University of Calgary
Winter semester 2017

PHYS 543: Quantum Mechanics II

Homework assignment 5

Due November 27, 2017

Problem 4.1. Calculate the Wigner functions of the states below. Calculate the probability densities $\mathrm{pr}(X)$ and $\tilde{\mathrm{pr}}(P)$ for the position and momentum from the states' wavefunctions or density matrices. Verify that these probability densities are obtained when the Wigner function is integrated over the other quadrature. For each case, plot the Wigner function and the two marginal distributions.

Do not use computers for calculations. Do use computers to generate graphics. Use $\alpha = 3$ *for all plots.*

a) vacuum state;

b) the single-photon state;

c) coherent state with a real, positive α;

d) coherent state with an imaginary α;

e) Schrödinger cat state $|\alpha\rangle - |-\alpha\rangle$ with a real α (neglect normalization).

f) the state with the density matrix $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$ with a real α (neglect normalization).

Useful integrals:

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ikx} e^{-x^2/4} \mathrm{d}x = \sqrt{2} e^{-k^2};$$

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} x^2 e^{ikx} e^{-x^2/4} \mathrm{d}x = -2\sqrt{2} e^{-k^2} \left(2k^2 - 1\right);$$

$$\int\limits_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

$$\int\limits_{-\infty}^{+\infty} x^2 e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Problem 4.2. Consider the states $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1| = \mathcal{N}_1(|\alpha\rangle + |-\alpha\rangle)(\langle\alpha| + \langle-\alpha|)$ and $\hat{\rho}_2 = \mathcal{N}_2(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)$ of a harmonic oscillator, where $|\pm\alpha\rangle$ are coherent states ($\alpha \in \mathbb{R}$) and $\mathcal{N}_{1,2}$ are normalization factors.

a) Find $\mathcal{N}_{1,2}$.
Hint: $\langle\alpha|\,\alpha'\rangle = e^{-|\alpha|^2/2 - |\alpha'|^2/2 + \alpha'\alpha^*}$.

b) Find the representation of the density operator of these states

- in the Fock basis;
- in the position basis;
- in the momentum basis.

Use software to make density plots of real parts of these density matrices/functions for $\alpha = 3$ in range $n \in [0, 10]$ in the Fock basis, $x, p \in [-5, 5]$ in the position and momentum bases.

Problem 4.3. As discussed in class, the relaxation time T_2 can be as high as $2T_1$ if the mechanism of thermalization is different from that of decoherence, i.e. if it cannot be modeled as gradual admixture of the thermal state to the spin ensemble. Here is an example to this effect.

The system is a two-level atom. The energy of the transition is far above the room temperature, so absolute zero temperature can be assumed. The environment consists of multiple electromagnetic modes, of which we take into account only those that are resonant with the atom. The environment modes are initially in the vacuum state and interact with the system via the energy exchange Hamiltonian

$$\hat{H} = \hbar\gamma(|\uparrow, 0\rangle\langle\downarrow, 1| + |\downarrow, 1\rangle\langle\uparrow, 0|). \tag{1}$$

Here $\{|\uparrow\rangle, |\downarrow\rangle\}$ are the excited and ground states of the atom and $\{|0\rangle, |1\rangle\}$ are the vacuum and single-photon states of the electromagnetic mode.

a) Find the evolution operator associated with this Hamiltonian.

b) Let the initial state of the system be

$$\hat{\rho}(0) \simeq \begin{pmatrix} \rho_{\uparrow\uparrow}(0) & \rho_{\uparrow\downarrow}(0) \\ \rho_{\downarrow\uparrow}(0) & \rho_{\downarrow\downarrow}(0) \end{pmatrix}$$

Find the state $\hat{\rho}(\Delta t)$ of the system assuming that the interaction has ended after a short time interval $\Delta t \ll \gamma^{-1}$, the environment mode has propagated away from the atom and all information about it is lost.
Hint: This is easier to solve if you write the initial state as an ensemble of pure states: $\hat{\rho}(0) = \sum_i p_i \hat{\rho}_i(0)$, where $\hat{\rho}_i(0) = (a_i |\uparrow\rangle + b_i |\downarrow\rangle)(a_i^* \langle\uparrow| + b_i^* \langle\downarrow|)$. Each such state will transform into a non-pure state $\hat{\rho}_i(\Delta t)$ after Δt. Find $\hat{\rho}_i(\Delta t)$ in terms of a_i and b_i, and then in terms of $\rho_{i,\uparrow\uparrow}(0), \ldots, \rho_{i,\downarrow\downarrow}(0)$. Finally, express $\hat{\rho}(\Delta t) = \sum_i p_i \rho_i(\Delta t)$ through $\rho_{\uparrow\uparrow}(0), \ldots, \rho_{\downarrow\downarrow}(0)$. Note that there is no need to explicitly find a_i and b_i.

c) Suppose that each environment mode repeatedly "propagates away" after a short interval $\tau \ll \Delta t$ and is replaced by a new one. Find $\hat{\rho}(\Delta t)$ under these conditions, still assuming $\Delta t \ll \gamma^{-1}$.

d) Based on the solution of part (c), write a set of differential equations describing the evolution of the density matrix elements as a function of time. Find T_1 and T_2.