/2} \sum_n \begin{cases} 2\alpha^n/\sqrt{n!}\,|n\rangle & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1| = N_1 e^{-\alpha^2} \sum_{m,n} \begin{cases} 4\alpha^n\alpha^m/\sqrt{n!m!}\,|n\rangle\langle m| & n \text{ and } m \text{ both even} \\ 0 & n \text{ or } m \text{ odd} \end{cases}$$

$$\hat{\rho}_2 = N_2 e^{-\alpha^2} \sum_{m,n} \frac{\alpha^n\alpha^m + (-\alpha)^n(-\alpha)^m}{\sqrt{n!\,m!}} |n\rangle\langle m|$$

$$= N_2 e^{-\alpha^2} \sum_{m,n} \begin{cases} 2\alpha^n\alpha^m/\sqrt{n!\,m!}\,|n\rangle\langle m| & n,m \text{ both odd or both even} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle x|\alpha\rangle = \pi^{-1/4} e^{-(x-\alpha\sqrt{2})^2/2}$$

$$\langle x|-\alpha\rangle = \pi^{-1/4} e^{-(x+\alpha\sqrt{2})^2/2}$$

$$\langle x|\rho_1|x'\rangle = \frac{N_1}{\sqrt{\pi}}\left(e^{-(x-\alpha\sqrt{2})^2/2} + e^{-(x+\alpha\sqrt{2})^2/2}\right)\left(e^{-(x'-\alpha\sqrt{2})^2/2} + e^{-(x'+\alpha\sqrt{2})^2/2}\right)$$

$$\langle x|\rho_2|x'\rangle = \frac{N_2}{\sqrt{\pi}}\left(e^{-(x-\alpha\sqrt{2})^2/2-(x'-\alpha\sqrt{2})^2/2} + e^{-(x+\alpha\sqrt{2})^2/2-(x'+\alpha\sqrt{2})^2/2}\right)$$

$$\langle p|\alpha\rangle = \pi^{-1/4} e^{i\alpha p\sqrt{2}} e^{-p^2/2}$$

$$\langle p|-\alpha\rangle = \pi^{-1/4} e^{i\alpha p\sqrt{2}} e^{-p^2/2}$$

$$\langle p|\psi_1\rangle = 2\pi^{-1/4} \cos(\alpha p\sqrt{2})\, e^{-p^2/2}$$

$$\langle p|\rho_1|p'\rangle = 4 N_1\sqrt{\pi} \cos(\alpha p\sqrt{2}) \cos(\alpha p'\sqrt{2})\, e^{-p^2/2-p'^2/2}$$

$$\langle p|\rho_2|p'\rangle = N_2\sqrt{\pi}\left(e^{-i\alpha(p-p')\sqrt{2}} + e^{i\alpha(p-p')\sqrt{2}}\right) e^{-p^2/2-p'^2/2}$$

$$= 2N_2\sqrt{\pi}\cos\alpha(p-p')\sqrt{2}\, e^{-p^2/2-p'^2/2}$$

3

a) $\hat{H} = \hbar\gamma\left(|\uparrow 0\rangle\langle\downarrow 1| + |\downarrow 1\rangle\langle\uparrow 0|\right)$

$$e^{-\frac{i}{\hbar}\hat{H}t} = \cos\gamma t\left(|\uparrow 0\rangle\langle\uparrow 0| + |\downarrow 1\rangle\langle\downarrow 1|\right)$$
$$+ i\sin\gamma t\left(|\uparrow 0\rangle\langle\downarrow 1| + |\downarrow 1\rangle\langle\uparrow 0|\right)$$
$$+ |\uparrow 1\rangle\langle\uparrow 1| + |\downarrow 0\rangle\langle\downarrow 0|$$

b) For i'th element of ensemble, system + environment at $t=0$:

$$|\Psi_i(0)\rangle = a_i|\uparrow 0\rangle + b_i|\downarrow 0\rangle$$

$$e^{-\frac{i}{\hbar}H\Delta t}|\Psi_i(0)\rangle = a_i \underbrace{\cos\gamma\Delta t}_{\equiv c}|\uparrow 0\rangle + i a_i \underbrace{\sin\gamma\Delta t}_{\equiv s}|\downarrow 1\rangle + b_i|\downarrow 0\rangle = |\Psi_i(\Delta t)\rangle$$

Trace over environment:

$$\rho_i(\Delta t) = Tr_E |\Psi_i(0)\rangle\langle\Psi_i(0)|$$
$$= \begin{pmatrix} |a_i|^2 c^2 & a_i b_i^* c \\ a_i^* b_i c & |b_i|^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & |a_i|^2 s^2 \end{pmatrix}$$

For $\gamma\Delta t \ll 1$, $c = \cos\gamma\Delta t \approx 1 - \frac{(\gamma\Delta t)^2}{2} \approx 1 - s^2/2$

$$c^2 \approx 1 - s^2$$

$$\Rightarrow \rho_i(\Delta t) \approx \begin{pmatrix} |a_i|^2(1-s^2) & a_i b_i^*(1-s^2/2) \\ a_i^* b_i(1-s^2/2) & |b_i|^2 + |a_i|^2 s^2 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{i\uparrow\uparrow}(1-s^2) & \rho_{i\uparrow\downarrow}(1-s^2/2) \\ \rho_{i\downarrow\uparrow}(1-s^2/2) & \rho_{i\downarrow\downarrow} + \rho_{\uparrow\uparrow}s^2 \end{pmatrix}$$

$$\Rightarrow \rho(\Delta t) = \begin{pmatrix} \rho_{\uparrow\uparrow}(1-s^2) & \rho_{\uparrow\downarrow}(1-s^2/2) \\ \rho_{\downarrow\uparrow}(1-\frac{s^2}{2}) & \rho_{\downarrow\downarrow} + \rho_{\uparrow\uparrow}s^2 \end{pmatrix}$$

c) For each interval of length τ:

$$\rho(t+\tau) = \begin{pmatrix} \rho_{\uparrow\uparrow}(t)(1-s^2) & \rho_{\uparrow\downarrow}(t)(1-s^2/2) \\ \rho_{\downarrow\uparrow}(t)(1-s^2/2) & \rho_{\downarrow\downarrow}(t) + \rho_{\uparrow\uparrow}(t)s^2 \end{pmatrix}$$

where $s = \sin\gamma\tau \approx \gamma\tau$.

For small $\Delta t \gg \tau$ ("small" means $\Delta t \ll 1/\gamma$)

$$\Delta \rho(\Delta t) = \frac{\Delta t}{\tau} \Delta \rho(\tau) = \begin{pmatrix} \rho_{\uparrow\uparrow}(t)\left(-\frac{\Delta t}{\tau}\right)(\gamma\tau)^2 & \rho_{\uparrow\downarrow}(t)\left(-\frac{\Delta t}{\tau}\right)\frac{(\gamma\tau)^2}{2} \\ \rho_{\downarrow\uparrow}(t)\left(-\frac{\Delta t}{\tau}\right)\frac{(\gamma\tau)^2}{2} & \rho_{\uparrow\uparrow}(t)\left(\frac{\Delta t}{\tau}\right)(\gamma\tau)^2 \end{pmatrix}$$

d) $$\frac{\Delta \rho}{\Delta t} = \begin{pmatrix} \rho_{\uparrow\uparrow}(t)(-\gamma^2\tau) & \rho_{\uparrow\downarrow}(t)(-\gamma^2\tau/2) \\ \rho_{\downarrow\uparrow}(t)(-\gamma^2\tau/2) & \rho_{\uparrow\uparrow}(t)(\gamma^2\tau) \end{pmatrix}$$

$$T_1 = \frac{1}{\gamma^2\tau} \qquad T_2 = \frac{2}{\gamma^2\tau}$$